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ON THE PRIME SUBMODULES OF MULTIPLICATION MODULES

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By considering the notion of multiplication modules over a commutative ring with identity, first we introduce the notion product of two submodules of such modules. Then we use this notion to characterize the prime submodules of a multiplication module. Finally, we state and prove a version of Nakayama lemma for multiplication modules and find some related basic results.

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1. Introduction. Let R be a commutative ring with identity and let M be a unitary R-module. Then, M is called a multiplication R-module provided for each submodule N of M; there exists an ideal I of R such that $N = IM$. Note that our definition agrees with that of [1, 2], but in [6] the term *multiplication module* is used in a different way. (In this paper, an R-module M is a multiplication if and only if every submodule of M is a multiplication module in the above sense.) Recently, prime submodules have been studied in a number of papers; for example, see [3, 4, 5]. Now in this paper, first we define the notion of product of two submodules of a multiplication module and then we obtain some related results. In particular, we give some equivalent conditions for prime submodules of multiplication submodules. Finally, we state and prove a version of Nakayama lemma for multiplication modules.

2. Preliminaries. Throughout this paper, R denotes a commutative ring with identity and all related modules are unitary R-modules.

Definition 2.1. A proper submodule K of M is called *prime* if $rm \in K$, for $r \in R$ and $m \in M$, then $r \in (K : M)$ or $m \in K$, where $(K : M) = \{r \in R \mid rM \subseteq M\}$.

Theorem 2.2 (see [5]). *Let K be a submodule of M. Then the following statements are satisfied:*

(i) *K is prime if and only if $P = (K : M)$ is a prime ideal of R and R/P-module M/K is torsion-free,*

(ii) *if $(K : M)$ is a maximal ideal of R, then K is a prime submodule of M.*

For any R-module M, let $\operatorname{Spec}(M)$ denote the collection of all prime submodules of M. Note that some modules M have no prime submodules (i.e., $\operatorname{Spec}(M)$

is empty); such modules are called *primeless*. For example, the zero-module is primeless. In [5], some nontrivial examples are shown and some conditions for primeless modules are given.

Definition 2.3. An R-module M is a multiplication module if for every submodule N of M, there is an ideal I of R such that $N = IM$.

Lemma 2.4 (see [1]). *Let M be a multiplication module and let N be a submodule of M. Then $N = (\text{ann}(M/N))M$.*

Lemma 2.5 (see [1, Proposition 1.1]). *An R-module M is a multiplication if and only if for each m in M, there exists an ideal I of R such that $Rm = IM$.*

Lemma 2.6 (see [1]). *An R-module M is a multiplication if and only if*

$$\cap_{\lambda\in\Lambda}(I_\lambda M) = (\cap_{\lambda\in\Lambda}[I_\lambda + \text{ann}(M)])M \tag{2.1}$$

for any collection of ideals I_λ $(\lambda \in \Lambda)$ of R.

Theorem 2.7 (see [1, Theorem 2.5]). *Let M be a nonzero multiplication R-module. Then,*

(i) *every proper submodule of M is contained in a maximal submodule of M;*

(ii) *K is a maximal submodule of M if and only if there exists a maximal ideal P of R such that $K = PM \neq M$.*

Theorem 2.8 (see [1, Corollary 2.11]). *The following statements are equivalent for a proper submodule N of M:*

(i) *N is a prime submodule of M;*

(ii) *$\text{ann}(M/N)$ is a prime ideal of R;*

(iii) *$N = PM$ for some prime ideal P of R with $\text{ann}(M) \subseteq P$.*

Theorem 2.9 (see [1, Theorem 3.1]). *Let M be a faithful multiplication R-module. Then the following statements are equivalent:*

(i) *M is finitely generated;*

(ii) *$AM \subseteq BM$ if and only if $A \subseteq B$;*

(iii) *for each submodule N of M, there exists a unique ideal I of R such that $N = IM$;*

(iv) *$M \neq AM$ for any proper ideal A of R;*

(v) *$M \neq PM$ for any maximal ideal P of R.*

Definition 2.10. Let N be a proper submodule of M. Then, the radical of N denoted by M-rad(N) or $r(N)$ is defined in [1] to be the intersection of all prime submodules of M containing N.

