



University of Barcelona

Research Group on Risk in Insurance and Finance www.ub.edu/riskcenter

Working paper 2015/07 \\ Number of pages 18

On the practical implementation of retirement gains by using an upside and a downside terminal wealth constraint

Catherine Donnelly, Montserrat Guillén, Jens Perch Nielsen and Ana María Pérez-Marín



On the practical implementation of retirement gains by using an upside and a downside terminal wealth constraint

Catherine Donnelly*
Montserrat Guillén†
Jens Perch Nielsen‡
Ana Maria Pérez-Marín§

Abstract

We analyze an investment strategy for an investor with a savings plan for retirement consisting on constraining the terminal wealth accumulated after the savings period by setting an upper and lower bound. We carry out a simulation of the terminal wealth after a savings period of thirty years by using daily, monthly, weekly and yearly updates. We calculate the percentiles of the final wealth and the corresponding lifetime annuity that the pension saver will receive during the consumption period. We observe how that the simulated values converge to the theoretical values of the percentiles when the frequency of update increases. Finally, in the numerical example the effect of inflation is also considered.

1 Introduction

We carry out a practical implementation of a new investment strategy for retirement where the investor establishes an upper and a lower bound on the terminal wealth. An investment plan is usually characterized by a period of savings followed by a period of consumptions. We analyze the problem of setting a dynamic investment strategy where an initial wealth is invested in order to reach a target capital at the end of the savings period which is bounded by some guaranteed upper and lower bounds. The proposed mechanism results in a transparent and automatic investment product where the portfolio is rebalanced automatically so that the accumulated wealth at any moment is constraint by the lower and upper bounds.

*Department of Actuarial Mathematics and Statistics, and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom. Email: C.Donnelly@hw.ac.uk. Tel: +44 131 451 3251. Fax: +44 131 451 3249.

†Department of Econometrics, Riskcenter-IREA, University of Barcelona, Avinguda Diagonal 690, 08034 Barcelona, Spain. Email: mguillen@ub.edu

‡Cass Business School, City University London, 106 Bunhill Row, London EC1Y 8TZ, United Kingdom. Email: Jens.Nielsen.1@city.ac.uk

§Department of Econometrics, Riskcenter-IREA, University of Barcelona, Avinguda Diagonal 690, 08034 Barcelona, Spain. Email: amperez@ub.edu

Donnelly et al. (2015) recently solved the portfolio selection problem of an investor with a deterministic savings plan who is constrained to have no more than a target wealth at retirement (an upper bound). Here we extend the results of Donnelly et al. (2015) by adding also a lower bound to the terminal wealth and implementing a numerical application. So, we assume that investors are willing to give up large gains if a minimum terminal wealth is guaranteed. Therefore, our contribution is about the reduction of the uncertainty of the terminal wealth being too large or too low, by providing a smoothing mechanism which includes an embedded guarantee on the terminal wealth.

The paper is organized as follows. In section 2 the background is presented. In section 3 we present the mathematical problem to solve. In section 4 we carry out the numerical illustration. Section 5 concludes.

2 Background

It is necessary to analyze the stochastic distribution of retirement wealth for proposing investment strategies (Greninger et al., 2000; Basu et al., 2011; Grossman and Zhou, 1996; Browne, 1999).

There are many authors introducing some constraint on the portfolio or the terminal wealth. Namely, Grossman and Zhou (1996) impose the constraint that the terminal wealth must be at least some fraction of the initial wealth. On the other hand, Korn and Trautmann (1995) set a constraint on the expected value of the final wealth.

Recently, Donnelly et al. (2015) found that by constraining the final wealth by using an upper bound, the investor increases their chance of attaining the desired target retirement wealth, and even if he fails to reach it, he still has a higher wealth than if he has no such upper bound. Note that Donnelly et al. (2015) proposed a different formulation compared to Dhaene et al. (2005), in which at least the target capital is attained with maximum probability. Donnelly et al. (2015) have also a different approach compared to Browne (1999), as Browne (1999) maximizes directly the probability of reaching the target retirement wealth. Here we consider the same approach as Donnelly et al. (2015) but adding also a lower bound for the final wealth. Note that our approach is also different from Gerrard et al. (2014) who analyzed the lowest part of the terminal wealth distribution after savings and consumption.

Here we concentrate on the savings phase (by choosing a saving period of thirty years) and we constrain the terminal wealth by using an upper and a lower bound. Additionally, we also provide the corresponding values of the lifetime annuities that the pension saver could receive during the consumption period. Other relevant contributions where some constrain on the terminal wealth is introduced can be found in Van Weert et al. (2010); Bouchard et al. (2010); Gaibh et al. (2009); Boyle and Tian (2007); Cuoco (1997); Zariphopoulou (1994), among others.

3 Presentation of mathematical problem to solve

We assume investment in a continuous-time market model over a finite time horizon $[0, T]$ for an integer $T > 0$. We also refer to T as the *terminal time*.

The market consists of one risky stock and one risk-free bond. The price of the stock is driven by an 1-dimensional, standard Brownian motion $W = \{W(t); t \in [0, T]\}$. The Brownian motion is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The risk-free bond has price process $\{S_0(t); t \in [0, T]\}$ and the risky stock has price process $\{S_1(t); t \in [0, T]\}$ with dynamics

$$dS_0(t) = r_N S_0(t) dt, \quad dS_1(t) = S_1(t) (\mu dt + \sigma dW(t)), \quad (3.1)$$

with $\sigma > 0$, $S_0(0) = 1$ and $S_1(0)$ being a fixed, strictly positive constant. We assume that $\mu > r_N$.

The information available to investors is represented by the filtration

$$\mathcal{F}_t := \sigma\{W(s), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0, T], \quad (3.2)$$

where $\mathcal{N}(\mathbb{P})$ denotes the collection of all \mathbb{P} -null events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Define the usual \mathbb{R} -valued market price of risk

$$\theta := \frac{\mu - r_N}{\sigma}.$$

3.1 Investor

An investor starts with a fixed non-random initial wealth $x_0 > 0$ and plans to make a sequence of known future savings $a > 0$. Define $C(t)$ to be the sum from time 0 to time t of the investor's planned discrete savings, with

$$dC(t) = \begin{cases} a & \text{if } t = 1, 2, \dots, T-1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, at the end of each unit time period, the investor pays an amount $a > 0$ into their fund.

