ON THE PRIME GRAPH OF A FINITE GROUP WITH UNIQUE NONABELIAN COMPOSITION FACTOR

MARIA A. GRECHKOSEEVA

Sobolev Institute of Mathematics
Koptyuga 4, Novosibirsk 630090, Russia

ANDREY V. VASIL'EV

Sobolev Institute of Mathematics
Koptyuga 4, Novosibirsk 630090, Russia
Novosibirsk State University
Pirogova 1, Novosibirsk 630090, Russia

Abstract. We say that finite groups are isospectral if they have the same sets of orders of elements. It is known that every nonsolvable finite group G isospectral to a finite simple group has a unique nonabelian composition factor, that is, the quotient of G by the solvable radical of G is an almost simple group. The main goal of this paper is prove that this almost simple group is a cyclic extension of its socle.

To this end, we consider a general situation when G is an arbitrary group with unique nonabelian composition factor, not necessarily isospectral to a simple group, and study the prime graph of G, where the prime graph of G is the graph whose vertices are the prime numbers dividing the order of G and two such numbers r and s are adjacent if and only if $r \neq s$ and G has an element of order rs. Namely, we establish some sufficient conditions for the prime graph of such a group to have a vertex adjacent to all other vertices. Besides proving the main result, this allows us to refine a recent result by P. Cameron and N. Maslova concerning finite groups almost recognizable by prime graph.

Keywords: almost simple group, group of Lie type, order of an element, recognition by spectrum, prime graph

1. Introduction

Given a finite group G, we denote the set of prime divisors of the order of G by $\pi(G)$. The set of element orders of G is called the spectrum of G and denoted by $\omega(G)$. If $\omega(G) = \omega(H)$, then G and H are said to be isospectral.

Suppose that G is a finite group isospectral to a finite nonabelian simple group L. Then G is either solvable, in which case L is one of $L_3(3)$, $U_3(3)$, $S_4(3)$, or has exactly

E-mail addresses: grechkoseeva@gmail.com, vasand@math.nsc.ru.

This work was supported by the Program of Fundamental Scientific Researches of the Siberian Branch of Russian Academy of Sciences, I.1.1, project 0314-2019-0001.

one nonabelian composition factor (see [5, Theorem 2]). In what follows, we assume that G is not solvable, and so G has a normal series

$$1 \leqslant K < H \leqslant G, \tag{1.1}$$

where K is the solvable radical of G, H/K is a nonabelian simple group and G/K is an almost simple group with socle H/K. Denoting H/K by S, we may identify G/K with a subgroup of $\operatorname{Aut} S$, and then G/H with a subgroup of $\operatorname{Out} S = \operatorname{Aut} S/S$. Observe that G/H is solvable.

If L is sufficiently 'large', more precisely, if L is a classical group of dimension at least 38 or a non-classical group other than Alt_6, Alt_{10}, J_2, ${}^3D_4(2)$, then $K = 1$ and $H \simeq L$ (see [9,13]). Furthermore, it follows that G/H is cyclic (see [8] and the references therein). In general case, K is not always trivial and H/K is not always isomorphic to L but in all known examples, G/H is cyclic. This observation suggests us to conjecture that G/H is always cyclic and the main goal of this paper is to prove this conjecture.

Theorem 1. *Let L be a finite nonabelian simple group and let G be a nonsolvable finite group with $\omega(G) = \omega(L)$. Suppose that $1 \leqslant K < H \leqslant G$ is the normal series of G as in (1.1). Then G/H is cyclic. Furthermore, if H/K is a simple group of Lie type other than $L_2(q)$, then G/H does not contain diagonal automorphisms.*

If L is sporadic or alternating, Theorem 1 is a direct consequence of the known description of groups isospectral to L. If L is a group of Lie type, the proof has several ingredients. The first is the well-known property of spectra of groups of Lie type stated in Lemma 2.1 in Section 2. The second is the nilpotency of the solvable radical of G established in [18]. The third is the following Theorem 2 which concerns all finite groups of some specific structure, not only those isospectral to simple groups.

