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On the Pricing of American Options*

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Abstract. The problem of valuation for contingent claims that can be exercised at any time before or at maturity, such as American options, is discussed in the manner of Bensoussan [1]. We offer an approach which both simplifies and extends the results of existing theory on this topic.

1. Introduction

In an important and relatively recent article, Bensoussan [1] presents a rigorous treatment for *American* contingent claims, that can be exercised at any time before or at maturity (in contradistinction to *European* contingent claims which are exercisable only at maturity). He adapts the Black and Scholes [3] methodology of duplicating the cash flow from such a claim to this situation by skillfully managing a self-financing portfolio that contains only the basic instruments of the market, i.e., the stocks and the bond, and that entails no arbitrage opportunities before exercise. Under a condition on the market model called *completeness* (due to Harrison and Pliska [7], [8] in its full generality and rendered more transparent in [1]), Bensoussan shows that the pricing of such claims is indeed possible and characterizes the exercise time by means of an appropriate optimal stopping problem.

In the study of the latter, Bensoussan employs the so-called "penalization method," which forces rather stringent boundedness and regularity conditions on the payoff from the contingent claim. Such conditions are not satisfied, however, by the prototypical examples of such claims, i.e., American call options.

The aim of the present paper is to offer an alternative methodology for this problem, which is actually simpler and manages to remove the above restrictions;

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it is based on a "martingale" treatment of the optimal stopping problem as in Fakeev [6], Bismut and Skalli [2], or El Karoui [4]. Furthermore, it seems to be well-suited for handling claims that are *perpetual*, i.e., exercisable at any time before the end of the age.

We present a suitably modified version of the Bensoussan model in Section 2 and the beginning of Section 4, including condition (2.6) which will guarantee the completeness of the model in our context. Section 3 introduces the concepts of *portfolio* and *consumption* processes, necessary for the treatment of valuation (pricing) questions. In order to motivate later developments, we present in Section 4 the treatment of European contingent claims, as in Karatzas and Shreve [10].

The notion of "hedging strategy" for an American contingent claim is introduced in Section 5, as a portfolio/consumption process pair which makes it possible to attain, by suitably investing in the market, the same wealth as the payoff achievable from the possession of the contingent claim. The *fair price* of the latter at time $t=0$ is defined as the smallest value of the initial wealth that permits the construction of a hedging strategy, and is related to a problem of optimal stopping. The analysis for the latter leads to the valuation formulae (5.10) and (5.11); some elementary consequences of these formulae are discussed, and the perpetual case is taken up in Section 6.

Problems of option pricing have a long history; see, for instance, Samuelson [15], [16], McKean [12], Black and Scholes [3], Merton [13] and the review article by Smith [17]. Harrison and Pliska [7], [8] developed a theory of "continuous trading," based on stochastic calculus, and demonstrated that the pricing of European contingent claims is possible under quite general market models.

2. The Market Model

Let us consider a market in which $d+1$ assets (or "securities") are traded continuously. One of them, called the *bond*, has a price P_0 which evolves according to the equation

$$dP_0(t)=r(t)P_0(t)\,dt, \qquad P_0(0)=p_0=1, \tag{2.1}$$

and determines the *discount factor*

$$\beta(t)\triangleq\frac{1}{P_0(t)}=\exp\left\{-\int_0^t r(s)\,ds\right\}, \qquad 0\le t<\infty. \tag{2.2}$$

The remaining d assets, called *stocks*, are risky; their prices are modeled by the linear stochastic differential equations

$$\begin{aligned} dP_i(t)&=P_i(t)\left[b_i(t)\,dt+\sum_{j=1}^d \sigma_{ij}(t)\,dW_j(t)\right], \qquad 0\le t<\infty,\\ P_i(0)&=p_i>0, \qquad 1\le i\le d. \end{aligned} \tag{2.3}$$

The *discounted prices* βP_i of the stocks obey the equations

$$d[\beta(t)P_i(t)] = \beta(t)P_i(t)\left[(b_i(t)-r(t))\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\right], \qquad 0 \le t < \infty. \tag{2.4}$$

Here $\{W(t) = (W_1(t), \ldots, W_d(t))^*, \mathcal{F}_t;\ 0 \le t < \infty\}$ is a d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t\}$ is the augmentation under P of the filtration

$$\mathcal{F}_t^W = \sigma(W(s);\ 0 \le s \le t), \qquad 0 \le t < \infty, \tag{2.5}$$

generated by the Brownian motion. It is well known (e.g., Karatzas and Shreve [10, Section 2.7]) that $\{\mathcal{F}_t\}$ satisfies the "usual conditions": it is right-continuous, and $\mathcal{F}_o$ contains the P-null events in $\mathcal{F}_\infty^W$. The integer d represents *the number of independent, exogenous sources of uncertainty in the market model.*

Throughout Sections 2-5, the *interest rate* $\{r(t);\ 0 \le t < \infty\}$ of the bond, the *appreciation* $\{b_i(t);\ 0 \le t < \infty\}$ and *dividend* $\{\mu_i(t);\ 0 \le t < \infty\}$ rates of the d stocks, as well as the *dispersion coefficients* $\{\sigma_{ij}(t);\ 0 \le t < \infty\}$, $1 \le i, j \le d$, are assumed to be measurable processes, adapted to the filtration $\{\mathcal{F}_t\}$, and uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$, for every finite $T > 0$. They will be referred to collectively as the *coefficients of the market model.*

Let us now consider the random matrices

$$\sigma(t) = \{\sigma_{ij}(t)\}_{i \le i, j \le d}, \qquad D(t) = \sigma(t)\sigma^*(t),$$

where * denotes transposition. It will be assumed throughout the paper that there exists a positive number ε for which

$$\xi^* D(t, \omega)\xi \ge \varepsilon \|\xi\|^2, \qquad \forall \xi \in \mathcal{R}^d, \tag{2.6}$$

holds for every $(t, \omega) \in [0, \infty) \times \Omega$.

3. Investment and Consumption

Let us now consider a "small investor," i.e., an agent whose actions cannot influence the prices. This agent starts with an initial endowment $x \ge 0$, and invests it in the $d+1$ assets of the market. If he decides to hold $N_i(t)$ shares from the asset $i = 0, 1, \ldots, d$ at time t, his wealth is then

$$X_t = \sum_{i=0}^{d} N_i(t)P_i(t), \qquad 0 \le t < \infty. \tag{3.1}$$

Now suppose that the trading of shares, as well as the payment of dividends (at the rate $\mu_i(t)$ per unit time and per dollar invested in the ith stock), takes place at discrete time points like $\ldots, t-h, t, t+h, \ldots$. If C_t denotes the cumulative amount of withdrawals for consumption, made up to time t, then

$$X_{t+h} - X_t = \sum_{i=0}^{d} N_i(t)[P_i(t+h) - P_i(t)] + h \sum_{i=1}^{d} N_i(t)P_i(t)\mu_i(t) - (C_{t+h} - C_t). \tag{3.2}$$

The continuous-time analogue of (3.2) is

$$dX_t = \sum_{i=0}^{d} N_i(t)\, dP_i(t) + \sum_{i=1}^{d} N_i(t) P_i(t) \mu_i(t) \cdot dt - dC_t,$$

which can be rewritten, with the help of (2.1), (2.3), and (3.1) in the form

$$dX_t = \left[r(t) X_t + \sum_{i=1}^{d} \pi_i(t)(b_i(t) + \mu_i(t) - r(t)) \right] dt - dC_t$$
$$+ \sum_{i=1}^{d} \sum_{j=1}^{d} \pi_i(t) \sigma_{ij}(t)\, dW_j(t); \qquad 0 \le t < \infty. \tag{3.3}$$

Here, $\pi_i(t) \triangleq N_i(t) P_i(t)$ is the amount invested in the ith stock, for $1 \le i \le d$.

Definition 3.1. A *portfolio process* $\pi = \{\pi(t) = (\pi_1(t), \ldots, \pi_d(t))^*, \mathcal{F}_t; 0 \le t \le \infty\}$ is measurable, $\mathcal{R}^d$-valued, adapted, and satisfies

$$\sum_{i=1}^{d} \int_0^T \pi_i^2(s)\, ds < \infty \quad \text{a.s. } P \tag{3.4}$$

for every finite $T > 0$.

Any component of $\pi(t)$ may become negative, and this is interpreted as short-selling that particular stock. The amount $\pi_0(t) \triangleq X_t - \sum_{i=1}^{d} \pi_i(t)$ invested in the bond may also become negative, and this amounts to borrowing at the interest rate $r(t)$.

Definition 3.2. A *consumption process* $C = \{C_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ is progressively measurable with respect to $\{\mathcal{F}_t\}$, takes values in $[0, \infty)$, and satisfies

(i) $C_o(\omega) = 0$,
(ii) the path $t \mapsto C_t(\omega)$ is nondecreasing and right-continuous

for P-a.e. $\omega \in \Omega$.