A portfolio process $\pi = \{\pi(t); t \in [0, T]\}$ is a \mathbb{R} -valued, square-integrable, $\{\mathcal{F}_t\}$ -progressively measurable process. The investor follows a self-financed strategy, investing at each instant $t \in [0, T]$ a monetary amount $\pi(t)$ in the stock such that the $\pi = \{\pi(t); t \in [0, T]\}$ is a portfolio process.

The wealth process $X^\pi = \{X^\pi(t); t \in [0, T]\}$ corresponding to a portfolio process is the $\{\mathcal{F}_t\}$ -adapted, \mathbb{R} -valued process given by the *wealth equation*

$$dX^\pi(t) = (r_N X^\pi(t) + \pi(t)\sigma\theta) dt + \pi(t)\sigma dW(t) + dC(t), \quad X^\pi(0) = x_0 \text{ a.s.} \quad (3.3)$$

Define the *savings plan* g of the investor, i.e. the discounted sum of the future savings by the investor by

$$g(t) := \int_t^T e^{-r_N(s-t)} dC(s), \quad \forall t \in [0, T]. \quad (3.4)$$

Then the set of *admissible portfolios* for the investor's initial wealth $x_0 > 0$ is defined to be

$$\mathcal{A} := \{\pi : \Omega \times [0, T] \rightarrow \mathbb{R} : X^\pi(0) = x_0, \text{ a.s. and } X^\pi(t) + g(t) \geq 0, t \in (0, T] \text{ a.s.}\}.$$

We say that a portfolio process π is *admissible* if $\pi \in \mathcal{A}$.

Define the state price density process H as $H(t) := \exp\left(-\left(r_N + \frac{1}{2}\theta^2\right)t - \theta W(t)\right)$, for each $t \in [0, T]$. A portfolio π must satisfy the *budget constraint* that

$$\mathbb{E}(H(T)X^\pi(T)) \leq x_0 + g(0). \quad (3.5)$$

The utility function of the investor is the power utility function

$$U(x) := \frac{1}{\gamma}x^\gamma, \quad x > 0,$$

for a fixed constant $\gamma \in (-\infty, 1) \setminus \{0\}$. The investor seeks to maximise the expected utility of their terminal wealth, subject to constraints on the range of values of the terminal wealth.

Define the constant

$$A := \frac{\theta}{\sigma(1-\gamma)}$$

and the process

$$Z(t) = \exp\left(\left(r_N + \theta\sigma A - \frac{1}{2}\sigma^2 A^2\right)t + \sigma AW(t)\right), \quad \forall t \in [0, T]. \quad (3.6)$$

3.2 Problem with an upper bound

Donnelly et al. (2015) introduced the constrained problem with an upper bound only, in which the investor seeks to maximize the expected utility of their terminal wealth, subject to the wealth being bounded above by the upper bound $K_U > 0$.

In order to avoid the uninteresting case that the investor can immediately be assured of maximizing the terminal utility, Donnelly et al. (2015) assume that $(x_0 + g(0))e^{r_N T} < K_U$ and solve the following problem (Problem 4.1 in Donnelly et al. (2015)):

Problem 3.1. Find $\pi^* \in \mathcal{A}$ such that

$$\mathbb{E}\left(U(X^{\pi^*}(T))\right) = \sup_{\pi \in \mathcal{A}} \{\mathbb{E}(U(X^\pi(T)))\},$$

and $X^{\pi^*}(T) \in [0, K_U]$, a.s.

Proposition 3.2. *Proposition 4.5 in Donnelly et al. (2015). An optimal investment strategy for Problem 3.1 is to invest the amount*

$$\pi^*(t) := A[1 - \Phi(d_+(t, P(t); K_U))] P(t) \quad (3.7)$$

in the risky stock and the amount $X^{\pi^}(t) - \pi^*(t)$ in the risk-free bond, in which $P(t) = (z_0 + g(0))Z(t)$ and the function d_+ is defined by*

$$d_\pm(t, y; K_U) := \frac{1}{\sigma A \sqrt{T-t}} \left(\ln\left(\frac{y}{K_U}\right) + \left(r_N \pm \frac{1}{2}\sigma^2 A^2\right)(T-t) \right), \quad \forall y > 0. \quad (3.8)$$

The wealth process corresponding to this optimal investment strategy is

$$X^{\pi^*}(t) = P(t) - g(t) - c(t, P(t); K_U), \quad (3.9)$$

in which

$$c(t, y; K_U) := y\Phi(d_+(t, y; K_U)) - K_U e^{-r_N(T-t)}\Phi(d_-(t, y; K_U)),$$

and $\Phi(z)$ denotes the cumulative standard normal distribution function at $z \in \mathbb{R}$.

In particular, the relationship between the investor's initial wealth $X^{\pi^*}(0) = x_0$ and the shadow initial wealth z_0 is

$$x_0 = z_0 - c(0, z_0 + g(0); K_U). \quad (3.10)$$

The proof is found in Donnelly et al. (2015).

3.2.1 Problem with a lower and an upper bound

Here we extend of the problem to include a lower bound $K_L \in (0, K_U)$, below which the terminal wealth must not fall. Combined with the upper bound K_U , this means that the investor's terminal wealth lies in the range $[K_L, K_U]$.

The addition of a lower bound has already been well studied in the literature (for example, see (Basak, 1995)).

In order to avoid both the uninteresting case that the investor can immediately be assured of maximizing the terminal utility and the breaching of the non-arbitrage condition, we assume that

Assumption 3.3. $K_L < (x_0 + g(0)) e^{r_N T} < K_U$.

Problem 3.4. Find $\pi^* \in \mathcal{A}$ such that

$$\mathbb{E} \left(U(X^{\pi^*}(T)) \right) = \sup_{\pi \in \mathcal{A}} \{ \mathbb{E} (U(X^\pi(T))) \},$$

and $X^{\pi^*}(T) \in [K_L, K_U]$, a.s.