Theorem 2. *Suppose that a finite group G has a normal series $1 \leqslant K < H \leqslant G$, where K is the solvable radical of G, $S = H/K$ is a finite simple group of Lie type, and $G/K \leq \operatorname{Aut} S$. Suppose also that K is nilpotent.*

(i) *If $S \neq L_2(q)$ and G/H contains a diagonal automorphism of S of prime order r, then $rs \in \omega(G)$ for all $s \in \pi(G) \setminus \{r\}$.*
(ii) *If G/H is not cyclic, then there is $r \in \pi(G/H)$ such that $rs \in \omega(G)$ for all $s \in \pi(G) \setminus \{r\}$.*

The set $\omega(G)$ defines the prime graph of G as follows: the vertex set of this is $\pi(G)$ and two primes $r, s \in \pi(G)$ are adjacent if and only if $r \neq s$ and $rs \in \omega(G)$. The prime graph is also known as the Gruenberg–Kegel graph and we denote it by $GK(G)$. It is not hard to see that Theorem 2 states a property of the graph $GK(G)$ rather than of the whole set $\omega(G)$. This allows us to apply this theorem to the problem of recognition of simple groups by prime graph. Recently, P. Cameron and N. Maslova [1] proved several new results relating to this problem. In Theorem 3, we slightly refine Theorem 1.4 of [1].

Theorem 3. *There exists a function $F(x) = O(x^5)$ such that for each labeled graph Γ, the following conditions are equivalent:*

(i) *there exist infinitely many groups H such that $GK(H) = \Gamma$;*
(ii) *there exist more than $F(|V(\Gamma)|)$ groups H such that $GK(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.*

In fact, Theorem 1.4 of [1] states exactly the same as Theorem 3 but with x^7 in place of x^5.

2. Proofs of Theorems 1 and 2

We begin this section with notation and preliminary results. We write $L_n^\varepsilon(q)$ and $E_6^\varepsilon(q)$ assuming that $\varepsilon \in \{+,-\}$, $L_n^+(q) = L_n(q)$, $L_n^-(q) = U_n(q)$, $E_6^+(q) = E_6(q)$, and $E_6^-(q) = {}^2E_6(q)$. If r is a prime and a is an integer, then $(a)_r$ is the highest power of r dividing a. If S is a group of Lie type, then $\operatorname{Inndiag} S$ is the subgroup of $\operatorname{Aut} S$ generated by inner and diagonal automorphisms, and $\operatorname{Outdiag} S$ is the image of $\operatorname{Inndiag} S$ in $\operatorname{Out} S$. Also we use the terms 'field automorphism' and 'graph automorphism' of S according to [4, Definition 2.5.13].

Lemma 2.1. *If S is a finite simple group of Lie type, then for every $r \in \pi(S)$ there is $s \in \pi(S)$ such that $r \neq s$ and $rs \notin \omega(S)$.*

Proof. This follows from [16, 17] (see, for example, [6, Lemma 2.2]). □

Lemma 2.2. *Let S be a finite simple group of Lie type in characteristic p. If r divides $|\operatorname{Outdiag} S|$ and $rp \notin \omega(S)$, then either $S = L_2(q)$, or $S = L_3^\varepsilon(q)$ and $(q-\varepsilon)_3 = 3$.*

Proof. This follows, for example, from [16, Propositions 3.1 and 3.2]. □

Lemma 2.3. *Let S be a finite simple group of Lie type in characteristic p. If $r \in \pi(S)$, r is odd and $2r \notin \omega(S)$, then either a Sylow r-subgroup of S is cyclic, or $S = L_2(q)$ and $r = p$, or $S = L_3^\varepsilon(q)$, $p = 2$, $r = 3$ and $(q-\varepsilon)_3 = 3$.*

Proof. This follows from the results of [16, Sections 3 and 4] and the cross-characteristic Sylow structure of groups of Lie type [3, (10-2)]. □

Lemma 2.4. *Let $S = {}^t\Sigma(q)$ be a finite simple group of Lie type, not a Suzuki–Ree group, and let φ be a field automorphism of S of prime order r. Then $r\cdot\omega({}^t\Sigma(q^{1/r})) \subseteq \omega(S \rtimes \langle\varphi\rangle)$.*