The unique solution of the linear stochastic differential equation (3.3) is given, for every pair (π, C) as above, by

$$X_t = P_o(t) \left[x + \int_0^t \beta(s) \pi^*(s)(b(s) + \mu(s) - r(s)\mathbf{1})\, ds - \int_0^t \beta(s)\, dC_s \right.$$
$$\left. + \int_0^t \beta(s) \pi^*(s) \sigma(s)\, dW(s) \right], \qquad 0 \le t < \infty, \tag{3.5}$$

and is called the *wealth process corresponding to the portfolio/consumption process pair* (π, C). Here $\mathbf{1}$ denotes the vector in $\mathcal{R}^d$ with every component equal to 1, and all the vectors are column vectors.

Definition 3.3. Given an initial endowment $x \ge 0$ and a finite time-horizon $T > 0$, we say that a pair of portfolio and consumption processes (π, C) is *admissible on* $[0, T]$ *for the initial endowment* $x \ge 0$, and write $(\pi, C) \in \mathcal{A}(T, x)$, if

$$X_t \ge 0, \qquad 0 \le t \le T, \tag{3.6}$$

holds almost surely. We also introduce the notation

$$\mathscr{A}(x) \triangleq \bigcap_{T>0} \mathscr{A}(T, x), \tag{3.7}$$

and say that a pair $(\pi, C) \in \mathscr{A}(x)$ is *admissible for the initial endowment* $x \geq 0$.

Let us now define the measurable, adapted, $\mathscr{R}^d$-valued process

$$\theta(t) \triangleq \sigma^*(t) D^{-1}(t)(b(t) + \mu(t) - r(t)\mathbf{1}), \mathscr{F}_t, \qquad 0 \leq t < \infty, \tag{3.8}$$

which is also bounded in $(t, \omega) \in [0, T] \times \Omega$ for every finite $T > 0$, because of our assumptions on the coefficients of the market model and (2.5). The components of this process satisfy the identities

$$\sum_{j=1}^{d} \sigma_{ij}(t)\theta_j(t) = b_i(t) + \mu_i(t) - r(t), \qquad 0 \leq t < \infty, \quad 1 \leq i \leq d, \tag{3.9}$$

P-almost surely.

Because of the local boundedness of $\|\theta(t, \omega)\|$, the exponential supermartingale

$$Z_t = \exp\left\{-\int_0^t \theta^*(s)\, dW(s) - \tfrac{1}{2}\int_0^t \|\theta(s)\|^2\, ds\right\}, \mathscr{F}_t, \qquad 0 \leq t < \infty, \tag{3.10}$$

is actually a martingale. If one fixes a finite $T > 0$ and introduces the probability measure

$$\tilde{P}_T(A) \triangleq E(Z_T 1_A), \qquad A \in \mathscr{F}_T, \tag{3.11}$$

then by the Girsanov theorem (Chapter 6 in Liptser and Shiryaev [11] or Section 3.5 in Karatzas and Shreve [10]) we have that

(i) P and $\tilde{P}_T$ are mutually absolutely continuous on $\mathscr{F}_T$, and
(ii) the process

$$\tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)\, ds, \mathscr{F}_t, \qquad 0 \leq t \leq T, \tag{3.12}$$

is an $\mathscr{R}^d$-valued Brownian motion on $(\Omega, \mathscr{F}_T, \tilde{P}_T)$.

In terms of this process, and thanks to (3.9), equations (3.3) and (3.5) can be written equivalently as

$$dX_t = r(t)X_t\, dt - dC_t + \sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i(s)\sigma_{ij}(s)\, d\tilde{W}_j(s), \tag{3.13}$$

$$\beta(t)X_t + \int_0^t \beta(s)\, dC_s = x + \int_0^t \beta(s)\pi^*(s)\sigma(s)\, d\tilde{W}(s), \tag{3.14}$$

respectively.

The right-hand side of (3.14) is a $\tilde{P}_T$-local martingale on $[0, T]$, whereas the left-hand side is, for every $(\pi, C) \in \mathscr{A}(T, x)$, a nonnegative process. Therefore, the latter is a continuous *supermartingale* on $[0, T]$, for which the optional sampling theorem yields

$$\tilde{E}_T\left[\beta(\tau)X_\tau + \int_0^\tau \beta(s)\, dC_s\right] \leq x, \qquad \forall \tau \in \mathscr{S}_{0,T}. \tag{3.15}$$

Remark on Notation 3.4. For fixed $0 \le u < v \le \infty$, we denote by $\mathcal{S}_{u,v}$ the collection of stopping times τ of $\{\mathcal{F}_t\}$ with values in $[u, v]$. We shall write

$$\mathcal{S}_t^* \triangleq \mathcal{S}_{t,\infty}, \qquad \mathcal{S}_t \triangleq \{\tau \in \mathcal{S}_t^*;\ \tau < \infty, \quad \text{a.s. } P\} \tag{3.16}$$

for any given $t \in [0, \infty)$.

From (3.15), a rather trite necessary condition for $(\pi, C) \in \mathcal{A}(T, x)$ is

$$\tilde{E}_T \int_0^T \beta(s)\, dC_s \le x. \tag{3.17}$$

It turns out that (3.17) is also, in a certain sense, "sufficient" for admissibility on $[0, T]$.

Proposition 3.5. *Let C be a consumption process satisfying* (3.17) *for a given $x \in [0, \infty)$. There exists then a portfolio process π, such that $(\pi, C) \in \mathcal{A}(T, x)$.*

Proof. With $D \triangleq \int_0^T \beta(t)\, dC_t$ we define the nonnegative process

$$\xi_t \triangleq \tilde{E}_T\left(\int_t^T \exp\left(-\int_t^s r(u)\, du \right) dC_s \middle| \mathcal{F}_t \right) + (x - \tilde{E}_T D) P_0(t), \tag{3.18}$$

so that

$$\beta(t)\xi_t + \int_0^t \beta(s)\, dC_s = x + m_t, \qquad 0 \le t \le T, \tag{3.19}$$

with $m_t \triangleq \tilde{E}_T(D|\mathcal{F}_t) - \tilde{E}_T D = E(DZ_T|\mathcal{F}_t)/Z_t - E(DZ_T)$, by virtue of the Bayes rule (Lemma 3.5.3 in Karatzas and Shreve [10]). It may be assumed that the paths of the P-martingale

$$N_t \triangleq E(DZ_T|\mathcal{F}_t), \qquad 0 \le t \le T,$$

are right-continuous, so from the fundamental martingale representation theorem (Ikeda and Watanabe [9, p. 80] or Karatzas and Shreve [10, Section 3.4]) we have

$$N_t = E(DZ_T) + \sum_{j=1}^d \int_0^t \varphi_j(s)\, dW_j(s), \qquad 0 \le t \le T, \tag{3.20}$$

a.s. P, for suitable measurable and adapted processes $\{\varphi_j(t), \mathcal{F}_t;\ 0 \le t \le T\}$ such that

$$\sum_{j=1}^d \int_0^T \varphi_j^2(t)\, dt < \infty \quad \text{a.s.} \tag{3.21}$$

But now $m_t = N_t/Z_t - E(DZ_T)$, and an application of Itô's rule, in conjunction with (3.20), (3.12), and $dZ_t = -Z_t\theta^*(t)\, dW(t)$, yields

$$m_t = \sum_{j=1}^d \int_0^t \psi_j(s)\, d\tilde{W}_j(s), \qquad 0 \le t \le T, \tag{3.22}$$

with

$$\psi(t) \triangleq \frac{1}{Z_t}(\varphi(t)+\theta(t)N_t). \tag{3.23}$$

If we set

$$\pi(t) \triangleq P_0(t)D^{-1}(t)\sigma(t)\psi(t), \tag{3.24}$$

equation (3.19) becomes (3.14) with $\xi \equiv X$. Condition (3.4) follows from (3.21), (2.6), the boundedness of $\|\theta\|$ on $[0, T]$, and the path continuity of both Z and N. □

Remark 3.6. In terms of the process $\tilde{W}$ of (3.12), equations (2.3) and (2.4) for the stock prices and their discounted counterparts can be written as

$$dP_i(t) = P_i(t)\left[(r(t)-\mu_i(t))\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,d\tilde{W}_j(t)\right] \tag{3.25}$$

$$d[\beta(t)P_i(t)] = \beta(t)P_i(t)\left[\sum_{j=1}^{d} \sigma_{ij}(t)\,d\tilde{W}_j(t) - \mu_i(t)\,dt\right], \qquad 1 \le i \le d. \tag{3.26}$$

From (3.26) it becomes obvious that the discounted price process $\{\beta(t)P_i(t), \mathscr{F}_t; 0 \le t \le T\}$, for a stock which pays no dividends, is a *martingale* under the measure $\tilde{P}_T$. In fact, the latter was constructed with an eye toward this property; see Harrison and Pliska [7], [8] for an amplification of this point. The existence of a probability measure, under which the discounted prices become martingales, plays a central role in the theory of continuous trading developed by these authors.