The next proposition gives an expression for the optimal terminal wealth for Problem 3.4, when there is both a lower and upper bound constraint on the terminal wealth.

Proposition 3.5. A solution to the constrained problem at the terminal time T is

$$\begin{aligned} X^*(T) = & (z_0 + g(0))Z(T) - \max \{0, (z_0 + g(0))Z(T) - K_U\} \\ & + \max \{0, K_L - (z_0 + g(0))Z(T)\}, \end{aligned} \quad (3.11)$$

with the shadow wealth $z_0 > 0$ chosen so that the budget constraint (3.5) is satisfied with equality by X^* , given the investor's initial wealth $X^*(0) = x_0$, a.s. and savings plan g .

The proof is found in Appendix A. Next we derive the value and replicating portfolio of the put option with maturity value $\max\{0, K_L - (z_0 + g(0))Z(T)\}$.

Lemma 3.6. *The price at time $t \in [0, T]$ of a European put option with maturity value $\max\{0, K_L - (z_0 + g(0))Z(T)\}$ is given by $p(t, P(t); K_L)$ with*

$$p(t, y; K_L) := K_L e^{-r_N(T-t)} \Phi(-d_-(t, y; K_L)) - y \Phi(-d_+(t, y; K_L)).$$

with $d_{\pm}(t, y; K_L)$ defined by equation (3.8).

The replicating portfolio for the put option is to hold in the risky asset at time t the amount $\pi_p(t, P(t); K_L)$, with

$$\pi_p(t, y; K_L) := -Ay \Phi(-d_+(t, y; K_L)), \quad \forall t \in [0, T], \quad y > 0 \quad (3.12)$$

and the remaining amount $p(t, P(t); K_L) - \pi_p(t, P(t); K_L)$ in the risk-free bond.

The proof is found in Appendix A. The optimal strategy for Problem 3.4 is given next.

Proposition 3.7. *An optimal investment strategy for Problem 3.4 is to invest the amount*

$$\pi^*(t) := A[1 - \Phi(d_+(t, P(t); K_U) - \Phi(-d_+(t, P(t); K_L)))] P(t) \quad (3.13)$$

in the risky stock and the amount $X^{\pi^*}(t) - \pi^*(t)$ in the risk-free bond, in which $P(t) = (z_0 + g(0))Z(t)$ and the function d_+ is defined by equation (3.8).

The wealth process corresponding to this optimal investment strategy is

$$X^{\pi^*}(t) = P(t) - g(t) - c(t, P(t); K_U) + p(t, P(t); K_L). \quad (3.14)$$

In particular, the relationship between the investor's initial wealth $X^{\pi^*}(0) = x_0$ and the shadow initial wealth z_0 is

$$x_0 = z_0 - c(0, z_0 + g(0); K_U) + p(0, z_0 + g(0); K_L). \quad (3.15)$$

The proof follows trivially from the previous lemmas.

The relative value of the shadow initial wealth z_0 over the investor's actual initial wealth x_0 has a concrete interpretation. For the p -quantiles of the constrained terminal wealth that fall below the target wealth K_U , it gives their uplift over those for the unconstrained terminal wealth.

To see this, we calculate the p -quantiles under the constrained strategy. For the constrained strategy, there is a probability mass at the target wealth K_U . For this reason we use the following generalised definition of the p -quantile.

Definition 3.8. The p -quantile for a random variable X is

$$\mathcal{Q}_p(X) = \inf \{y \in \mathbb{R} : \mathbb{P}[X \leq y] \geq p\},$$

with the convention that $\inf \{\emptyset\} = \infty$.

Proposition 3.9 (*p*-quantiles). *Suppose an investor has initial wealth $x_0 > 0$ and follows the savings plan g . Define*

$$\beta_p := \sigma A \sqrt{T} \Phi^{-1}(p) + \left(r_N + \theta \sigma A - \frac{1}{2} \sigma^2 A^2 \right) T. \quad (3.16)$$

If the investor follows the optimal constrained strategy, i.e. the terminal wealth is constrained to lie in the range $[K_L, K_U]$, then the p -quantile of the investor's terminal wealth $X(T)$ is

$$\mathcal{Q}_p(X(T); (K_L, K_U)) = \max \{ K_L, \min \{ K_U, (z_0 + g(0))e^{\beta_p} \} \}. \quad (3.17)$$

The proof is found in Appendix A.

4 Numerical illustration

In this example, we fix the parameter values $r_N = 0$, $\mu = 0.0343$, $\sigma = 0.1544$, $A = 1$, $T = 30$, $g = 0$ and $x_0 = 300$. Note that the choice of the parameters implies that the investor's risk aversion constant is $\gamma = -0.44$.

Here we describe the algorithm that calculates the real wealth accumulated by the investor after the investment period. The values of K_L and K_U are set to 225 and 450 respectively. For these values of K_L and K_U we calculate the shadow initial wealth z_0 satisfying (3.15), which results $z_0 = 302.2626$.

We do 10000 simulations of a T -dimensional vector of standard normal random values $W(t)$, $t = 1, \dots, T$. Then, we simulate the process $S(t)$ according to $S(t) = S(t-1) \exp((r_N + \theta \sigma A - \frac{1}{2} \sigma^2 A^2)t + \sigma A W(t))$ for $t = 2, \dots, T$ where $\theta = \frac{\mu - r_N}{\sigma}$ and $S(1) = 1$. The real wealth process X_t is initialized with the value of $x_0 = 300$. We calculate the amount invested in stocks in the first investment period, $\pi(1)$ by using expression (3.13) for $t = 1$ where $P(1) = (z_0 + g(0))S(1)$ and with the two following restrictions: 1) this amount cannot be higher than the initial level of real wealth ($x_0 = 300$), and 2) this amount must be positive. Then, the simulation loop starts and for each moment in time $j = 2, \dots, 30$ we calculate which is the amount in stocks just before time j (updating the amount in stock in the previous period $\pi(j-1)$ by using the factor $S(j)/S(j-1)$). The amount in bonds just before time j is given by the difference between the real wealth and the amount in stocks at that time. Then, the amount invested in stocks for the next period is calculated by using expression (3.13) but with the following two restrictions: 1) this amount cannot be larger than the current real wealth, and 2) this amount must be positive. Then, the loop jumps to the next period. When the algorithm is finished, we calculate the quantiles of the real wealth at $T = 30$. The quantiles, obtained by simulating the final wealth, can be compared to the theoretical ones, those resulting from expression 3.17.

