Tôhoku Math, Journ.
Vol. 19, No. 3, 1967

ON THE PREDUALS OF W^*-ALGEBRAS

KAZUYUKI SAITÔ

(Received May 22, 1967)

In the present paper ,we shall show some properties of weakly relatively compact subsets of predual of W^*-algebra, which were also discussed in [1], [10] and [12].

Let M be a W^*-algebra (namely, C^*-algebra with a dual structure as a Banach space [7]), M^* (resp. M_*) be the dual (resp. predual) of M, and let M_h, M_p, and M_{pi} be the set of all Hermitian elements, projections, and partial isometries in M, respectively.

The weak topology on M_* is $\sigma(M_*, M)$-topology in the sense of [3; p. 50].

For any linear functional φ in M, we define the functionals φa, $a\varphi$, φ^* and $|\varphi|$ on M as follows: $\varphi a(b) = \varphi(ab)$, $a\varphi(b) = \varphi(ba)$, $\varphi^*(b) = \overline{\varphi(b^*)}$ for all b $\in M$, where $\overline{\varphi(b^*)}$ is the complex conjugate of $\varphi(b^*)$. $|\varphi|$ is said the absolute value of φ [8]. If φ is in M_*, then φa, $a\varphi$, and φ^* are also in M_*. We denote the set $\{|\varphi|; \varphi \in K\}$ by $|K|$.

A functonal φ on M is positive if $\varphi(a^*a) \geqq 0$ for all $a \in M$. Denote the set of all positive functionals in M^* (resp. M_*) by M^{*+} (resp. M_*^+).

We may consider the following five typical topologies on M:

(1) The norm topology as a Banach space, (2) The Mackey topology τ on M, namely, the togology of uniform convergence on the weakly relatively compact sets of M_*, (3) The topology s^* defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^*; \varphi \in M_*^+\}$, where $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$, and $\alpha_\varphi^*(x) = \varphi(xx^*)^{1/2}$ for $x \in M$, (4) The topology s defined by a family of semi-norms $\{\alpha_\varphi; \varphi \in M_*^+\}$, (5) The weak topology on M as point, which is merely called σ-topology. The topology s^*(resp. s and σ) coincides with strong $*$-operator topology, namely the operator topology defined by a family of semi-norms $\{\|x\xi\| \ \|x^*\xi\| ; \xi \in \mathfrak{H}\}$ (resp. the strong operator topology and the weak operator topology) on bounded spheres, when M is faithfully represented as a von Neumann algebra on a Hilbert apace $\mathfrak{H}$. The τ-topology is equivalent to the s^*-topology on bounded spheres. [1]

In the followings, theorem 1 shows a characterization of the finiteness of W^*-algebras. Theorem 2 and the following remark concern with a weak convergence property in the predual of an atomic W^*-algebra, which is a non-commutative generalization of a well known theorem in the Lebesgue L^1, and the last theorem 3 deals with weakly relatively compact subsets lying in the positive portion of the predual M_*.

Firstly, we state and prove the following

THEOREM 1. *Let M be a W^*-algebra, then M is finite if and only if, for any weakly relatively compact subset K of M_*, $|K|$ is also weakly relatively compact.*

PROOF. Necessity: By Eberlein-Šmulian theorem, it sufficies to prove only the case $K=\{\varphi_n\}_{n=1}^\infty$. By [1; Theorem 2], it is sufficient to prove that, for any orthogonal sequence of projections $\{e_k\}_{k=1}^\infty$ $\lim_{k\to\infty} |\varphi_n|(e_k)=0$ uniformly for n.