More generally, if the dividend rate process μ_i is nonnegative, then the discounted price process βP_i is a supermartingale under $\tilde{P}_T$.

4. Contingent Claims

In order to fix ideas, let us take $d = 1$ in the market model of Section 2, and suppose that at time $t = 0$ we sign a contract which gives us the option to buy, at any time τ between $t = 0$ and an "expiration date" $t = T$, one share of the stock at a specified price of c dollars (the contractual "exercise price"). If at time $t = \tau$ the price $P_1(\tau)$ of the stock is below the exercise price, the contract is worthless to us; but if $P_1(\tau) > c$, we can exercise our option (i.e., to buy one share at the pre-assigned price c) and then sell the share immediately in the market, thus making a net profit of $(P_1(\tau)-c)^+$ dollars. Because clairvoyance has to be excluded, τ is restricted to be a *stopping time* of $\{\mathscr{F}_t\}$, with values in $[0, T]$.

Such a contract is commonly called an *American option*, in contradistinction to a *European option* which allows exercise only on the expiration date, i.e., $\tau = T$. Both European and American options are financial instruments and can be traded in their own right (e.g., at the Chicago Board Options Exchange and other organized secondary markets for options).

The following definitions provide a generalization of the concept of option.

Definition 4.1. An *American Contingent Claim* (or ACC) (T, f, g) is a financial instrument consisting of

(i) an expiration date $T \in (0, \infty]$,
(ii) the selection of an exercise time $\tau \in \mathcal{S}_{0,T}$,
(iii) a payoff rate g_t per unit time on $(0, \tau)$, and
(iv) a terminal payoff f_τ at the exercise time.

The processes $F = \{f_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ and $G = \{g_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ are assumed throughout Sections 2–5 to be nonnegative, progressively measurable, and to satisfy, for some fixed $\mu > 1$,

$$E\left(\sup_{0 \le s \le t} f_s + \int_0^t g_s\, ds \right)^{\mu} < \infty \qquad \text{for every} \quad 0 < t < \infty. \tag{4.1}$$

Furthermore, F is assumed to have continuous paths.

Remark 4.2. An ACC with an expiration date $T = \infty$ is called *perpetual*; in this case, (iv) above is to be understood with the convention

$$f_\infty(\omega) \triangleq \overline{\lim_{t \to \infty}} f_t(\omega), \qquad \omega \in \Omega. \tag{4.2}$$

Definition 4.3. A *European Contingent Claim* (or ECC) (T, f_T, g) is a financial instrument consisting of

(i) a maturity date $T \in (0, \infty)$,
(ii) a payoff rate of g_t per unit time on $(0, T)$, and
(iii) a terminal payoff f_T at maturity.

Example 4.4. An American (European) option is a special case of an ACC (resp., ECC) with $d = 1$, $g_t = 0$, and $f_t = (P_1(t) - c)^+$. The number $c \ge 0$ is called the *exercise price* of the option.

The central question of option pricing can be formulated as follows: *What is a fair price to pay at time $t = 0$ for the (European or American) Contingent Claim?* For simplicity, preparation and motivation, we shall address first the case of a European Contingent Claim, following Karatzas and Shreve [10, Section 5.8]. American Contingent Claims will be taken up again in the next sections.

Definition 4.5. Let $x \ge 0$ and $T > 0$ be two given finite numbers; a pair $(\pi, C) \in \mathcal{A}(T, x)$ with corresponding wealth process X is called a *hedging strategy against the ECC* of Definition 4.3, if

(i) $C_t = \int_0^t g_s\, ds$, $0 \le t \le T$, and
(ii) $X_T = f_T$

hold almost surely.

Clearly, a hedging strategy "duplicates" the payoff from the ECC, by managing a portfolio that consists of the basic instruments in the market (i.e., the stocks and the bond) and by appropriate, absolutely continuous consumption as in (i) above at the payoff rate of the ECC. If there exists such a hedging strategy for an initial endowment $X_0 = x$, then the agent, instead of buying at time $t=0$ the ECC (T, f_T, g) for the price x, can invest in the market according to the portfolio π and consume his wealth according to the process C in such a way as to duplicate the payoff from the ECC. Consequently, the price of the latter should not be greater than x.

Definition 4.6. The smallest value of $x \geq 0$ for which there exists a hedging strategy $(\pi, C) \in \mathscr{A}(T, x)$ against the ECC of Definition 4.3, is called the *fair price* (or *value*) at $t=0$ of the ECC.

Let us now introduce the continuous, nonnegative process

$$Q_t \triangleq \beta(t) f_t + \int_0^t \beta(s) g_s \, ds, \mathscr{F}_t, \qquad 0 \leq t \leq T, \tag{4.3}$$

and denote by K_T an upper bound on both $\|\theta(t, \omega)\|$ and $\beta(t, \omega)$, $(t, \omega) \in [0, T] \times \Omega$. For every finite $\alpha > 1$ we obviously have

$$Z_T^\alpha = \exp\left\{-\int_0^T \alpha\theta^*(s)\, dW(s) - \tfrac{1}{2}\int_0^T \|\alpha\theta(s)\|^2 \, ds\right\}$$
$$\cdot \exp\left\{\frac{\alpha(\alpha-1)}{2}\int_0^T \|\theta(s)\|^2 \, ds\right\}$$

and thus

$$EZ_T^\alpha \leq \exp\left\{\frac{\alpha(\alpha-1)}{2} T K_T^2\right\} < \infty. \tag{4.4}$$

We then obtain from the Hölder inequality and (4.3), with $p \triangleq \sqrt{\mu} > 1$ and $1/p + 1/q = 1$,

$$\tilde{E}_T\left(\max_{0 \leq t \leq T} Q_t\right)^p \leq \left[K_T^\mu \cdot E\left(\max_{0 \leq t \leq T} f_t + \int_0^T g_t \, dt\right)^\mu\right]^{1/p} \cdot (EZ_T^q)^{1/q} < \infty \tag{4.5}$$

by virtue of (4.1) and (4.4).

Theorem 4.7. *The fair price for the ECC (T, f_T, g) is given by the finite number*

$$\tilde{E}_T(Q_T) = \tilde{E}_T\left[f_T \exp\left(-\int_0^T r(u)\, du\right) + \int_0^T g_t \exp\left(-\int_0^t r(u)\, du\right) dt\right].$$

Moreover, there exists a hedging strategy $(\pi, C) \in \mathscr{A}(T, x)$, whose corresponding wealth process $X = \{X_t, \mathscr{F}_t; 0 \leq t \leq T\}$ is continuous and satisfies

$$X_t = \tilde{E}_T\left[f_T \exp\left(-\int_t^T r(u)\, du\right) + \int_t^T g_s \exp\left(-\int_t^s r(u)\, du\right) ds \,\Big|\, \mathscr{F}_t\right] \quad a.s. \tag{4.6}$$

for every fixed $t \in [0, T]$.

Proof. For every $x \geq 0$, for which there exists a hedging strategy $(\pi, C) \in \mathscr{A}(T, x)$ against the ECC, we have, from (3.15) with $\tau = T$, $\tilde{E}_T(Q_T) \leq x$. Therefore, the fair price cannot be smaller than the number $\tilde{E}_T(Q_T)$. It remains to find a hedging strategy with this number as its initial wealth. For this we consider, by analogy with (3.18) and (3.19), the nonnegative process

$$\zeta_t \triangleq P_0(t) \cdot \left[\tilde{E}_T(Q_T) + m_t - \int_0^t \beta(s) g_s \, ds \right], \qquad 0 \leq t \leq T, \tag{4.7}$$

where m is a right-continuous version of the $\tilde{P}_T$-martingale $\tilde{E}(Q_T | \mathscr{F}_t) - \tilde{E}(Q_T)$, with $m_0 = 0$ a.s. $\tilde{P}_T$. It is easily checked that, for every fixed $t \in [0, T]$, ζ_t agrees almost surely with the right-hand side of (4.6). Proceeding exactly as in the proof of Proposition 3.5 with D replaced by Q_T, we conclude that m admits a stochastic integral representation of the form (3.22), and that ζ can actually be taken to have continuous paths. Then $\pi(t)$ can be defined as in (3.24) and C_t by $\int_0^t g_s \, ds$, so that (4.7) becomes (3.14) with the identifications $X \equiv \zeta$, $x \equiv \tilde{E}_T(Q_T)$. □