Proof. This follows from the Lang–Steinberg theorem [14, Section 10] (see, for example, [7, Lemma 2.8]). □

Lemma 2.5. *Suppose that G is a finite group, K is a normal subgroup of G and every $g \in G \setminus K$ acts fixed-point-freely on K. Then every odd order Sylow subgroup of G/K is cyclic and a Sylow 2-subgroup of G/K is cyclic or generalized quaternion.*

Proof. This is a well-known property of fixed-point-free automorphisms (see, for example, [10, Satz 8.7]). □

Proof of Theorem 2. Denote the defining characteristic of S by p, G/K by $\overline{G}$ and G/H by $\widehat{G}$. As we remarked in the introduction, $\widehat{G}$ can be regarded as a subgroup of $\operatorname{Out} S$.

Clearly, we may assume that either $\operatorname{Outdiag} S \neq 1$ or $\operatorname{Out} S$ is not cyclic, in particular, we may assume that S is not a Suzuki–Ree group and so $3 \in \pi(S)$.

(i) Suppose that $r \in \pi(\widehat{G} \cap \operatorname{Outdiag} S)$. Observe that $r \in \pi(S)$ and $r \neq p$. By Lemma 2.2, it follows that $rp \in \omega(S)$ unless $S = L_3^\varepsilon(q)$, $r = 3$ and $(q-\varepsilon)_3 = 3$. In this case $PGL_3^\varepsilon(q) \leqslant \overline{G}$, and since $PGL_3^\varepsilon(q)$ has an element of order $p(q-\varepsilon)$, we see that $rp \in \omega(\overline{G})$.

Suppose that $s \in \pi(S)$ and $s \neq p$. If $s \in \pi(\operatorname{Outdiag} S)$, then $rs \in \omega(S)$ since $\operatorname{Outdiag} S$ is abelian. So we may assume that $s \notin \pi(\operatorname{Outdiag} S)$. The maximal tori of $\operatorname{Inndiag} S$ are isomorphic to those of the universal version $\tilde{S}_u$ of $\tilde{S}$, where $\tilde{S} = S$ if S is not of type B_n or C_n, and $\tilde{B}_n(q) = C_n(q)$, $\tilde{C}_n(q) = B_n(q)$ (see [2, Section 4.4]). Since every maximal torus of $\tilde{S}_u$ contains the center $Z(\tilde{S}_u)$ of $\tilde{S}_u$ and $|Z(\tilde{S}_u)| = |\operatorname{Outdiag} S|$, we see that $\operatorname{Inndiag} S$ includes a maximal torus whose order is divisible by $s|\operatorname{Outdiag} S|$. So $\overline{G}$ contains an abelian subgroup of order sr.

Let $s \in \pi(\overline{G}) \setminus \pi(S)$. Since $s \neq 2, 3$ and $s \notin \pi(\operatorname{Outdiag} S)$, it follows that G/K contains a field automorphism of S of order s. By Lemma 2.4, we have $s \cdot \omega(S_0) \subseteq \omega(G/K)$, where S_0 is a group of the same Lie type as S. If $r = 2, 3$, then it is clear that $r \in \pi(S_0)$. If $r \neq 2, 3$, then $S = L_n^\varepsilon(q)$, r divides $(n, q-\varepsilon)$ and $S_0 = L_n^\varepsilon(q^{1/s})$. Since r divides $p^{r-1} - \varepsilon^{r-1}$ and $r - 1 \leqslant n - 1$, we see that $r \in \pi(S_0)$.

Let $s \in \pi(K) \setminus \pi(\overline{G})$. If $r = 2$, then s is adjacent to r in $GK(G)$ by [15, Proposition 2]. So we may assume that r is odd. If $S = E_6^\varepsilon(q)$ or $S = L_n^\varepsilon(q)$ with $n \geqslant 4$, then S includes a torus of the form $\mathbb{Z}_{q-\varepsilon} \times \mathbb{Z}_{q-\varepsilon}$, and hence S includes an elementary abelian group of order r^2. If $L = L_3^\varepsilon(q)$, then $PGL_3^\varepsilon(q) \leqslant \overline{G}$ and so $\overline{G}$ includes an elementary abelian group of order r^2. Now we apply Lemma 2.5 to conclude that $rs \in \omega(G)$.