Theorem 2.11 (see [1, Corollary 2.11]). *Let N be a proper submodule of a multiplication R-module M. Then $M\text{-rad}(N) = \sqrt{A}M$, where $A = \text{ann}(M/N)$.*

Definition 2.12. Let M be an R-module. Then, the radical of M denoted by $\text{rad}(M)$ is defined to be the intersection of the maximal submodules of M if such exists, and M otherwise.

Let $\mathcal{M}$ denote the collection of all maximal ideals of R. Define $P_1(M) = \{P \in \mathcal{M} \mid M \neq PM\}$ and $P_2(M) = \{P \in \mathcal{M} \mid \text{ann}(M) \subseteq P\}$. Now, define $J_1(M) = \cap\{P \mid P \in P_1(M)\}$ and $J_2(M) = \cap\{P \mid P \in P_2(M)\}$.

Theorem 2.13 (see [1, Theorem 2.7]). *Let M be a multiplication R-module. Then* $\text{rad}(M) = J_1(M)M = J_2(M)M$.

3. The product of multiplication submodules

Definition 3.1. Let M be an R-module and let N be a submodule of M such that $N = IM$ for some ideal I of R. Then, we say that I is a *presentation ideal* of N or, for short, a *presentation* of N. We denote the set of all presentation ideals of N by $\text{Pr}(N)$.

Note that it is possible that for a submodule N, no such presentation ideal exists. For example, if V is a vector space over an arbitrary field with a proper subspace W ($\neq 0$ and V), then W does not have any presentations. By Lemma 2.4, it is clear that every submodule of M has a presentation ideal if and only if M is a multiplication module. In particular, for every submodule N of a multiplication module M, $\text{ann}(M/N)$ is a presentation for N.

Let $L(R)$ and $L(M)$ denote the lattices of ideals of R and submodules of M, respectively. Define the relation $\sim$ on $L(R)$ as follows:

$$I \sim J \Longleftrightarrow IM = JM. \tag{3.1}$$

It is easy to verify that this relation is an equivalence relation on $L(R)$. We denote the equivalence class of $I \in L(R)$ by $[I]$.

Theorem 3.2. *Let M be a faithful multiplication R-module. Then, the following statements are equivalent:*

(i) *M is finitely generated;*

(ii) *each equivalence class of the relation $\sim$ is a singleton;*

(iii) *the map*

$$\varphi : L(R) \longrightarrow L(M) \tag{3.2}$$

defined by $\varphi(I) = IM$ is a lattice isomorphism;

(iv) *for every proper ideal I of R, $[I] = \{I\}$;*

(v) *for any maximal ideal P of R, $[P] = \{P\}$.*

Proof. (i)⇒(ii) follows from Theorem 2.8, Definition 3.1, and Theorem 2.9.

(ii)⇒(iii). By Theorem 2.8, we conclude that φ is bijective and order-preserving. Obviously, $(I+J)M = IM + JM$ and by Lemma 2.5, $(I \cap J)M = IM \cap JM$ since M is faithful. Therefore, ϕ is a lattice isomorphism.

(iii)⇒(iv), (iv)⇒(v), and (v)⇒(i) are an immediate consequence of Theorem 2.8. □

Definition 3.3. Let $N = IM$ and $K = JM$ for some ideals I and J of R. The product of N and K is denoted by $N \cdot K$ or NK is defined by IJM.

Clearly, NK is a submodule of M and contained in $N \cap K$. Now, we show that the product of two submodules is defining an operation on submodules of M.