Remark 4.8. Let $(\hat{\pi}, \hat{C}) \in \mathscr{A}(T, x)$ be any other hedging strategy against the ECC, with $x = \tilde{E}_T(Q_T)$. Denoting by $\hat{X}$ the wealth process corresponding to this strategy, and by

$$x + \hat{M}_t \triangleq x + \int_0^t \beta(s) \hat{\pi}^*(s) \sigma(s) \, d\tilde{W}(s) = \beta(t) \hat{X}_t + \int_0^t \beta(s) \, d\hat{C}_s \tag{4.8}$$

the nonnegative supermartingale of (3.14), we have $\tilde{E}_T(x + \hat{M}_0) = x = \tilde{E}_T(Q_T) = \tilde{E}_T(x + \hat{M}_T)$. It follows that $\{\hat{M}_t, \mathscr{F}_t; 0 \leq t \leq T\}$ is actually a martingale under $\tilde{P}_T$, and thus for every $t \in [0, T]$ we obtain from (4.8)

$$\begin{aligned}
\hat{X}_t &= P_0(t) \left\{ x + \tilde{E}_T(\hat{M}_T | \mathscr{F}_t) - \int_0^t \beta(s) g_s \, ds \right\} \\
&= \tilde{E}_T \left[f_T \exp\left(- \int_t^T r(u) \, du \right) + \int_t^T g_s \exp\left(- \int_t^s r(u) \, du \right) ds \,\Big|\, \mathscr{F}_t \right] \\
&= X_t \quad \text{a.s. } \tilde{P}_T.
\end{aligned}$$

We conclude that $\hat{X}$ is indistinguishable from the process X of Theorem 4.6; this latter is thus called *the valuation process* of the ECC.

It is also easily checked that the portfolio processes π, $\hat{\pi}$ agree on a subset of $[0, T]$ with full Lebesgue measure, almost surely.

Example 4.9. Consider a market model with constant coefficients $r(t) \equiv r \geq 0$, $\mu_i(t) \equiv \mu_i$, $\sigma_{ij}(t) \equiv \sigma_{ij}$, $1 \leq i, j \leq d$, and a contingent claim with $f_t = \varphi(P(t))$, $g_t \equiv 0$. Here, φ: $\mathscr{R}_+^d \to [0, \infty)$ is a continuous function and

$$P(t) = (P_1(t), \ldots, P_d(t))^* \tag{4.9}$$

is the vector of stock price processes which satisfy, in this case, the equations (3.25) in the form

$$dP_i(t) = P_i(t)\left[(r-\mu_i)\, dt + \sum_{j=1}^{d} \sigma_{ij}\, d\tilde{W}_j(t) \right], \qquad 1 \le i \le d. \tag{4.10}$$

The solution of these equations is given by

$$P_i(t) = p_i \exp\left[(r-\mu_i - \tfrac{1}{2}D_{ii})t + \sum_{j=1}^{d} \sigma_{ij} \tilde{W}_j(t) \right]. \tag{4.11}$$

We now introduce the function $h(t, p, y): [0, \infty) \times \mathscr{R}_+^d \times \mathscr{R}^d \to \mathscr{R}_+^d$, via

$$h_i(t, p, y) \triangleq p_i \exp[(r-\mu_i - \tfrac{1}{2}D_{ii})t + y_i], \qquad 1 \le i \le d,$$

and observe that (4.11) can be written in the vector form

$$P(t) = h(t, p, \sigma\tilde{W}(t)). \tag{4.12}$$

Coming now to the ECC $(T, f_T, 0)$ with $f_T = \varphi(P(T))$, we see from (4.6) and (4.12) that its valuation process is given by

$$\begin{aligned} X_t &= \tilde{E}_T[e^{-r(T-t)}\varphi(P(T))|\mathscr{F}_t] \\ &= \tilde{E}_T[e^{-r(T-t)}\varphi(h(T-t, P(t), \sigma(\tilde{W}(T)-\tilde{W}(t))))|\mathscr{F}_t] \\ &= e^{-r(T-t)} \int_{\mathscr{R}^d} \varphi(h(T-t, P(t), \sigma z))\Gamma_{T-t}(z)\, dz \end{aligned}$$

a.s. $\tilde{P}_T$, for every $t \in [0, T)$, where

$$\Gamma_t(z) \triangleq (2\pi t)^{-d/2} \exp\left\{ -\frac{\|z\|^2}{2t} \right\}, \qquad z \in \triangleq \mathscr{R}^d, \quad t > 0,$$

is the fundamental Gaussian kernel. It follows that, with

$$G(t, p) \triangleq \begin{cases} e^{-r(T-t)} \int_{\mathscr{R}^d} \varphi(h(T-t, p, \sigma z))\Gamma_{T-t}(z)\, dz, & 0 \le t < T, \quad p \in \mathscr{R}_+^d, \\ \varphi(p), & t = T, \quad p \in \mathscr{R}_+^d, \end{cases} \tag{4.13}$$

the valuation process of the ECC is given by

$$X_t = G(t, P(t)). \tag{4.14}$$

In this case it is even possible to "compute" the portfolio $\pi(t)$ that achieves the valuation process of (4.14). Indeed, under appropriate growth conditions on φ, the function $G(t, p)$ of (4.12) is the unique solution of the Cauchy problem

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sum_{k=1}^{d}\sum_{k=1}^{d} D_{ik} p_i p_k \frac{\partial^2 G}{\partial p_i \partial p_k} + \sum_{i=1}^{d} (r-\mu_i) p_i \frac{\partial G}{\partial p_i} - rG = 0 \qquad \text{on} \quad [0, T) \times \mathscr{R}_+^d,$$

$$G(T, p) = \varphi(p), \qquad p \in \mathscr{R}_+^d,$$

by the Feynman-Kac theorem. Applying Itô's rule to the process X of (4.14) and using the above equation and (4.10), we arrive at

$$dX_t = rX_t\,dt + \sum_{i=1}^{d}\sum_{j=1}^{d} \sigma_{ij}P_i(t)\frac{\partial}{\partial p_i}G(t, P(t))\,d\tilde{W}_j(t).$$

A comparison with (3.13) gives then

$$\pi_i(t) = P_i(t)\cdot\frac{\partial}{\partial p_i}G(t, P(t)), \qquad 0\le t\le T, \quad 1\le i\le d, \tag{4.15}$$

for the portfolio process of Theorem 4.7. This example is adapted from Harrison and Pliska [7].

Remark 4.10. In the particular case of a European option as in Example 4.4, with $d=1$, $\varphi(p)=(p-c)^+$ and exercise price $c>0$, the integration in (4.13) can be carried out in a somewhat more explicit form. Indeed, with

$$\Phi(z)\triangleq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left(-\frac{x^2}{2}\right)dx$$

and

$$v\pm(t,p)\triangleq\frac{1}{\sigma_{11}\sqrt{t}}\left[\log\left(\frac{p}{c}\right)+(r-\mu_1\pm\tfrac{1}{2}\sigma_{11}^2)t\right],$$

we have

$$G(t,p)=\begin{cases} pe^{-\mu(T-t)}\Phi(v_+(T-t,p))-ce^{-r(T-t)}\Phi(v_-(T-t,p)), & 0\le t<T,\ 0<p<\infty,\\ (p-c)^+, & t=T,\ 0<p<\infty. \end{cases} \tag{4.16}$$

Together with

$$X_t = G(t, P_1(t)), \qquad 0\le t\le T, \tag{4.17}$$

(4.16) constitutes the celebrated *Black and Scholes* [3] *formula.*

5. American Contingent Claims on a Finite Horizon

We now broach the *valuation* (or *pricing*) problem for an American Contingent Claim as in Definition 4.1. An analogue of Definition 4.5 will be needed, for the notion of hedging strategy.

Definition 5.1. For given finite horizon $T>0$ and level of initial wealth $x\ge 0$, consider a pair $(\pi, C)\in\mathscr{A}(T,x)$ and let X denote the corresponding wealth process. We say that (π, C) is a *hedging strategy against the ACC* (T, f, g) of Definition 4.1, and write $(\pi, C)\in\mathscr{H}(T,x)$, if for $\tilde{P}_T$-a.e., $\omega\in\Omega$, the following requirements hold:

$$A_t(\omega)\triangleq C_t(\omega)-\int_0^t g_s(\omega)\,ds, \qquad 0\le t\le T,$$

is a continuous, nondecreasing function. (5.1)

$$X_t(\omega) \ge f_t(\omega), \qquad \forall t \in [0, T]. \tag{5.2}$$

$$X_T(\omega) = f_T(\omega). \tag{5.3}$$

$$A_t(\omega) = A_{\tau_t(\omega)}(\omega) \qquad \text{for every fixed number} \quad t \in [0, T], \tag{5.4}$$

where we are using the notation

$$\tau_t \triangleq \inf\{t \le s \le T; \quad X_s = f_s\}. \tag{5.5}$$

The continuity of A is equivalent to that of the consumption process C, which in turn implies the continuity of the wealth X in (3.5); it then easily follows that the random variable in (5.5) is a stopping time: $\tau_t \in \mathcal{S}_{t,T}$.