(ii) Let $S \neq L_2(q)$. By (i), we may assume that $\widehat{G} \cap \operatorname{Outdiag} S = 1$. Then either $\widehat{G}$ includes an elementary abelian group of order 2^2, or $S = O_8^+(q)$ and, up to conjugation in $\operatorname{Out} S$, $\widehat{G}$ contains the image of the graph automorphism γ of S induced by the symmetry of the Dynkin diagram of order 3.

In the first case, $S = L_n(q)$, $O_{2n}^+(q)$, or $E_6(q)$, and we claim that 2 is adjacent to all odd primes in $GK(G)$. By [15, Proposition 2], every $s \in \pi(K) \cup \pi(\widehat{G})$ is adjacent to 2. Now let $t \in \pi(S)$ and suppose that $2t \notin \omega(S)$. Excluding for a while the case when $t = 3$, $S = L_3(q)$, $p = 2$, $(q-1)_3 = 3$ and applying Lemma 2.3, we conclude that a Sylow t-subgroup T of S is cyclic, and hence $N_{\overline{G}}(T)/C_{\overline{G}}(T)$ is cyclic. On the other hand, by the Frattini argument, $N_{\overline{G}}(T)/(N_{\overline{G}}(T) \cap S) \simeq \widehat{G}$, and so a Sylow 2-subgroup of $N_{\overline{G}}(T)$ is not cyclic. Thus $2 \in C_{\overline{G}}(T)$, and $2t \in \omega(\overline{G})$.

Suppose that $t = 3$, $S = L_3(q)$, $p = 2$, and $(q-1)_3 = 3$. Since $\widehat{G}$ includes an elementary abelian group of order 2^2, it follows that $\overline{G}$ contains a field automorphism of S of order 2, and so $6 \in \omega(\overline{G})$.

Now suppose that $S = O_8^+(q)$ and $\overline{G}$ contains the graph automorphism γ. The centralizer of γ in S is isomorphic to $G_2(q)$ [3, (9-1)] and so $3s \in \omega(G)$ for all $3 \neq s \in G_2(q)$. Since S includes an elementary abelian group of order 9, we conclude that $3s \in \omega(G)$ for all $s \in \pi(K) \setminus \{3\}$. Also a $2'$-Hall subgroup of $\operatorname{Out} S$ is abelian, and hence 3 is adjacent to every $s \in \pi(\widehat{G}) \setminus \{2, 3\}$ in $GK(\widehat{G})$.. Let $s \in \pi(S) \setminus \{3\}$ and $3s \notin \omega(S)$. Then s divides $q^2 + q + 1$ or $q^2 - q + 1$, therefore, $s \in \pi(G_2(q))$ and, as we remarked, $3s \in \omega(G)$. Thus 3 is adjacent to all vertices in $GK(G)$.

Let $S = L_2(q)$, where $q = p^l$. We claim that 2 is adjacent to all odd primes in $GK(G)$. Since $\operatorname{Out} S$ is a direct product of cyclic groups of orders $(2, q-1)$ and l, it follows that p is odd, l is even and $\overline{G} = PGL_2(q) \rtimes \langle \varphi \rangle$, where φ is a field automorphism of S of even order. Since $PGL_2(q)$ contains elements of orders $q \pm 1$ and $2p \in \omega(\overline{G})$ by Lemma 2.4, we see that 2 is adjacent to every odd $s \in \pi(\overline{G})$. Let $s \in \pi(K)$ be odd. A Sylow 2-subgroup of $PGL_2(q)$ is dihedral, and so it cannot act fixed-point-freely on a Sylow s-subgroup of K by Lemma 2.5. Hence $2s \in \pi(G)$, and the proof of Theorem 2 is complete. □

Now we are able to prove Theorem 1. Let $S = H/K$. Clearly, we may assume that $\operatorname{Out} S$ is not cyclic. In particular, we may assume that S is neither sporadic nor alternating with the following convention: if $S = Alt_6 \simeq L_2(9)$, we regard S as a group of Lie type.