Theorem 3.4. *Let $N = IM$ and $K = JM$ be submodules of a multiplication R-module M. Then, the product of N and M is independent of presentations of N and K.*

Proof. Let $N = I_1M = I_2M = N'$ and $K = J_1M = J_2M = K'$ for ideals I_i and J_i of R, $i = 1,2$. Consider $rsm \in NK = I_1J_1M$ for some $r \in I_1$, $s \in J_1$, and $m \in M$. From $J_1M = J_2M$, we have

$$sm = \sum_{i=1}^{n} r_i m_i, \quad r_i \in J_2,\ m_i \in M. \tag{3.3}$$

Then,

$$rsm = \sum_{i=1}^{n} r_i (rm_i). \tag{3.4}$$

From $rm_i \in I_1M = I_2M$, we conclude that

$$rm_i = \sum_{j=1}^{k} t_{ij} m'_{ij}, \quad t_{ij} \in I_2,\ m'_{ij} \in M. \tag{3.5}$$

Thus,

$$rsm = \sum_{i=1}^{n} \sum_{j=1}^{k} r_i t_{ij} m'_{ij}. \tag{3.6}$$

Therefore, $rsm \in I_2J_2M$, and hence $I_1J_1M \subseteq I_2J_2M$. Similarly, we have $I_2J_2M \subseteq I_1J_1M$. This completes the proof. □

Proposition 3.5. *Let M be a multiplication module N, and let K and L be submodules of M. Then the following statements are satisfied:*

(i) *$L(M)$, the lattice of submodules of M with operation product on submodules, is a semiring;*

(ii) *the product is distributive with respect to the sum on $L(M)$;*

(iii) *$(K+L)(K\cap L) \subseteq KL$;*

(iv) *$K \cap L = KL$ provided $K + L = M$ (in this case, K and L are said to be* coprime *or* comaximal*).*

Proof. (i), (ii), (iii) are obtained from Definition 3.3, Lemma 2.5, the well-known related results of the ideals theory, and the fact that $\sum_{k\in K} I_k M = (\sum_{k\in K} I_k)M$.

(iv) $K+L=M$ implies that $M(K\cap L)\subseteq KL$ by (iii), and hence $K\cap L\subseteq KL$. Clearly $KL\subseteq K\cap L$. Therefore $KL=K\cap L$. □

Lemma 3.6. *Let N and K be submodules of a multiplication module M. Then,*

(i) *the ideals $\operatorname{ann}(M/N)\cdot\operatorname{ann}(M/K)$ and $\operatorname{ann}(M/NK)$ are presentations of NK;*

(ii) *if M is finitely generated, then $\operatorname{ann}(M/N)\cdot\operatorname{ann}(M/K)=\operatorname{ann}(M/NK)$.*

Proof. (i) By Lemma 2.4 and Theorem 3.4, $\operatorname{ann}(M/N)$ and $\operatorname{ann}(M/K)$ are presentations for N and K, respectively. Thus, by Definition 3.3, $MN = [\operatorname{ann}(M/N)\cdot\operatorname{ann}(M/K)]M$. Therefore, $(\operatorname{ann}(M/N)\cdot\operatorname{ann}(M/K))$ is a presentation for MN.

(ii) By Lemma 2.4, we have $MN=\operatorname{ann}(M/NK)$ and hence by Theorem 2.8 and (i), we conclude that

$$\operatorname{ann}(M/N)\cdot\operatorname{ann}(M/K)=\operatorname{ann}(M/NK). \tag{3.7}$$

□

Remark 3.7. (i) Recall that by Lemma 2.5, for any $m\in M$, we have $Rm=IM$ for some ideal I of R. In this case, we say that I is a presentation ideal of m or, for short, a presentation of m and denote it by $\Pr(m)$. In fact, $\Pr(m)$ is equal to $\Pr(Rm)$.

(ii) For $m,m'\in M$, by mm', we mean the product of Rm and Rm', which is equal to IJM for every presentation ideals I and J of m and m', respectively.

Proposition 3.8. *Let M be a multiplication R-module. Let $N,K,N_i\in I$ be submodules of M, $s\in R$, and k any positive integer. Then the following statements are satisfied:*

(i) $\Pr(\sum_{i\in I} N_i)=\sum_{i\in I}\Pr(N_i)$;

(ii) $\Pr(\cap_{i\in I} N_i)=(\cap_{i\in I}[\Pr(N_i)+\operatorname{ann}(M)])M$;

(iii) $\Pr(\sum_{i=1}^k m_i)\subseteq\sum_{i=1}^k\Pr(m_i)$;

(iv) $\Pr(sm)=s\Pr(m)$;

(v) $\Pr(NK)=\Pr(N)\cdot\Pr(K)$;

(vi) $\Pr(N^k)=(\Pr(N))^k$;

(vii) $\Pr(m^k)=(\Pr(m))^k$;

(viii) $\Pr(M\text{-rad}(N))=M\text{-rad}(\Pr(N))$.