Let

$$|\varphi_n|(e_k) = \varphi_n(e_k u_n^*) = \varphi_n^*(u_n e_k) \quad (u_n \in M_{p.i.})$$

be the polar decomposition of φ_n in the sense of [7]. If the statement that $\lim_{k\to\infty} |\varphi_n|(e_k) = 0$ uniformly for n is false, then there exists an $\varepsilon > 0$ such that for each n there is some φ_n' in K such that

$$(*) \qquad |\varphi_n'|(e_n) \geqq \varepsilon.$$

As the $*$-operation is continuous for the weak topology, K^* (the set $\{\varphi^*; \varphi \in K\}$) is also weakly relatively compact. Setting

$$a_n = u_n e_n,\ \|a_n\| \leqq 1, \text{ and } a_n^* a_n = e_n e(|\varphi_n'|) e_n$$

where $e(|\varphi_n'|)$ is the carrier projection of $|\varphi_n'|$ [5], and a_n converges strongly to 0. Since M is a finite algebra, τ is equivalent to s on S, the unit sphere. Hence $\lim_{n\to\infty} a_n = 0$ for τ- topology and then $\lim_{m\to\infty} \varphi_n^*(a_m) = 0$ uniformly for n, contradicting the inequality $(*)$.

Sufficiency: By [5], there exists a central projection e such that $M(1-e)$ is finite algebra, $e=0$ or Me is properly infinite algebra and $M = Me \oplus M(1-e)$. If $e \neq 0$, then Me is a properly infinite W^*-algebra and there is a family of orthogonal projections $\{e_i\}_{i=1}^\infty$ such that

$$e = \sum_{i=1}^\infty e_i,\ e_i \sim e_j (e_i \in M). \qquad [5].$$

Let ψ be a σ-continuous state on $e_1 M e_1$ and putting

$$\varphi(a) = \psi(e_1 a e_1),\ a \in M,$$

φ is a σ-continuous positive functional on M such that $\varphi(e_1)=1$. Setting $\varphi_n(a)=\varphi(v_n^*a)$, where v_n is a partial isometry in M such that $v_n^*v_n=e_1$, $v_nv_n^*=e_n$, φ_n is σ-continuous. Then we have

$$\varphi_n(a)=\varphi(v_n^*a)=\varphi(v_n^*av_n^*v_n)=(v_n\varphi v_n^*)(av_n^*)=v_n^*(v_n\varphi v_n^*)(a).$$

Putting $\psi_n=v_n\varphi v_n^*$, ψ_n is positive and we have

$$\psi_n(e_n)=\varphi(v_n^*e_nv_n)=\varphi(e_1)=1,$$
$$\psi_n(1)=\|\psi_n\|=\varphi(v_n^*v_n)=\varphi(e_1)=1.$$

Hence we have

$$|\varphi_n(a)|^2\leqq\psi_n(aa^*)\cdot\psi_n(v_nv_n^*)=\psi_n(aa^*)\cdot\psi_n(e_n)$$
$$=\psi_n(aa^*)\cdot\|\psi_n\|.$$

By the unicity of polar decomposition [11], $\varphi_n=v_n^*\psi_n$ is the polar decomposition of φ_n and ψ_n is the absolute value of φ_n, that is, $|\varphi_n|=\psi_n$. By [9], we have

$$\varphi_n(a)=\varphi(v_n^*a)=(\pi_\varphi(a)\,\eta_\varphi(1),\,\eta_\varphi(v_n)),$$

where π_φ is a cyclic representation on $\mathfrak{H}_\varphi$ induced by φ and $\eta_\varphi(a)$ is an element of $\mathfrak{H}_\varphi$ corresponding to a in M in the sense of I. E. Segal [9]. As $\{\eta_\varphi(v_n)\}$ is an orthogonal system in $\mathfrak{H}_\varphi$, we have

$$\lim_{n\to\infty}(\pi_\varphi(a)\eta_\varphi(1),\,\eta_\varphi(v_n))=0,$$

that is, φ_n is weakly convergent to 0. Hence $\{\varphi_n\}_{n=1}^\infty$ is a weakly relatively compact subset of M_*.