If L is sporadic and $L \neq J_2$, or if $L = Alt_n$ and $n \neq 6, 10$, then $G \simeq L$ (see [11] and [5] respectively). If $L = J_2$, then $G \simeq L$ or $S = Alt_8$ by [11]. If $L = Alt_{10}$, then $G \simeq L$ or $S = Alt_5$ by [12]. If $L = Alt_6$, we regard L as a group of Lie type.

Let L be a group of Lie type. By [18, Theorem 1], it follows that K is nilpotent, and so G satisfies the hypothesis of Theorem 2. If G/H is not cyclic or if $S \neq L_2(q)$ and

G/H contains a diagonal automorphism of S, then there is $r \in \pi(G)$ adjacent to all other vertices in $GK(G)$. But this is impossible by Lemma 2.1 since $GK(G) = GK(L)$. This contradiction completes the proof of Theorem 1.

3. Groups almost recognizable by prime graph

Given a positive integer k, a finite group G is said to be k-recognizable by prime graph if there are exactly k pairwise nonisomorphic finite groups H with $GK(H) = GK(G)$ and almost recognizable by prime graph if it is k-recognizable for some k.

By [1, Theorem 1.3], if G is almost recognizable by prime graph, then G is almost simple and each group H with $GK(H) = GK(G)$ is almost simple. So if G is a k-recognizable group, then k is at most the number of almost simple groups H such that $\pi(H) = \pi(G)$. By [1, Proposition 4.2], this number is at most $O(|\pi(G)|^7)$. A direct corollary of this discussion is the following theorem.

Theorem A [1, Theorem 1.4]. *There exists a function $F(x) = O(x^7)$ such that for each labeled graph Γ, the following conditions are equivalent:*

(i) *there exist infinitely many groups H such that $GK(H) = \Gamma$;*
(ii) *there exist more then $F(|V(\Gamma)|)$ groups H such that $GK(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.*

It is clear that estimating k we do not need to calculate all almost simple groups H such that $\pi(H) = \pi(G)$. It is sufficient to calculate those H whose prime graph satisfies some necessary conditions for H to be almost recognizable by prime graph. One of these conditions is stated in [1, Theorem 1.3]: 2 is nonadjacent to at least one odd prime in $GK(H)$. But in fact this condition can be strengthened: every $r \in \pi(H)$ is nonadjacent to at least one prime $s \neq r$ in $GK(H)$. Indeed, otherwise $GK(H) = GK(H \times \mathbb{Z}_r^k)$ for all positive integers k. Applying Theorem 2, we see that it sufficient to calculate H such that H/S is cyclic, where S is the socle of H.

Lemma 3.1. *There is a function $F(x) = O(x^2)$ such that if S is a finite simple group of Lie type, then there are at most $F(|\pi(S)|)$ almost simple groups H with socle S such that H/S is cyclic.*

Proof. Let n be the Lie rank of S and $q = p^l$ the order of the base field of S. Denote the number of divisors of l by $d(l)$. By [1, Lemma 2.7], we have $n \leqslant 2|\pi(S)| + 3$ and $d(l) \leqslant |\pi(S)| + 1$.

Steinberg's theorem [4, Theorem 2.5.12] states that $\operatorname{Out} S = \operatorname{Outdiag} S \rtimes \Phi_S\Gamma_S$, where $|\operatorname{Outdiag} S| \leqslant n + 1$ and $\Phi_S\Gamma_S$ is either a subgroup in $\mathbb{Z}_l \times Sym_3$ or a cyclic group of order $2l$ or $3l$. In any case the number of cyclic subgroups of $\Phi_S\Gamma_S$ is at most $6d(l)$. Thus the number of cyclic subgroups of $\operatorname{Out} S$ is at most $6(n + 1)d(l)$, which is $O(|\pi(S)|^2)$ by the preceding paragraph. □

Now we are ready to prove Theorem 3 (in fact we follow the lines of the proof of [1, Theorem 1.4] but Theorem 2 allows us to use the bound of Lemma 3.1 instead of that of [1, Proposition 4.1]). It is sufficient to show that there exists a function $F(x) = O(x^5)$ such that for every finite group G, if G is almost recognizable by prime graph, then there are at most $F(|\pi(G)|)$ pairwise nonisomorphic groups H with $GK(H) = GK(G)$.