Discussion 5.2. Suppose that an agent buys the ACC (T, f, g) at $t = 0$ for the price $x \ge 0$, and that there exists a pair $(\pi, C) \in \mathcal{H}(T, x)$. Then it makes no sense for the agent to exercise the claim at any time in $\{0 \le t \le T; \ X_t > f_t\}$, because he could have done strictly better in terms of terminal wealth, and at least as well in terms of consumption, by investing instead in the market and consuming his wealth according to the pair (π, C). A similar reasoning shows that it makes no sense to exercise the claim at any time in $\{0 \le t \le T; \ X_t = f_t\}$ other than τ_0; then on $[0, \tau_0]$ the portfolio/consumption pair (π, C) *duplicates exactly the payoff stream from the contingent claim.*

In particular, on $[0, \tau_0)$ the consumption is absolutely continuous, with rate exactly equal to the running payoff from the ACC (as in the case of an ECC):

$$C_t = \int_0^t g_s \, ds, \qquad 0 \le t < \tau_0,$$

and the portfolio "hedges", i.e., maintains a level of wealth strictly above the corresponding terminal payoff from the ACC:

$$X_t > f_t, \qquad 0 \le t < \tau_0,$$

almost surely. On $(\tau_0, T]$ we have to allow for consumption at a rate greater than g, or even for singular consumption, in order to satisfy both $X_{\tau_0} = f_{\tau_0}$ and (5.3).

By analogy with Definition 4.5, the fair price for an ACC is defined as the smallest value of the initial endowment $x \ge 0$, which permits the construction of a hedging strategy.

Definition 5.3. The *fair price* (or *value*) at $t = 0$ for the ACC (T, f, g) of Definition 4.1, is the number

$$V_0 \triangleq \inf\{x \ge 0; \ \exists (\pi, C) \in \mathcal{H}(T, x)\}. \tag{5.6}$$

Let $x \ge 0$ be any number for which there exists a hedging strategy $(\pi, C) \in \mathcal{H}(T, x)$; the optional sampling theorem applied to the nonnegative supermartingale of (3.14) then gives, in conjunction with properties (5.1) and (5.2),

$$\tilde{E}_T(Q_\tau) = \tilde{E}_T\left[\beta(\tau) f_\tau + \int_0^\tau \beta(s) g_s \, ds\right] \le \tilde{E}_T\left[\beta(\tau) X_\tau + \int_0^\tau \beta(s) \, dC_s\right] \le x \tag{5.7}$$

for every $\tau \in \mathcal{S}_{0,T}$. Therefore, with the notation

$$u(t) \triangleq \sup_{\tau \in \mathcal{S}_{t,T}} \tilde{E}_T(Q_\tau), \qquad 0 \le t \le T, \tag{5.8}$$

we have $u(0) \le x$, and from (5.6) we deduce

$$u(0) \le V_0. \tag{5.9}$$

Theorem 5.4. *The fair price at $t=0$ for the ACC (T, f, g) is given by*

$$V_0 = u(0) \triangleq \sup_{\tau \in \mathcal{S}_{0,T}} \tilde{E}_T\left[f_\tau \exp\left(-\int_0^\tau r(u)\,du\right) + \int_0^\tau g_s \exp\left(-\int_0^s r(u)\,du\right) ds \right]. \tag{5.10}$$

Moreover, there exists a strategy $(\pi, C) \in \mathcal{H}(T, u(0))$ with corresponding wealth process $X = \{X_t, \mathcal{F}_t;\ 0 \le t \le T\}$ which is continuous and satisfies

$$X_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} \tilde{E}_T\left[f_\tau \exp\left(-\int_t^\tau r(u)\,du\right) + \int_t^\tau g_s \exp\left(-\int_t^s r(u)\,du\right) ds \,\middle|\, \mathcal{F}_t \right] \quad a.s. \tag{5.11}$$

for every fixed $t \in [0, T]$.

In view of (5.9), only the second claim needs verification. For this purpose we have to recall some theory for the *optimal stopping* problem of (5.8), e.g., from Fakeev [5] (see also Xue [19]) or Bismut and Skalli [2]. According to these references, there exists a nonnegative supermartingale $Y = \{Y_t, \mathcal{F}_t;\ 0 \le t \le T\}$ with RCLL (Right Continuous with finite Left-hand Limits) paths, such that

$$u(t) = \tilde{E}_T(Y_t), \tag{5.12}$$

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} \tilde{E}_T(Q_\tau \mid \mathcal{F}_t) \quad \text{a.s. } \tilde{P}_T \tag{5.13}$$

hold for every given $t \in [0, T]$. In particular,

$$Y_0 = u(0), \qquad Y_T = Q_T \quad \text{a.s. } \tilde{P}_T. \tag{5.14}$$

This process is the *Snell envelope* of Q, i.e., the smallest supermartingale with RCLL paths which majorizes Q, and the stopping time

$$\rho_t \triangleq \inf\{t \le s \le T;\ Y_s = Q_s\} \tag{5.15}$$

is optimal for the problem of (5.8):

$$u(t) = \tilde{E}_T(Q_{\rho_t}), \qquad \forall t \in [0, T]. \tag{5.16}$$

Bismut and Skalli [2] also show that the supermartingale Y is *regular*:

$$\left.\begin{array}{l}\text{for every monotone sequence } \{\sigma_n\}_{n=1}^\infty \subseteq \mathcal{S}_{0,T} \text{ converging a.s. } \tilde{P}_T\\ \text{to a stopping time } \sigma \in \mathcal{S}_{0,T}, \text{ we have } \lim_{n\to\infty} \tilde{E}_T(Y_{\sigma_n}) = \tilde{E}_T(Y_\sigma).\end{array}\right\} \tag{5.17}$$

Lemma 5.5. *The supermartingale Y of* (5.12) *and* (5.13) *is of class $D[0, T]$, i.e., the family $\{Y_\tau\}_{\tau\in\mathscr{S}_{0,T}}$ is uniformly integrable, under $\tilde{P}_T$.*

Proof. We have $0\le Y_t\le m_t$: $\forall 0\le t\le T$ a.s. $\tilde{P}_T$, where m is an RCLL modification of the martingale $\tilde{E}_T(\max_{0\le\theta\le T} Q_\theta|\mathscr{F}_t)$. In the notation of (4.5) and with the help of the Doob and Jensen inequalities, we obtain

$$\tilde{E}_T\left(\sup_{0\le t\le T} Y_t\right)^p \le \tilde{E}_T\left(\sup_{0\le t\le T} m_t^p\right)\le q^p\cdot \tilde{E}_T(m_Y^p)$$

$$\le q^p\cdot \tilde{E}_T\left(\max_{0\le t\le T} Q_t\right)^p.$$

But this last expectation is finite, thanks to (4.5), and the conclusion follows. □

Lemma 5.5 and the regularity condition (5.17) allow us to invoke the Doob–Meyer decomposition (Meyer [14, Chapter VII] or Karatzas and Shreve [10, Section 1.4]) and to write Y in the form

$$Y_t = u(0)+M_t-\Lambda_t,\qquad 0\le t\le T, \tag{5.18}$$

a.s. $\tilde{P}_T$, where Λ is a *continuous* nondecreasing process and M is a $\tilde{P}_T$-martingale with RCLL paths and $M_0=\Lambda_0=0$, $\tilde{E}_T(\Lambda_T)=u(0)-\tilde{E}_T(Q_T)$.

The same reasoning as in the proof of Proposition 3.5, employing the "Bayes rule" and the stochastic integral representation of Brownian martingales, allows us to write M as

$$M_t=\sum_{j=1}^d\int_0^t \psi_j(s)\,d\tilde{W}_j(s),\qquad 0\le t\le T, \tag{5.19}$$

for suitable measurable and adapted processes $\{\psi_j(t), \mathscr{F}_t; 0\le t\le T\}$ which satisfy

$$\sum_{j=1}^d\int_0^T \psi_j^2(t)\,dt<\infty\quad\text{a.s. } \tilde{P}_T. \tag{5.20}$$

In particular, M and therefore also Y can be supposed to have continuous paths.

Lemma 5.6. *For every fixed $t\in[0, T]$ we have*

$$\Lambda_t=\Lambda_{\rho_t}\quad \textit{a.s. } \tilde{P}_T. \tag{5.21}$$

Proof. Thanks to (5.12), (5.16), and the continuity of Q and Y, we have

$$\tilde{E}_T(Y_t)=u(t)=\tilde{E}_T(Q_{\rho_t})=\tilde{E}_T(Y_{\rho_t}).$$

But then from (5.18) and the optional sampling theorem we conclude $\tilde{E}_T(\Lambda_t)=\tilde{E}_T(\Lambda_{\rho_t})$, and (5.21) follows. □

We are now in a position to establish the basic result of this section.