If $\{|\varphi_n|\}_{n=1}^\infty$ is weakly relatively compact, then by [1], for the above family of orthogonal projections $\{e_i\}_{i=1}^\infty$, $\lim_{k\to\infty}|\varphi_n|(e_k)=0$ uniformly for n. On the other hand, we have

$$|\varphi_n|(e_n)=\psi_n(e_n)=\varphi(e_1)=1,\text{ for each } n.$$

This is a contradiction, that is, $\{|\varphi_n|\}_{n=1}^\infty$ is not weakly relatively compact. Therefore, if M is not finite, then there exists a weakly relatively compact subset $\{\varphi_n\}_{n=1}^\infty$ of M_* such that $\{|\varphi_n|\}_{n=1}^\infty$ is not weakly relatively compact.

This completes the proof.

REMARK. If M is an abelian W^*-algebra, then $M=L^\infty(\Omega, \mu)$ where Ω is a locally compact Hausdorff space and μ is a positive Radon measure on Ω by [5]. Therefore, in the abelian case, the necessary condition of the above theorem is a well known result in the classical measure theory.

By an atomic W^*-algebra M we mean a W^*-algebra such that for every projection e in M, there exists a minimal subprojection f of e in M.

Then we obtain

THEOREM 2. *Let M be an atomic W^*-algebra and $\{\varphi_n\}_{n=1}^\infty$ be a sequence in M_* such that $\lim_{n\to\infty} \varphi_n(e)$ exists and is finite for each e in M_p and that $\{|\varphi_n|\}_{n=1}^\infty$, $\{|\varphi_n^*|\}_{n=1}^\infty$ are weakly relatively compact, then there exists φ in M_* such that $\lim_{n\to\infty} \|\varphi_n-\varphi\|=0$.*

In the proof of our theorem we shall use the following lemma due to C.Akemann [1].

LEMMA. *Let M be a W^*-algebra and $\{e_\theta\}_{\theta\in\Theta}$ be an increasing net of projections in M such that $\sup_{\theta\in\Theta} e_\theta=1$, then for bounded subset K of M_*, K is weakly relatively compact if and only if for every positive ε, there exists an e in $\{e_\theta\}_{\theta\in\Theta}$ such that $\|e^\perp\varphi e^\perp\|\leqq\varepsilon$ for each φ in K, where $e^\perp$ means the projection $1-e$.*

PFOOF OF THEOREM 2. From the result of Aarnes [2], there is a real number $r>0$ such that $\|\varphi_n\|\leqq r$ for each n. Then by the spectral theory and Banach-Steinhaus theorem, there exists φ in M_* such that φ_n converges weakly to φ.

Denoting

$$K=\{|\varphi_n|, |\varphi_n^*|, |\varphi|, |\varphi^*|; n=1,2,\cdots\},$$

K is weakly relatively compact. By the hypothesis and Zorn's lemma, there exists a family of projections $\{e_\theta\}_{\theta\in\Theta}$ in M as follows;

(1) The algebra $e_\theta M e_\theta$ is finite dimensional for each θ.

(2) The $\{e_\theta\}_{\theta\in\Theta}$ are increasing net.

(3) $\sup_{\theta\in\Theta} e_\theta=1$.

By scalar multiplication, we may assume that $\sup_k \|\varphi_k\|=1$ without loss of generality.

By the above lemma, for $\varepsilon > 0$, there is a projection e in $\{e_\theta\}_{\theta\in\Theta}$ such that, $\|e^\perp \varphi e^\perp\| \leqq \varepsilon$ for all φ in K. Since eMe is finite dimensional, the weak and the norm topologies coincide on eMe, so that there exists an integer k_0 such that we have, for each a in S,

$$|(\varphi_k - \varphi)(eae)| < \varepsilon,$$

for $k > k_0$.