Assume that G is k-recognizable by prime graph. By [1, Theorem 1.3], each group H with $GK(H) = GK(G)$ is almost simple. Furthermore, as we remarked, every $r \in \pi(H)$ is nonadjacent to at least one prime $s \neq r$ in $GK(H)$. By Theorem 2, it follows that H is a cyclic extension of its socle. By [1, Proposition 3.1], the number of nonabelian simple groups S such that $\pi(S) \subseteq \pi(G)$ is bounded by $F_1(|\pi(G)|)$ with $F_1(x) = O(x^3)$. Applying Lemma 3.1, we see that the number of almost simple groups H with socle S such that H/S is cyclic is at most $F_2(|\pi(S)|)$, where $F_2(x) = O(x^3)$. Thus $k \leqslant F_1(|\pi(G)|)F_2(|\pi(G)|) = O(|\pi(G)|^5)$, and this completes the proof of Theorem 3.

References

[1] P. J. Cameron and N. V. Maslova, Criterion of unrecognizability of a finite group by its Gruenberg-Kegel graph, 2021, arXiv:2012.01482 [math.GR].

[2] R. W. Carter, Finite groups of Lie type. Conjugacy classes and complex characters, John Wiley & Sons, New York, 1985.

[3] D. Gorenstein and R. Lyons, Local structure of finite groups of characteristic 2 type, Memoirs Amer. Math. Soc. **276**, 1983.

[4] D. Gorenstein, R. Lyons, and R. Solomon, The classification of the finite simple groups. Number 3, Amer. Math. Soc. Surveys and Monographs **40.3**, 1998.

[5] I. B. Gorshkov, Recognizability of alternating groups by spectrum, *Algebra Logic* **52** (2013), no. 1, 41–45.

[6] M. A. Grechkoseeva, On element orders in covers of finite simple groups of Lie type, *J. Algebra Appl.* **14** (2015), 1550056 [16 pages].

[7] M. A. Grechkoseeva, On spectra of almost simple groups with symplectic or orthogonal socle, *Siberian Math. J.* **57** (2016), no. 4, 582–588.

[8] M. A. Grechkoseeva, On spectra of almost simple extensions of even-dimensional orthogonal groups, *Siberian Math. J.* **59** (2018), no. 4, 623–640.

[9] M. A. Grechkoseeva and A. V. Vasil'ev, On the structure of finite groups isospectral to finite simple groups, *J. Group Theory* **18** (2015), no. 5, 741–759.

[10] B. Huppert, Endliche Gruppen. I, Springer-Verlag, Berlin, 1967.

[11] V. D. Mazurov and W. J. Shi, A note to the characterization of sporadic simple groups, *Algebra Colloq.* **5** (1998), no. 3, 285–288.

[12] A. M. Staroletov, Groups isospectral to the degree 10 alternating group, *Siberian Math. J.* **51** (2010), no. 3, 507–514.

[13] A. Staroletov, On almost recognizability by spectrum of simple classical groups, *Int. J. Group Theory* **6** (2017), no. 4, 7–33.

[14] R. Steinberg, Endomorphisms of linear algebraic groups, Memoirs Amer. Math. Soc. **80**, 1968.

[15] A. V. Vasil'ev, On connection between the structure of a finite group and the properties of its prime graph, *Siberian Math. J.* **46** (2005), no. 3, 396–404.

[16] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic* **44** (2005), no. 6, 381–406.

[17] A. V. Vasil'ev and E. P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, *Algebra Logic* **50** (2011), no. 4, 291–322.

[18] N. Yang, M. A. Grechkoseeva, and A. V. Vasil'ev, On the nilpotency of the solvable radical of a finite group isospectral to a simple group, *J. Group Theory* **23** (2020), no. 3, 447–470.