The algorithm is summarized as follows:

1. Initialize parameters ($r_N = 0$, $\mu = 0.0343$, $\sigma = 0.1544$, $A = 1$, $T = 30$, $g = 0$, $x_0 = 300$, $K_L = 250$ and $K_U = 415$.)

2. Compute z_0 from $x_0 = z_0 - c(0, z_0 + g(0); G) + p(0, z_0 + g(0); F)$.
3. Do 10000 replications.
4. Simulate $S(t)$ from $S(t) = S(t-1)exp((r_N + \theta\sigma A - \frac{1}{2}\sigma^2 A^2)t + \sigma AW(t))$ for $t = 2, \dots, T$ where $\theta = \frac{\mu - r_N}{\sigma}$ and $S(1) = 1$.
5. Calculate $\pi(1) = max(min(A(1 - \Phi(d_+(1, P(1)); K_U) - \Phi(-d_+(1, P(1); K_L)))P(1), x_0)0)$, where $P(1) = z_0 S(1)$.
6. Compute real wealth as the initial wealth x_0 .
7. For $t = 2, \dots, 30$
 - 7.1 Calculate the value invested in bonds as current real wealth minus $\pi(t-1)$.
 - 7.2 Compute the value of $\pi(t-1)$ after period $t-1$.
 - 7.3 Update real wealth at t as the sum of the two.
 - 7.4 Calculate $\pi(t)$ from $\pi(t) = max(min(A(1 - \Phi(d_+(t, P(t)); K_U) - \Phi(-d_+(t, P(t); K_L)))P(t), RealWealth(t), 0)$, where $P(t) = (z_0 + g(0))S(t)$.
8. End of loop.
9. End of loop.
10. Compute quantiles of the real wealth at $T = 30$.

Table 1 shows the distribution of the real wealth at $T = 30$ for $K_L = 250$ and $K_U = 415$. The first column shows the values of p for the p -quantiles. The second column shows the theoretical quantiles obtained by using expression 3.17. The next four columns show the quantiles of the simulated final wealth at $T = 30$ for different updates: yearly, monthly, weekly and daily. Note how the values of the quantiles converge to the theoretical values as the frequency of the update increases.

We have also calculated the life-long annuity payable every year that a 65 year old investor will receive corresponding the final wealth of $X(T)$ accumulated after the investment period. The expression is:

$$Annuity(Age = 65, X(T)) = \frac{X(T)}{\sum_{t=66}^{111} (1+r)^{-(t-65)} {}_{t-65}P_{65}}$$

where $r = 0$ and the values for the survival probabilities ${}_{t-65}P_{65}$ have been calculated by using the Society of Actuaries Life Table available in the *lifecontingencies* R package. The denominator in the previous expression is obtained by using the *axn* function of the *lifecontingencies* R package.

R code for $X(T) = 415$:

```
library(lifecontingencies)
data(soaLt)
soa08Act=with(soaLt, new("actuarialtable",interest=0,x=x,lx=Ix,name="SOA2008"))
```

Table 1: Table showing the distribution of the real wealth at $T = 30$ for $K_L = 250$ and $K_U = 415$.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$Q_p(X(T))$				
1%	250.0000	241.7170	246.1227	248.0315	249.0233
2.5%	250.0000	246.4381	248.1896	248.9851	249.5117
5%	250.0000	249.7004	249.3909	249.6616	249.8114
10%	250.0000	254.9273	250.8434	250.3659	250.1304
15%	250.0000	267.4020	254.2965	252.4664	251.3356
20%	290.3133	292.8727	283.4717	284.7746	285.8970
25%	334.3877	324.8488	328.2672	328.3217	328.3625
30%	379.6421	354.7174	371.8317	373.9506	376.4171
35%	415.0000	377.6660	402.7025	409.3999	412.5361
40%	415.0000	392.5784	409.1703	412.2969	413.8254
45%	415.0000	400.6617	411.3917	413.2157	414.2071
50%	415.0000	406.0062	412.6297	413.7881	414.4618
55%	415.0000	409.7468	413.5108	414.2082	414.6521
60%	415.0000	412.5056	414.2754	414.5923	414.8100
65%	415.0000	415.1566	414.9827	414.9076	414.9505
70%	415.0000	417.7468	415.6342	415.2297	415.0977
75%	415.0000	420.3110	416.3125	415.5666	415.2401
80%	415.0000	423.2772	417.0502	415.9280	415.4017
85%	415.0000	426.4701	417.9061	416.3479	415.5859
90%	415.0000	430.3906	419.1283	416.9107	415.8299
95%	415.0000	437.4809	421.1314	417.8511	416.2652
97.5%	415.0000	444.0110	423.4497	418.8456	416.6837
99%	415.0000	453.9527	426.0071	420.1559	417.3608

```

axn(soa08Act, x=65, payment="arrears")
15.02172
415/axn(soa08Act, x=65, payment="arrears")
27.62666

```

The results are shown in Table 3. in Appendix B.

4.1 Inflation

Critically, when income streams are presented in real terms, retirement investors choose flat or increasing *real* income streams over decreasing ones (Beshears et al., 2014). However, in practice, income streams are mostly presented in nominal terms rather than in real terms. It is difficult for the average person to understand the potential heavy toll of inflation on their retirement income, even though expected lifetimes are increasing.

We believe that the standard investment strategies followed by retirement investors are sub-optimal. The investment strategy followed by individual who wants a real income stream in retirement can have a considerable weighting of inflation-indexed assets. Today, the only inflation-indexed assets that are widely traded in the market are government-issued index-linked bonds (e.g. TIPS, OATis, ILGs). However, index-linked bonds give a negative real return to investors. They are highly demanded and in very short supply.