Thus, for any a in S and $k > k_0$, we have the inequalities:

$$\begin{aligned}|(\varphi_k - \varphi)(a)| &< |(\varphi_k - \varphi)(eae)| + |(\varphi_k - \varphi)(eae^\perp)| \\ &+ |(\varphi_k - \varphi)(e^\perp ae)| + |(\varphi_k - \varphi)(e^\perp ae^\perp)| \\ &< \varepsilon + |\varphi_k^*(e^\perp a^* e)| + |\varphi^*(e^\perp a^* e)| \\ &+ |\varphi_k(e^\perp ae)| + |\varphi(e^\perp ae)| + |\varphi_k(e^\perp ae^\perp)| \\ &+ |\varphi(e^\perp ae^\perp)|.\end{aligned}$$

Now let $\varphi_k = u_k|\varphi_k|$, (resp. $\varphi_k^* = v_k|\varphi_k^*|$) be the polar decomposition of φ_k (resp. φ_k^*); then, by the Schwarz inequality, we have

$$\begin{aligned}|\varphi_k(e^\perp ae)| &= ||\varphi_k|(e^\perp aeu_k)| \\ &< \{|\varphi_k|(e^\perp)\}^{1/2} \cdot \{|\varphi_k|((aeu_k)^*(aeu_k))\}^{1/2} \\ &< \{|\varphi_k|(e^\perp)\}^{1/2} < \varepsilon^{1/2}.\end{aligned}$$

Similarly we have

$$|\varphi_k^*(e^\perp a^* e)| < \varepsilon^{1/2}.$$

Combining the above estimations, we get

$$|(\varphi_k - \varphi)(a)| < \varepsilon + 6\varepsilon^{1/2}$$

for $k > k_0$ and $a \in S$. Since ε is arbitrary and a is an arbitrary element of S, we have that $\lim_{k\to\infty} \|\varphi_k - \varphi\| = 0$. This completes the proof.

REMARK. This theorem can be considered as a non-commutative version of [6; p. 295] and includes the result of C.Akemann [1;Theorem IV 1.]. In finite case, by Theorem 1, we can drop the condition that $\{|\varphi_n|\}_{n=1}^\infty, \{|\varphi_n^*|\}_{n=1}^\infty$ are weakly relatively compact, but in general case, we cannot drop it, as the following example shows. Let $\mathfrak{H}$ be an infinite dimensional separable Hilbert space, $\{\xi_i\}_{i=1}^\infty$ an orthonormal basis for it, and define functionals $\{\omega_n\}$ in $\boldsymbol{B}(\mathfrak{H})$ by;

$$\omega_n(a) = (a\xi_1, \xi_n), \text{ for } a \in \boldsymbol{B}(\mathfrak{H}).$$

(Note that $\boldsymbol{B}(\mathfrak{H})$ is an atomic W^*-algebra.) and we have

$$\omega_n^*(a) = (a\xi_n, \xi_1).$$

Then by the definition of ω_n, both ω_n and ω_n^* converge weakly to 0. Let v_n be a partial isometry defined by $v_n\xi = (\xi, \xi_1)\xi_n$ for $\xi \in \mathfrak{H}$.

Putting $\varphi_n(a) = (a\xi_n, \xi_n)$ and $\omega_n(a) = \varphi_n(av_n^*)$, we have

$$|\omega_n(a)|^2 \leqq \varphi_n(aa^*)\|\varphi_n\|, \ \|\varphi_n\| = 1 = \|\omega_n\|.$$

By the unicity of polar decomposition of functionals, we have

$$|\omega_n| = \varphi_n \text{ and } |\omega_n^*| = \varphi_1.$$

Hence $\{|\omega_n^*|\}_{n=1}^{\infty}$ is weakly relatively compact. On the other hand, $\{|\omega_n|\}_{n=1}^{\infty}$ is not weakly relatively compact. If otherwise, putting $e_n = p_{[\xi_n]}$, for the family of orthogonal projections $\{e_n\}_{n=1}^{\infty}$, we have

$$\lim_{k\to\infty} \varphi_n(e_k) = 0 \text{ uniformly for } n.$$

This is a contradiction. And ω_n cannot converge to 0 uniformly. Hence either of the above conditions can not be dropped.