Proof of Theorem 5.4. We introduce the continuous, adapted process

$$X_t \triangleq \frac{1}{\beta(t)}\left[Y_t - \int_0^t \beta(s) g_s\, ds\right], \mathscr{F}_t, \qquad 0 \le t \le T. \tag{5.22}$$

From (5.13) it follows that this process satisfies (5.11) and also (5.2) and (5.3) thanks to the continuity of F; furthermore, the stopping times ρ_t of (5.15) and τ_t of (5.5) are actually the same for this choice of X.

On the other hand, the representations (5.18) and (5.19) allow us to cast (5.22) in the form

$$\beta(t)X_t + \int_0^t \beta(s) g_s\, ds + \Lambda_t = u(0) + \int_0^t \psi^*(s)\, d\tilde{W}(s),$$

or equivalently in the form (3.14), with $\pi(t)$ as in (3.24) and

$$C_t \triangleq \int_0^t g_s\, ds + \int_0^t P_0(s)\, d\Lambda_s \tag{5.23}$$

for $0 \le t \le T$. It develops that the process X of (5.22) gives the wealth corresponding to the portfolio/consumption pair (π, C) of (3.24) and (5.23); the requirement (5.1) is obviously satisfied, and (5.4) follows from Lemma 5.6. Consequently, $(\pi, C) \in \mathscr{H}(T, u(0))$. □

It can be shown (see Bismut and Skalli [2]) that the path $\Lambda(\omega)$ is flat off $\{0 \le s < \infty;\ Y_s(\omega) = Q_s(\omega)\}$, which means that, with the consumption process C defined by (5.23), the path $A(\omega)$ of (5.1) is actually flat off $\{0 \le s < \infty;\ X_s(\omega) = f_s(\omega)\}$, for $\tilde{P}_T$-a.e. $\omega \in \Omega$.

Remark 5.7. Lemma 5.5 shows that the process X *of* (5.22) *is of class* $D[0, T]$ *under* $\tilde{P}_T$. Under this additional condition it is possible to show that the wealth process $\hat{X}$ of any pair $(\hat{\pi}, \hat{C}) \in \mathscr{H}(T, u(0))$, not necessarily the same as the one constructed in the proof of Theorem 5.4, is given by (5.11) and is thus uniquely determined.

Indeed, for every fixed $t \in [0, T]$, the optional sampling theorem applied to the nonnegative supermartingale of (4.8) with $x = u(0)$ yields, in conjunction with (5.1) and (5.2),

$$\begin{aligned}\beta(t)\hat{X}_t &\ge \tilde{E}_T\left[\beta(\tau)\hat{X}_\tau + \int_t^\tau \beta(s)\, d\hat{C}_s \,\Big|\, \mathscr{F}_t\right]\\ &\ge \tilde{E}_T\left[\beta(\tau)f_\tau + \int_t^\tau \beta(s) g_s\, ds \,\Big|\, \mathscr{F}_t\right] \quad \text{a.s. } \tilde{P}_T\end{aligned} \tag{5.24}$$

for every $\tau \in \mathcal{S}_{t,T}$. On the other hand, suppose that $\hat{X}$ is of class $D[0, T]$ under $\tilde{P}_T$, consider $\hat{\tau}_t = \inf\{t \le s \le T;\ \hat{X}_s = f_s\}$ as in (5.5), and define

$$\sigma_m \triangleq \hat{\tau}_t \wedge \inf\left\{t \le s \le T;\ \int_t^s \|\pi(u)\|^2\, du \ge m\right\}$$

for every $m \ge 1$. We can now take conditional expectations in

$$\beta(\sigma_m)\hat{X}_{\sigma_m} + \int_t^{\sigma_m} \beta(s)\, d\hat{C}_s = \beta(t)\hat{X}_t + \int_t^{\sigma_m} \beta(s)\hat{\pi}^*(s)\sigma(s)\, d\tilde{W}(s)$$

with respect to $\mathcal{F}_t$, and obtain from (5.4)

$$\begin{aligned} \beta(t)\hat{X}_t &= \tilde{E}_T\left[\beta(\sigma_m)\hat{X}_{\sigma_m} + \int_t^{\sigma_m} \beta(s)\, d\hat{C}_s \,\middle|\, \mathcal{F}_t\right] \\ &= \tilde{E}_T\left[\beta(\sigma_m)\hat{X}_{\sigma_m} + \int_t^{\sigma_m} \beta(s) g_s\, ds \,\middle|\, \mathcal{F}_t\right] \quad \text{a.s. } \tilde{P}_T. \end{aligned} \tag{5.25}$$

Because of (3.4) we have $\lim_{m\to\infty} \sigma_m = \hat{\tau}_t$, a.s. $\tilde{P}_T$; the membership of $\hat{X}$ in $D[0, T]$ and the monotone convergence theorem allow us to conclude from (5.25), by letting $m \to \infty$, that

$$\begin{aligned} \beta(t)\hat{X}_t &= \tilde{E}_T\left[\beta(\hat{\tau}_t)\hat{X}_{\hat{\tau}_t} + \int_t^{\hat{\tau}_t} \beta(s) g_s\, ds \,\middle|\, \mathcal{F}_t\right] \\ &= \tilde{E}_T\left[\beta(\hat{\tau}_t) f_{\hat{\tau}_t} + \int_t^{\hat{\tau}_t} \beta(s) g_s\, ds \,\middle|\, \mathcal{F}_t\right] \quad \text{a.s. } \tilde{P}_T. \end{aligned} \tag{5.26}$$

It follows from (5.24) and (5.26) that $\hat{X}_t$ is given by the right-hand side of (5.11) for every $t \in [0, T]$, and therefore that $\hat{X}$ is indistinguishable from the process X of Theorem 5.4. This latter is thus called *the valuation process for the ACC* (T, f, g).

Remark 5.8. Theorem 5.4 was established in [1] under a regularity condition on the process F, and under the assumption that both processes F, G of Definition 4.1 are uniformly *bounded.* This condition is not satisfied, however, in the prototypical case of an American option (Example 4.4).

Let us now examine some elementary consequences of Theorem 5.4.

Example 5.9. Consider the case where the process Q of (4.3) is a *submartingale* under $\tilde{P}_T$ (equivalently, the process $\{Q_t Z_t, \mathcal{F}_t;\ 0 \le t \le T\}$ is a submartingale under P). Then it is easily seen from (5.12), (5.13), and the optional sampling theorem that

$$Y_t = \tilde{E}_T(Q_T \mid \mathcal{F}_t) \quad \text{a.s. } \tilde{P}_T \quad \text{and} \quad u(t) = \tilde{E}_T(Q_T)$$

hold for every given $t \in [0, T]$, i.e., $\tau_t = T$ is optimal in (5.8). It develops that the pricing problem is equivalent, in this case, to that of the ECC (T, f_T, g), and that the valuation process is given simply by (4.6).

For instance, in the case of Example 4.4 with $r(t) \geq 0$, $\mu_1(t) \equiv 0$, and $c > 0$, the process

$$Q_t = (\beta(t)P_1(t) - c\beta(t))^+$$

is easily seen to be a submartingale under $\tilde{P}_T$; cf. Remark 3.6. We recover a result of Merton [13] in the following form: *an American option with positive exercise price, written on a stock which pays no dividends, should not be exercised before the expiration date.* More specifically, if $r(t) \equiv r > 0$ and $\sigma_{11}(t) \equiv \sigma_{11} > 0$, the valuation process is given by the Black and Scholes formula (4.16) and (4.17).

Remark 5.10. In the context of Example 4.9, suppose that the function $\varphi: \mathscr{R}_+^d \to [0, \infty)$ is twice continuously differentiable and satisfies

$$\tfrac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} D_{ik}\, p_i p_k \frac{\partial^2 \varphi(p)}{\partial p_i \partial p_k} + \sum_{i=1}^{d} (r - \mu_i) p_i \frac{\partial \varphi(p)}{\partial p_i} \geq r\varphi(p)$$

in $\mathscr{R}_+^d$, as well as a polynomial growth condition in $\|p\|$. Then it is not hard to check that the process $Q_t = e^{-rt}\varphi(P(t))$; $0 \leq t \leq T$ is a submartingale under $\tilde{P}_T$, and the valuation process for the ACC $(T, f, 0)$ with $f_t = \varphi(P(t))$ is given by (4.14).