We believe that if inflation funds were introduced to the market, retirement investors would invest heavily in them. These funds could consist of real assets, such as infrastructure, commodities, equities, property. They would aim to broadly track price inflation. They would aim to give a positive real return over price inflation, although with some volatility around this.

The potential amount invested in inflation funds could be enormous, e.g. the total value of pension assets in the US was around 108% in 2012 (around \$16 851 billion in 2012) according to Towers Watson (2013). The US index-linked bond issuance is about 5% of this amount, e.g. in April 2012 it was \$866 billion (Krämer, 2013). Quite simply there is a massive potential demand for inflation-indexed assets that cannot be satisfied by the current volume of inflation-linked bonds.

Here we modify our algorithm so that some part of the wealth could be investing in an inflation fund. The algorithm is essentially the same, but now we assume that the amount not invested in stocks is giving some return. To do so, we first do 10000 simulations of a bivariate normal sample of size T , $(W_1(t), W_2(t))$, $t = 1, \dots, T$ with mean values equal to 0, standard deviations equal to 1 and correlation ρ equal to 0.5. Then, with these values we simulate the stock process $S_1(t)$ according to $S_1(t) = S_1(t-1)exp((r + \theta_1 \sigma_1 A - \frac{1}{2} \sigma_1^2 A^2)t + \sigma_1 A W_1(t))$ for $t = 2, \dots, T$ where $\mu_1 = 0.0343$, $\sigma_1 = 0.1544$, $\theta_1 = \frac{\mu_1 - rN}{\sigma_1}$ and $S_1(1) = 1$. Then, we also simulate the values of the inflation fund process $S_2(t)$ according to $S_2(t) = S_2(t-1)exp((r + \theta_2 \sigma_2 A - \frac{1}{2} \sigma_2^2 A^2)t + \sigma_2 A W_2(t))$ for $t = 2, \dots, T$ where $\mu_2 = 0.008$, $\sigma_2 = 0.02$, $\theta_2 = \frac{\mu_2 - rN}{\sigma_2}$ and $S_2(1) = 1$. Then, we use the values of $S_1(t)$, $t = 1, \dots, T$ to update the amount invested in stocks and $S_2(t)$, $t = 1, \dots, T$ to update the rest of money every year.

Namely, in the algorithm previously described, we introduce the following changes:

1. Initialize parameters ($r_N = 0$, $\mu_1 = 0.0343$, $\sigma_1 = 0.1544$, $\mu_2 = 0.008$, $\sigma_2 = 0.02$, $A = 1$, $T = 30$, $g = 0$, $x_0 = 300$, $K_L = 250$ and $K_U = 415$.)
4. Simulate $S_1(t) = S_1(t-1)\exp((r_N + \theta_1\sigma_1A - \frac{1}{2}\sigma_1^2 A^2)t + \sigma_1AW_1(t))$ for $t = 2, \dots, T$ and $S_2(t) = S_2(t-1)\exp((r_N + \theta_2\sigma_2A - \frac{1}{2}\sigma_2^2 A^2)t + \sigma_2AW_2(t))$ for $t = 2, \dots, T$ where $\theta_1 = \frac{\alpha_1 - r_N}{\sigma_1}$ and $\theta_2 = \frac{\alpha_2 - r_N}{\sigma_2}$.
5. Calculate $\pi(1) = \max(\min(A(1 - \Phi(d_+(1, P(1)); K_U) - \Phi(-d_+(1, P(1); K_L))))P(1), x_0)0$, where $P(1) = z_0S_1(1)$.
- 7.1. Subtract from the real wealth the amount $\pi(t-1)$ and update this difference by multiplying it by $S_2(t)/S_2(t-1)$.
- 7.4. Calculate $\pi(t)$ from $\pi(t) = \max(\min(A(1 - \Phi(d_+(t, P(t)); K_U) - \Phi(-d_+(t, P(t); K_L))))P(t), RealWealth(t), 0)$, where $P(t) = (z_0 + g(0))S_1(t)$.

Table 2 shows the quantiles obtained by simulating the final wealth at $T = 30$, with $K_L = 250$ and $K_U = 415$ assuming that the amount not invested in stocks is invested in an inflation fund with 0.8% return and 2% volatility. The correlation used to simulate the bivariate normal distribution (as explained previously in the algorithm) is 0.5.

We have also calculated the life-long annuity payable every year that a 65 year old investor will receive for the final wealth of $X(T)$ accumulated after the investment period. The results are shown in Table 4 in Appendix B where the value of r has been changed to 0.8% (the return assumed for the inflation fund).

5 Conclusion

We have shown the practical implementation of a new investment strategy that has the advantage of constraining the final wealth accumulated after the investment period between a lower and an upper bound. In this way, the saver is protected against extreme values, by providing a smoothing mechanism which includes an embedded guarantee on the terminal wealth.

Another advantage of the proposed strategy is that the portfolio is rebalanced automatically so that the accumulated wealth at any moment is constraint by the lower and upper bounds. We have also illustrated how the accumulated wealth can be translated into a life-long annuity that the investor will receive, which is easy to understand and communicate, increasing the transparency of the investment mechanism.

References

- Basak, S. (1995). A general equilibrium model of portfolio insurance. *The Review of Financial Studies*, 8(4):1059–1090.
- Basu, A., Byrne, A., and Drew, M. (2011). Dynamic lifecycle strategies for target date retirement funds. *Journal of Portfolio Management*, 37:83–96.
- Beshears, J., Choi, J., Laibson, D., Madrian, B., and Zeldes, S. (2014). What makes annuitization more appealing? *Journal of Public Economics*, 116:2–16.