THEOREM 3. *Let M be a W^*-algebra and K be a weakly relatively compact subset in M_*^+, then $\{a\varphi; \varphi \in K, a \in S\}$ is also weakly relatively compact.*

PROOF. By uniform boundedness theorem, $\Delta = \sup\{\|\varphi\|; \varphi \in K\} < \infty$. For any sequence of orthogonal projections $\{e_n\}_{n=1}^{\infty}$ in M, we have, by Schwarz inequality,

$$|\varphi(e_n a)| \leqq \varphi(a^*a)^{1/2}\varphi(e_n)^{1/2} \leqq \Delta^{1/2}\varphi(e_n)^{1/2}$$

for $a \in S$, $\varphi \in K$. Since K is weakly relatively compact,

$$\lim_{n\to\infty} \varphi(e_n) = 0 \text{ uniformly for } \varphi \in K.$$

Therefore $\{a\varphi; \varphi \in K, a \in S\}$ is weakly relatively compact. This completes the proof of Theorem 3.

REMARK. In the above theorem, we cannot drop the hypothesis that $K \subset M_*^+$. Considering $\boldsymbol{B}(\mathfrak{H})$ (where $\mathfrak{H}$ is a separable infinite dimensional Hilbert space), then by the above arguments, there is a weakly relatively compact subset K of $\boldsymbol{B}(\mathfrak{H})_*$ whose absolute value is not weakly relatively compact. $|\varphi| = v^*\varphi \in \{a\varphi\,;\ a \in S, \varphi \in K\}$ where v is in M_{pi}. Hence $|K| \subset \{a\varphi; a \in S, \varphi \in K\}$ and $\{a\varphi\,;\ a \in S, \varphi \in K\}$ is not weakly relatively compact.

Moreover, for a W^*-algebra M to be finite, it is necessary and sufficient that for every weakly relatively compact subset K of the predual M_* of M, $\{a\varphi; a \in S, \varphi \in K\}$ is also weakly relatively compact. Since $|K| \subset \{a\varphi; a \in S, \varphi \in K\}$, the proof is the same as that of Theorem 1, so we omit it.

COROLLARY. *Let M be a W^*-algebra and K a subset of M_* whose absolute value $|K|$ is weakly relatively compact, then K is also weakly relatively compact.*

PROOF. By the polar decomposition of functional, we have

$$K \subset \{a\varphi; a \in S, \varphi \in |K|\}.$$

Hence, by Theorem 2, K is weakly relatively compact. This completes the proof.

The author wishes to express his hearty thanks to Prof. M. Fukamiya and Dr. M.Takesaki for their many valuable suggestions in the presentation of this paper and he is grateful to Prof. S.Sakai for calling my attention to the reference [1], and is also grateful to Dr. C.Akemann for prepublication copy of [1].

REFERENCES

[1] C. A. AKEMANN, The dual space of an operator algebra, To appear in Trans. Amer. Math. Soc.,

[2] J. AARNES, The Vitali-Hahn-Saks theorem for von Neumann algebra, Math. Scand., 18(1966), 87–92.

[3] N. BOURBAKI, Espaces vectoriels topologiques II, (1955).

[4] J. DIXMIER, Formes linéaires sur un anneau d'opérateurs, Bull. Soc. Math. France, 81(1953), 9–39.

[5] J. DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars Paris, (1957).

[6] DUNFORD-SCHWARTZ, Linear operators I, (1958).

[7] S. SAKAI, The theory of W^*-algebra, Lecture note, Yale Univ., (1962)

[8] S. SAKAI, On topologies of finite W^*-algebras, Illinois Journ. Math., 9(1965), 236-241.

[9] I. E. SEGAL, Irreducible representations of operator algebras, Bull. Amer. Math. Soc., 53(1947), 73–88.

[10] M. TAKESAKI, On the conjugate space of operator algebra, Tôhoku Math. Journ., 10 (1958), 194-203.
[11] M. TOMITA, Spectral theory of operator algebras 1, Math. Journ. Okayama Univ., 9(1959), 63-98.
[12] H. UMEGAKI, Weak compactness in an operator space, Kôdai Math. Sem. Rep., 8 (1956), 145-151.

MATHEMATICAL INSTITUTE,
TÔHOKU UNIVERSITY
SENDAI, JAPAN.