Example 5.11. If the process Q of (4.3) is a *supermartingale* under $\tilde{P}_T$, then $Y = Q$, $u(t) = \tilde{E}_T(Q_t)$, $\tau_t = t$, and (5.22) gives

$$X = f. \tag{5.27}$$

Consider in this vein the situation in Example 4.4 with $c = 0$, $\mu_1(t) \geq 0$. Then $Q = \beta P_1$ is a supermartingale under $\tilde{P}_T$, and (5.27) gives $X = P_1$. In other words, *an American option with zero exercise price should be valued at the same amount as the stock.*

Remark 5.12. In the case of an American option with $\mu_1(t) \geq 0$ (Example 4.4), we have, from Remark 3.6 and (5.11),

$$\beta(t)X_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_{t,T}} \tilde{E}_T[\beta(\tau)(P_1(\tau) - c)^+ | \mathscr{F}_t] \leq \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_{t,T}} \tilde{E}_T[\beta(\tau)P_1(\tau) | \mathscr{F}_t]$$
$$\leq \beta(t)P_1(t) \quad \text{a.s. } \tilde{P}_T$$

for every $t \in [0, T]$, i.e. $X_t \leq P_1(t)$: the underlying stock is always at least as valuable as the option.

6. Perpetual Claims

When it comes to the valuation of perpetual American Contingent Claims (Remark 4.2), it becomes essential that the process $\tilde{W}$ of (3.12) be a Brownian motion on the entire of $[0, \infty)$ under an appropriate probability measure $\tilde{P}$, and be accompanied by a filtration which satisfies the usual conditions and measures all the processes in the model. These requirements cause some technical difficulties, which are resolved once it is assumed that all the coefficients in the market model are progressively measurable functionals of the driving, d-dimensional Brownian motion W.

Definition 6.1. A measurable process $\{\psi_t; 0 \le t < \infty\}$ on the space $(\Omega, \mathscr{F}, P)$, which is of the form

$$\psi_t(\omega) = \Psi(t, W.(\omega)), \qquad (t, \omega) \in [0, \infty) \times \Omega, \tag{6.1}$$

for some function $\Psi: [0, \infty) \times C^d[0, \infty) \to \mathscr{R}$, is called a *progressively measurable Brownian functional* if the mapping $(t, y) \mapsto \Psi(t, y)$: $[0, T] \times C^d[0, \infty) \to \mathscr{R}$ is $\mathscr{B}(\mathscr{R}) \backslash \mathscr{B}([0, T]) \otimes \mathscr{B}_T(C^d[0, \infty))$ measurable and bounded, for every finite $T > 0$. In (6.1) $W.(\omega)$ is the path of the Brownian motion W in Section 2.

We have denoted here by $C^d[0, \infty)$ the space of continuous functions $y: [0, \infty) \to \mathscr{R}^d$, equipped with the topology of uniform convergence on compact subsets of $[0, \infty)$, by $\mathscr{B}(C^d[0, \infty))$ the associated Borel σ-field, and by $\mathscr{B}_T(C^d[0, \infty))$ the σ-field $\varphi_T^{-1}(\mathscr{B}(C^d[0, \infty)))$, where φ_T: $C^d[0, \infty) \to C^d[0, \infty)$ is the truncation mapping $(\varphi_T y)(s) \triangleq y(T \wedge s)$; $0 \le s < \infty$, $y \in C^d[0, \infty)$.

Assumption 6.2. The coefficients $r, b_i, \mu_i, \sigma_{ij}$, $1 \le i, j \le d$, of the market model, as well as the processes F and G of Definition 4.1, will be assumed in this section to be progressively measurable Brownian functionals. Furthermore, the process F will be assumed to have continuous paths.

Then the components $\{\theta_i\}_{i=1}^d$ of the vector-valued process in (3.8) are also progressively measurable Brownian functionals, and by the consistency theorem one can construct a probability measure $\tilde{P}$ on $(\Omega, \mathscr{F}_\infty^W)$, such that

(i) the probability measures $\tilde{P}$ and $\tilde{P}_T$ agree on $\tilde{\mathscr{F}}_T$, for every $0 \le T < \infty$, and

(ii) the process (6.2)

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s)\, ds, \mathscr{F}_t^W, \qquad 0 \le t < \infty,$$

is standard, d-dimensional Brownian motion under $\tilde{P}$ (see Ikeda and Watanabe [9, pp. 176–180] or Karatzas and Shreve [10, Section 3.5]). Equations (3.13), (3.14) and (3.25), (3.26) now hold for $0 \le t < \infty$, a.s. $\tilde{P}$.

In order to endow the new Brownian motion of (6.2) with a filtration which satisfies the usual conditions, one could take the augmentation under $\tilde{P}$ of

$$\mathscr{F}_t^{\tilde{W}} = \sigma(\tilde{W}(s); 0 \le s \le t), \qquad 0 \le t < \infty,$$

but the resulting filtration will typically fail to measure the coefficients of the model. A more convenient and universal choice of filtration is obtained as follows: denote by $\{\mathscr{M}_t\}$ the augmentation under $\tilde{P}$ of $\{\mathscr{F}_t^W\}$, and define

$$\tilde{\mathscr{F}}_t \triangleq \mathscr{M}_{t+} = \bigcap_{\varepsilon > 0} \mathscr{M}_{t+\varepsilon}, \qquad 0 \le t < \infty. \tag{6.3}$$

This filtration obviously satisfies the usual conditions, and

$$\{\tilde{W}(t), \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\} \text{ is a standard Brownian motion under } \tilde{P}. \tag{6.4}$$

Indeed, it suffices to show that, for every function $f\colon \mathcal{R}^d \to \mathcal{R}$ which is twice continuously differentiable and has compact support, the process

$$M_t^f \triangleq f(\tilde{W}(t)) - \tfrac{1}{2}\int_0^t \Delta f(\tilde{W}(s))\, ds, \qquad 0 \le t < \infty,$$

is an $\{\tilde{\mathcal{F}}_t\}$-martingale under $\tilde{P}$. We know from (6.2) (ii) that M^f is an $\{\mathcal{F}_t^W\}$-martingale under $\tilde{P}$: thus with $0 < s < s + 1/n < t < \infty$, there exists for every given $F \in \mathcal{M}_{s+1/n}$ an event $G \in \mathcal{F}_{s+1/n}^W$ such that $\tilde{P}(F \triangle G) = 0$ and

$$\tilde{E}[(M_t^f - M_{s+1/n}^f)1_F] = \tilde{E}[(M_t^f - M_{s+1/n}^f)1_G] = 0. \tag{6.5}$$

By taking $F \in \mathcal{M}_{s+} = \tilde{\mathcal{F}}_s$ and then letting $n \to \infty$ in (6.5), we obtain $\tilde{E}[(M_t^f - M_s^f)1_F] = 0$, and therefore (6.4) as well.

In this section we shall take $(\Omega, \tilde{\mathcal{F}}_\infty, \tilde{P}), \{\tilde{\mathcal{F}}_t\}$ *as our basic probability space.* All processes under consideration will be adapted to $\{\tilde{\mathcal{F}}_t\}$. In Definitions 3.1-3.3 and Remark 3.4, the filtration $\{\tilde{\mathcal{F}}_t\}$ now replaces $\{\mathcal{F}_t\}$, and almost surely statements are understood with respect to $\tilde{P}$. By analogy with (4.2), we shall understand

$$\xi_\infty(\omega) \triangleq \overline{\lim_{t\to\infty}}\, \xi_t(\omega),\ \ \omega \in \Omega, \quad \text{and} \quad \tilde{E}\xi_\tau \triangleq \tilde{E}[\xi_\tau 1_{\{\tau<\infty\}} + \xi_\infty 1_{\{\tau=\infty\}}] \tag{6.6}$$

for any nonnegative, progressively measurable process $\{\xi_t, \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\}$ on this space and any $\tau \in \mathcal{S}_0^*$. Finally, it will be assumed that the process $Q = \{Q_t, \mathcal{F}_t^W;\ 0 \le t < \infty\}$ of (4.3) satisfies

$$\tilde{E}\Big(\sup_{0\le t<\infty} Q_t\Big) < \infty. \tag{6.7}$$

We now take up the valuation problem for perpetual American Contingent Claims.

Definition 6.3. For any given level $x \ge 0$ of initial endowment, we say that a pair $(\pi, C) \in \mathcal{A}(x)$ is a *hedging strategy against the perpetual ACC* (∞, f, g), and write $(\pi, C) \in \mathcal{H}(x)$, if the analogues of (5.1), (5.2), (5.4) on $[0, \infty)$ and

$$\overline{\lim_{t\to\infty}}\, \beta(t, \omega) X_t(\omega) = \overline{\lim_{t\to\infty}}\, \beta(t, \omega) f_t(\omega) \tag{5.3'}$$

hold for $\tilde{P}$-a.e. $\omega \in \Omega$.

By analogy with Definition 5.3, the number

$$V_0 \triangleq \inf\{x \ge 0;\ \exists(\pi, C) \in \mathcal{H}(x)\} \tag{6.8}$$

will be called the *fair price* (or *value*) at $t = 0$ for the perpetual ACC (∞, f, g).