Table 2: Table showing the distribution of the real wealth at $T = 30$ for $K_L = 250$ and $K_U = 415$. Yearly: with inflation 0.8% return, 2% volatility, $\rho = 0.5$

Update:	Yearly
p	$Q_p(X(T))$
1%	251.8884
2.5%	263.6504
5%	276.6988
10%	295.6553
15%	315.1019
20%	339.5316
25%	368.7824
30%	398.2215
35%	421.3348
40%	438.5349
45%	451.9557
50%	462.9167
55%	472.8325
60%	481.4466
65%	490.6310
70%	499.7588
75%	509.7144
80%	519.8703
85%	532.6464
90%	547.7482
95%	570.9136
97.5%	589.3045
99%	612.7947

- Bouchard, B., Elie, R., and Imbert, C. (2010). Optimal control under stochastic target constraints. *SIAM Journal on Control and Optimization*, 48:3501–3531.
- Boyle, P. and Tian, W. (2007). Portfolio management with constraints. *Mathematical Finance*, 17:319–343.
- Browne, S. (1999). Reaching goals by a deadline: digital options and continuous-time active portfolio management. *Advances in Applied Probability*, 31:551–577.
- Cuoco, D. (1997). Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. *Journal of Economic Theory*, 72:33–73.
- Dhaene, J., Vanduffel, S., Goovaerts, M., Kaas, R., and Vyncke, D. (2005). Comonotonic approximations for optimal portfolio selection problems. *Journal of Risk and Insurance*, 72:253–300.
- Donnelly, C., Gerrard, R., Guillen, M., and J.P., N. (2015). Less is more: increasing retirement gains by using an upside terminal wealth constraint. *Insurance: Mathematics and Economics*, in press.
- Gaïbh, A., Sass, J., and Wunderlich, R. (2009). Utility maximization under bounded expected loss. *Stochastic Models*, 25:375–407.
- Gerrard, R., Guillén, M., Nielsen, J., and Pérez-Marín, A. (2014). Long-run savings and investment strategy optimization. *The Scientific World Journal*. Article ID 510531.
- Greninger, S., Hampton, V., Kitt, K., and Jacquet, S. (2000). Retirement planning guidelines: a delphi study of financial planners and educators. *Financial Services Review*, 6:231–245.
- Grossman, S. and Zhou, Z. (1996). Equilibrium analysis of portfolio insurance. *Journal of Finance*, 51:1379–1403.
- Korn, R. and Trautmann, S. (1995). Continuous-time portfolio optimization under terminal wealth constraints. *Mathematical Methods of Operations Research*, 42:69–92.
- Krämer, W. (2013). An introduction to inflation-linked bonds. http://www.lazardnet.com/docs/sp0/6034/AnIntroductionToInflation-LinkedBonds_LazardResearch.pdf?pagename=Investment+Research.
- Towers Watson (2013). Global pensions asset study 2013. <http://www.towerswatson.com/en/Insights/IC-Types/Survey-Research-Results/2013/01/Global-Pensions-Asset-Study-2013>.
- Van Weert, K., Dhaene, J., and Goovaerts, M. (2010). Optimal portfolio selection for general provisioning and terminal wealth problems. *Insurance: Mathematics and Economics*, 47:90–97.
- Zariphopoulou, T. (1994). Consumption-investment models with constraints. *SIAM Journal on Control and Optimization*, 32:59–85.

A Proof of optimal investment strategy in simple financial market model

Proof. of Proposition 3.5.

The proof is an adaption of the proof of [cite relevant proposition in the first paper].

Assume that we have chosen $z_0 > 0$ chosen so that the budget constraint (3.5) is satisfied with equality by X^* .

For the investor's utility function, the first derivative $U'(x) = x^{\gamma-1}$, which is a strictly decreasing function, has a strictly decreasing inverse

$$I(y) := y^{\frac{1}{\gamma-1}}, \quad y > 0.$$

We can show that for the constant

$$y := (z_0 + g(0))^{\gamma-1} e^{(\gamma r_N + \frac{1}{2} \frac{\gamma}{1-\gamma} \theta^2)T},$$

we have

$$(z_0 + g(0))Z(T) = I(yH(T)).$$

We work with $I(y(z_0)H(T))$ in the proof, rather than with $(z_0 + g(0))Z(T)$ due to the properties of $I(x)$ and $U'(x)$: they are both strictly decreasing functions of x .

Let $X(T) \in [K_L, K_U]$, a.s. be any attainable final wealth so that $\mathbb{E}(H(T)X(T)) \leq x_0$. We show that

$$\mathbb{E}(U(X(T))) \leq \mathbb{E}(U(X^*(T))),$$

in which

$$X^*(T) = \begin{cases} K_L & I(yH(T)) \leq K_L \\ I(yH(T)) & \text{if } I(yH(T)) \in (K_L, K_U) \\ K_U & I(yH(T)) \geq K_U. \end{cases}$$

As I and U' are strictly decreasing functions we can write:

$$X^*(T) = \begin{cases} K_L & yH(T) \geq U'(K_L) \\ I(yH(T)) & \text{if } yH(T) \in (U'(K_L), U'(K_U)) \\ K_U & \text{if } yH(T) \leq U'(K_U) \end{cases}$$

As U is a concave function then for any $a, b \in \mathbb{R}$, $U(a) - U(b) \leq U'(b) \cdot (a - b)$. In particular,

$$U(X(T)) - U(X^*(T)) \leq U'(X^*(T)) \cdot (X(T) - X^*(T)), \quad \text{a.s.}$$

Take expectations:

$$\begin{aligned} & \mathbb{E}(U(X(T)) - U(X^*(T))) \\ & \leq \mathbb{E}(U'(X^*(T)) \cdot (X(T) - X^*(T))) \\ & \leq \mathbb{E}(U'(X^*(T)) \cdot (X(T) - X^*(T)) \mid yH(T) \geq U'(K_L)) \cdot \mathbb{P}[yH(T) \geq U'(K_L)] \\ & \quad + \mathbb{E}(U'(X^*(T)) \cdot (X(T) - X^*(T)) \mid yH(T) \in (U'(K_L), U'(K_U))) \cdot \mathbb{P}[yH(T) \in (U'(K_L), U'(K_U))] \\ & \quad + \mathbb{E}(U'(X^*(T)) \cdot (X(T) - X^*(T)) \mid yH(T) \leq U'(K_U)) \cdot \mathbb{P}[yH(T) \leq U'(K_U)]. \end{aligned}$$

Observe that on the event $[yH(T) \in (U'(K_L), U'(K_U))]$,

$$U'(X^*(T)) = U'(I(yH(T))) = yH(T)$$

so that

$$\begin{aligned} & \mathbb{E}(U'(X^*(T)) \cdot (X(T) - X^*(T)) | yH(T) > U'(K_U)) \\ & = \mathbb{E}(yH(T) \cdot (X(T) - X^*(T)) | yH(T) > U'(K_U)). \end{aligned}$$

Next observe that on the event $[yH(T) \leq U'(K_U)]$, as $X(T) \in [K_L, K_U]$ a.s, then

$$X(T) - X^*(T) = X(T) - K_U \leq 0$$

and

$$U'(X^*(T)) = U'(K_U) \geq yH(T).$$

The negative sign of $X(T) - X^*(T)$ reverses the inequality $U'(X^*(T)) \geq yH(T)$, giving that on the event $[yH(T) \leq U'(K_U)]$,

$$U'(X^*(T)) \cdot (X(T) - X^*(T)) \leq yH(T) \cdot (X(T) - X^*(T)).$$

On the event $[yH(T) \geq U'(K_L)]$, as $X(T) \in [K_L, K_U]$ a.s, then

$$X(T) - X^*(T) = X(T) - K_L \geq 0$$

and

$$U'(X^*(T)) = U'(K_L) \leq yH(T).$$

Due to the positive sign of $X(T) - X^*(T)$, the inequality $U'(X^*(T)) \leq yH(T)$ is maintained, giving

$$U'(X^*(T)) \cdot (X(T) - X^*(T)) \leq yH(T) \cdot (X(T) - X^*(T)).$$

In summary, we find that

$$\mathbb{E}(U(X(T)) - U(X^*(T))) \leq \mathbb{E}(yH(T) \cdot (X(T) - X^*(T))).$$

As both solutions satisfy the budget constraint (3.5), the last line in the above inequality can be evaluated as

$$\mathbb{E}(yH(T) \cdot (X(T) - X^*(T))) \leq y \cdot ((x_0 + g(0)) - (x_0 + g(0))) = 0,$$

which means

$$\mathbb{E}(U(X(T)) - U(X^*(T))) \leq 0.$$

Hence

$$\mathbb{E}\left(U(X^{\pi^*}(T))\right) = \sup_{\pi \in \mathcal{A}} \mathbb{E}(U(X^\pi(T))) \leq \mathbb{E}(U(X^*(T))) \leq \mathbb{E}\left(U(X^{\pi^*}(T))\right),$$

i.e. $X^{\pi^*}(T) = X^*(T)$, a.s.

□

Proof. of Lemma 3.6.

From [cite relevant lemma in our first paper], a European call option with maturity value $\max\{0, (z_0 + g(0))Z(T) - K_L\}$ is given by $c(t, P(t); K_L)$ with

$$P(t) := (z_0 + g(0))Z(t), \tag{A.1}$$

and

$$c(t, y; K_L) := y\Phi(d_+(t, y; K_L)) - K_L e^{-r_N(T-t)}\Phi(d_-(t, y; K_L)),$$

in which the functions $d_{\pm}(t, y; K_L)$ are defined by equation (3.8).

Thus by put-call parity, the value of the put option with the same strike price K_L satisfies

$$p(t, y; K_L) = c(t, y; K_L) + K_L e^{-r_N(T-t)} - y.$$

To find the replicating portfolio, we differentiate the put pricing function p to get

$$p_t(t, y; K_L) = -\frac{y\phi(d_+(t, y; K_L))\sigma A}{2\sqrt{T-t}} + r_N K_L e^{-r_N(T-t)} \Phi(-d_-(t, y; K_L))$$

$$p_y(t, y; K_L) = \Phi(d_+(t, y; K_L)) - 1 = -\Phi(-d_+(t, y; K_L)), \quad p_{yy}(t, y) = \frac{\phi(d_+(t, y; K_L))}{y\sigma A\sqrt{T-t}}.$$

By Ito's formula,

$$dp(t, P(t)) = p_t(t, P(t))dt + p_y(t, P(t))dP(t) + \frac{1}{2}p_{yy}(t, P(t))d[P](t),$$

in which

$$dP(t) = (r_N + \theta\sigma A)P(t)dt + \sigma AP(t)dW(t).$$

Substituting for: the derivatives of the pricing function p , the dynamics of P and the candidate replicating portfolio $\pi_p(t, P(t)) := -AP(t)\Phi(-d_+(t, P(t)))$, we find that the dynamics of the pricing function c satisfy the wealth equation (3.3). Hence $\pi_p(t, P(t))$ is the amount to be invested in the risky stock at time t in order to replicate the payoff of the European put option. \square

Proof. of Lemma 3.9. Fix $p \in (0, 1)$. From [cite relevant lemma in our first paper], with no lower bound on the terminal wealth,

$$\mathcal{Q}_p(X(T); (0, K_U)) = \min \{K_U, (z_0 + g(0))e^{\beta p}\}.$$

It is useful to consider another investor who has the same savings plan g and the same upper bound K_U as the first investor. However, this second investor has no lower bound on the terminal wealth, i.e. $K_L = 0$, and starts with an initial wealth \tilde{x}_0 that satisfies

$$\tilde{x}_0 = z_0 - c(0, z_0 + g(0); K_U).$$

. This second investor follows the optimal constrained strategy. Then, as $g(T) = 0$, the wealth at time T of the second investor is

$$\tilde{X}(T) = (z_0 + g(0))Z(T) - g(T) - c(T, P(T); K_U) = \min \{K_U, P(T)\}.$$

Thus the terminal wealth of the constrained investor, who has a lower bound K_L on their terminal wealth, is related to that of the second unconstrained investor by

$$X(T) = \begin{cases} \tilde{X}(T) & \text{if } \tilde{X}(T) \geq K_L \\ K_L & \text{if } \tilde{X}(T) < K_L. \end{cases}$$