Theorem 6.4. *For the number in* (6.8) *we have*

$$V_0 = u(0) \triangleq \sup_{\tau \in \mathcal{S}_0^*} \tilde{E}\left[f_\tau \exp\left(-\int_0^\tau r(u)\,dy\right) + \int_0^\tau g_s \exp\left(-\int_0^s r(u)\,du\right) ds\right]. \tag{6.9}$$

Moreover, there exists a strategy $(\pi, C) \in \mathcal{H}(u^*(0))$ *whose wealth process* $X = \{X_t, \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\}$ *is given by*

$$X_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t^*} \tilde{E}\left[f_\tau \exp\left(-\int_t^\tau r(u)\,du\right) + \int_t^\tau g_s \exp\left(-\int_t^s r(u)\,du\right) ds \,\middle|\, \mathcal{F}_t\right] \quad \text{a.s. } \tilde{P} \tag{6.10}$$

for every fixed $0 \le t \le \infty$.

The detailed development is omitted; as before, it uses the optional sampling theorem (which, applied to the nonnegative $\tilde{P}$-supermartingale of (3.14), leads ultimately to $u(0) \le V_0$) and the results of [5] and [2] for the optimal stopping problem

$$u(t) \triangleq \sup_{\tau \in \mathcal{S}_t^*} \tilde{E}(Q_\tau), \qquad 0 \le t < \infty. \tag{6.11}$$

The Snell envelope for this problem, i.e., the smallest supermartingale $Y = \{Y_t, \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\}$ which majorizes Q, satisfies

$$u(t) = \tilde{E}Y_t \quad \text{and} \quad Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t^*} \tilde{E}(Q_\tau \mid \mathcal{F}_t) \quad \text{a.s. } \tilde{P} \tag{6.12}$$

for every $t \in [0, \infty)$, as well as

$$Y_\infty = Q_\infty \quad \text{a.s. } \tilde{P} \tag{6.13}$$

with the convention of (6.6). This supermartingale is *regular* (i.e., (5.17) now holds, under $\tilde{P}$, for every finite $T > 0$) and *of class D* (i.e., the family $\{Y_\tau\}_{\tau \in \mathcal{S}_0^*}$ is uniformly integrable under $\tilde{P}$). For the last claim, it is useful to recall the strengthening of (6.12):

$$Y_\sigma = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\sigma^*} \tilde{E}(Q_\tau \mid \mathcal{F}_\sigma) \quad \text{a.s. } \tilde{P}$$

which is valid for every $\sigma \in \mathcal{S}_0^*$ with $\mathcal{S}_\sigma^* \triangleq \{\tau \in \mathcal{S}_0^*;\ \sigma \le \tau, \text{ a.s. } \tilde{P}\}$ (e.g., El Karoui [4, Section 2.15]). The stopping time $\rho_t = \inf\{s \ge t;\ Y_s = Q_s\} \in \mathcal{S}_t^*$ is optimal for the problem (6.11): $u(t) = \tilde{E}(Q_{\rho_t})$.

Now Y admits a Doob–Meyer decomposition of the form (5.18) on $[0, \infty)$, where $\{M_t, \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\}$ is a uniformly integrable martingale and $\{\Lambda_t, \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\}$ a continuous, integrable nondecreasing process, flat off $\{0 \le s < \infty;\ Y_s = Q_s\}$ and with $E(\Lambda_\infty) = u(0) - \tilde{E}(Q_\infty)$, $\Lambda_0 = M_0 = 0$ a.s. $\tilde{P}$. The martingale M admits the (Fujisaki–Kallianpur–Kunita) representation (5.19) on $[0, \infty)$, where now the integrands ψ_j are measurable, $\{\tilde{\mathcal{F}}_t\}$-adapted, and satisfy (5.20) a.s. $\tilde{P}$ for every finite $T > 0$; see, for instance, Theorem 5.20 in Liptser and Shiryaev [11], in conjunction with (6.2)(ii). Finally, the continuous process $\{X_t, \tilde{\mathscr{F}}_t;\ 0 \le t < \infty\}$ is defined as in (5.22) and is shown to be the wealth process of a strategy $(\pi, C) \in \mathscr{H}(u(0))$, just as in the proof of Theorem 5.4.

It is also observed that the process βX is of class D under $\tilde{P}$. For any other strategy $(\hat{\pi}, \hat{C}) \in \mathscr{H}(u(0))$ with wealth process $\hat{X}$ satisfying this condition, it is shown as in Remark 5.7 that $\hat{X}$ is indistinguishable from X, which thus earns the right to be called *the valuation process of the perpetual ACC* (∞, f, g).

Remark 6.5. In the case of perpetual American options as in Example 4.4 with $\sigma_{11}(t) \equiv \sigma > 0$, $\mu_1(t) \ge \mu > 0$, we have

$$Q_t = \beta(t)(P_1(t) - c)^+ \le \beta(t) P_1(t),$$

where the last process satisfies (3.26). It follows easily from this equation that

$$0 \le Q_t \le p_1\, e^{\sigma \eta_t}, \qquad 0 \le t < \infty,$$

holds a.s. $\tilde{P}$, where $\eta_t = \tilde{W}_1(t) - \nu t$ is a Brownian motion with negative drift and $\nu = \mu/\sigma + \sigma/2$. But now the law

$$\tilde{P}\left[\sup_{0 \le t < \infty} \eta_t \in db\right] = 2\nu\, e^{-2\nu b}\, db, \qquad b > 0,$$

is well known, and condition (6.7) follows from it. Consequently, *Theorem* 6.4 *applies to such American options* and we also have $Q_\infty = 0$, a.s. $\tilde{P}$.

For an American option as above but with $\mu_1(t) \equiv 0$ (i.e., on a stock which pays no dividends) and

$$\begin{aligned} &r(t) \ge 0, \qquad 0 \le t < \infty, \\ &\lim_{T\to\infty} \int_0^T r(t)\, dt = \infty \end{aligned} \tag{6.14}$$

valid a.s. $\tilde{P}$, we shall agree that the valuation process is given by

$$X_t \triangleq \lim_{\mu \downarrow 0} X_t(\mu), \qquad 0 \le t < \infty, \tag{6.15}$$

where $X(\mu)$ is the valuation process under a constant dividend rate, provided that the limit in (6.15) exists a.s. $\tilde{P}$. From the optional sampling theorem and (6.10) we then obtain

$$\begin{aligned} \beta(t)P_1(t) &\ge \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_t^*} \tilde{E}[\beta(\tau)P_1(\tau) \,|\, \tilde{\mathscr{F}}_t] \\ &\ge \beta(t)X_t(\mu) = \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_t^*} \tilde{E}[\beta(\tau)(P_1(\tau) - c)^+ \,|\, \tilde{\mathscr{F}}_t] \\ &\ge \tilde{E}[\beta(T)(P_1(T) - c)^+ | \tilde{\mathscr{F}}_t] \\ &\ge \beta(t)P_1(t)\exp\{-\mu(T-t)\} - c\tilde{E}[\beta(T) | \mathscr{F}_t| \end{aligned}$$

for every finite numbers $\mu > 0$ and $T > t$. Letting $\mu \downarrow 0$ and then $T \to \infty$ we obtain $X_t = P_1(t)$ from (6.15). In words, *a perpetual American option on a stock which pays no dividends, and in the presence of condition* (6.14), *must sell for the same amount as the stock.*

If the perpetual option has zero exercise price, the same conclusion holds without the restriction $\mu_1(t) \equiv 0$ (by analogy with Example 5.10).

Case 6.6 (The Markovian Case). Consider the market and contingent claim model of Example 4.9; under the conditions of this section, the valuation process X of Theorem 6.4 is obtained via

$$X_t = v(P(t)), \qquad 0 \le t < \infty, \tag{6.16}$$

a.s. $\tilde{P}$, where v: $\mathcal{R}_+^d \to [0, \infty)$ is the least r-excessive majorant of the function φ (see Fakeev [5]).

Example 6.7. In Case 6.6 (Markovian) with $d=1$, $c=1$ and $\varphi(x)=(x-1)^+$ the function v in (6.16) was computed by McKean [12] as

$$v(x) = \begin{cases} (\kappa - 1)\left(\dfrac{x}{\kappa}\right)^\gamma, & 0 < x < \kappa, \\ x - 1, & \kappa \le x < \infty, \end{cases}$$

with $\gamma = (1/\sigma^2)(\sqrt{\delta^2 + 2r\sigma^2} - \delta)$, $\alpha = r - \mu > 0$, $\delta = \alpha - \sigma^2/2$, and $\kappa = \gamma/(\gamma - 1) > 1$, and the optimal exercise time becomes $\tau_t = \inf\{s \ge t;\ P_1(s) \ge \kappa\}$. This stopping time belongs to $\mathcal{S}_t$. The finite-horizon version of this problem was studied by Van Moerbeke [18], along with the associated free-boundary problem.

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