The desired expression (3.17) follows by consideration of the last expression. \square

B Lifetime annuities

Table 3: Distribution of the life-long annuity at 65 years old for $K_L = 250$ and $K_U = 415$.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$Q_p(Annuity(65))$				
1%	16.64	16.09	16.38	16.51	16.58
2.5%	16.64	16.41	16.52	16.58	16.61
5%	16.64	16.62	16.60	16.62	16.63
10%	16.64	16.97	16.70	16.67	16.65
15%	16.64	17.80	16.93	16.81	16.73
20%	19.33	19.50	18.87	18.96	19.03
25%	22.26	21.63	21.85	21.86	21.86
30%	25.27	23.61	24.75	24.89	25.06
35%	27.63	25.14	26.81	27.25	27.46
40%	27.63	26.13	27.24	27.45	27.55
45%	27.63	26.67	27.39	27.51	27.57
50%	27.63	27.03	27.47	27.55	27.59
55%	27.63	27.28	27.53	27.57	27.60
60%	27.63	27.46	27.58	27.60	27.61
65%	27.63	27.64	27.63	27.62	27.62
70%	27.63	27.81	27.67	27.64	27.63
75%	27.63	27.98	27.71	27.66	27.64
80%	27.63	28.18	27.76	27.69	27.65
85%	27.63	28.39	27.82	27.72	27.67
90%	27.63	28.65	27.90	27.75	27.68
95%	27.63	29.12	28.03	27.82	27.71
97.5%	27.63	29.56	28.19	27.88	27.74
99%	27.63	30.22	28.36	27.97	27.78

Table 4: Distribution of the life-long annuity at 65 years old for $K_L = 250$ and $K_U = 415$. Yearly: final wealth simulated with inflation 0.8% return, 2% volatility, $\rho = 0.5$. Life-long annuity calculated with yearly interest rate of 0.8%

Update:	Yearly
p	$Q_p(\text{Annuity}(65))$
1%	18.17
2.5%	19.02
5%	19.96
10%	21.33
15%	22.73
20%	24.5
25%	26.61
30%	28.73
35%	30.4
40%	31.64
45%	32.61
50%	33.4
55%	34.11
60%	34.73
65%	35.4
70%	36.05
75%	36.77
80%	37.51
85%	38.43
90%	39.52
95%	41.19
97.5%	42.52
99%	44.21

UB-Riskcenter Working Paper Series

List of Published Working Papers

- [WP 2014/01]. Bolancé, C., Guillén, M. and Pitt, D. (2014) “Non-parametric models for univariate claim severity distributions – an approach using R”, UB Riskcenter Working Papers Series 2014-01.
- [WP 2014/02]. Mari del Cristo, L. and Gómez-Puig, M. (2014) “Dollarization and the relationship between EMBI and fundamentals in Latin American countries”, UB Riskcenter Working Papers Series 2014-02.
- [WP 2014/03]. Gómez-Puig, M. and Sosvilla-Rivero, S. (2014) “Causality and contagion in EMU sovereign debt markets”, UB Riskcenter Working Papers Series 2014-03.
- [WP 2014/04]. Gómez-Puig, M., Sosvilla-Rivero, S. and Ramos-Herrera M.C. “An update on EMU sovereign yield spread drivers in time of crisis: A panel data analysis”, UB Riskcenter Working Papers Series 2014-04.
- [WP 2014/05]. Alemany, R., Bolancé, C. and Guillén, M. (2014) “Accounting for severity of risk when pricing insurance products”, UB Riskcenter Working Papers Series 2014-05.
- [WP 2014/06]. Guelman, L., Guillén, M. and Pérez-Marín, A.M. (2014) “Optimal personalized treatment rules for marketing interventions: A review of methods, a new proposal, and an insurance case study.”, UB Riskcenter Working Papers Series 2014-06.
- [WP 2014/07]. Piulachs, X., Alemany, R. and Guillén, M. (2014) “A joint longitudinal and survival model with health care usage for insured elderly”, UB Riskcenter Working Papers Series 2014-07.
- [WP 2014/08]. Abad, P and Chuliá, H. (2014) “European government bond market integration in turbulent times”, UB Riskcenter Working Papers Series 2014-08.
- [WP 2014/09]. Belles-Sampera, J., Guillén, M. and Santolino, M. (2014) “The use of flexible quantile-based measures in risk assessment”, UB Riskcenter Working Papers Series 2014-09.
- [WP 2015/01]. Bolancé C., Guillén, M. and Padilla, A. (2015) “Estimación del riesgo mediante el ajuste de cópulas”, UB Riskcenter Working Papers Series 2015-01.
- [WP 2015/02]. Donnelly, C., Gerrard, R., Guillén, M. and Nielsen, J.P. (2015) “Less is more: increasing retirement gains by using an upside terminal wealth constraint”, UB Riskcenter Working Papers Series 2015-02.
- [WP 2015/03]. Chuliá, H., Guillén, G. and Uribe (2015) “Mortality and longevity risks in the United Kingdom: Dynamic factor models and copula-functions”, UB Riskcenter Working Paper Series 2015-03.

- [WP 2015/04]. Alaminos, E. and Ayuso, M. (2015) “Desarrollo metodológico del modelo actuarial de múltiples estados casado – viudo y cálculo actuarial del coste por pensiones de jubilación y viudedad”, UB Riskcenter Working Paper Series 2015-04.
- [WP 2015/05]. Belles-Sampera, J., Guillén, M. and Santolino, M. (2015) “What attitudes to risk underlie distortion risk measure choices?”, UB Riskcenter Working Paper Series 2015-05.
- [WP 2015/06]. Solé-Auró, A., and Alcañiz, M. (2015) “Is the educational health gap increasing for women? Results from Catalonia (Spain)”, UB Riskcenter Working Paper Series 2015-06.
- [WP 2015/07]. Donnelly, C., Guillén, M., Nielsen, J.P. and Pérez-Marín, A.M. (2015) “On the practical implementation of retirement gains by using an upside and a downside terminal wealth constraint”, UB Riskcenter Working Papers Series 2015-07.