Proof. (i) Let I_i be presentation ideals of N_i for every $i\in I$. Then it is easy to verify that

$$\sum_{i\in I} N_i=\sum_{i\in I}(M_i)=\left(\sum_{i\in I} I_i\right)M. \tag{3.8}$$

Thus, $\Pr(\sum_{i\in I} N_i)=\sum_{i\in I}\Pr(N_i)$.

(ii) It is an immediate consequence of Lemma 2.6.

(iii) By Remark 3.7(i), we have

$$\Pr\left(\sum_{i=1}^{k} m_i\right) = \Pr\left(R\sum_{i=1}^{k} m_i\right) \subseteq \Pr\left(R\sum_{i=1}^{k} Rm_i\right) = \Pr\left(\sum_{i=1}^{k} Rm_i\right) = \sum_{i=1}^{k} \Pr(m_i). \tag{3.9}$$

(iv), (v), (vi), and (vii) are an immediate consequence of Theorem 3.4 and Remark 3.7.

(viii) It follows from Theorem 2.11. □

Definition 3.9. Let M be a multiplication R-module and let N be a submodule of M. Then,

(i) N is called *nilpotent* if $N^k = 0$ for some positive integer k, where N^k means the product of N, k times;

(ii) an element m of M is called nilpotent if $m^k = 0$ for some positive integer k.

The set of all nilpotent elements of M is denoted by N_M.

Theorem 3.10. *Let M be a multiplication module. A submodule N of M is nilpotent if and only if for every presentation ideal I of N, $I^k \subseteq \text{ann}(M)$ for some positive integer $k \in \mathbb{N}$.*

Proof. Let I be a presentation ideal of N. If N is nilpotent, then $N^k = 0$ for some positive integer k, that is, $N^k = I^kM = 0$. Thus, $I^k \subseteq \text{ann}(M)$. Conversely, suppose that $I^k \subseteq \text{ann}(M)$ for some presentation ideal I of N. Then,

$$N^k = I^kM \subseteq \text{ann}(M)M = 0. \tag{3.10}$$

Therefore, N is nilpotent. □

Corollary 3.11. *Let M be a faithful R-multiplication module and let N be a submodule of M. Then, N is nilpotent if and only if every presentation ideal of N is a nilpotent ideal.*

Theorem 3.12. *Let M be a multiplication module. Then, N_M is a submodule of M and M/N_M has no nonzero nilpotent element.*

Proof. Let $x, y \in N_M$, say $x^m = 0$ and $y^n = 0$. Consider presentation ideals I and J of x and y, respectively. Then $x^m = I^mM = 0$ and $y^m = I^nM = 0$. Since $Rx = IM$ and $Ry = JM$, then by Lemma 2.5, we have $R(x+y) \subseteq Rx+Ry = IM+JM = (I+J)M$, then $I+J$ is a presentation ideal for $x+y$. Let $l = m+n$. Then,

$$(x+y)^{m+n} = (I+J)^{m+n}M = \left(\sum_{i=0}^{l}\binom{l}{i}(I)^i(J)^{l-i}\right)M = (0)M = (0), \tag{3.11}$$

and hence $x+y \in N_M$. Now, let $m \in N_M$ and $r \in R$. Consider presentation ideal I of m. Thus, $m^k = I^kM = 0$ since $Rrm = (rI)M \subseteq IM$. Thus, $(rm)^k = (rI)^kM \subseteq I^kM = (0)$ and hence $rm \in N_M$. Therefore, N_M is a submodule of M.

Let $\overline{x} \in M/N_M$ be represented by x. Then, $\overline{x^n}$ is represented by x^n so that $\overline{x^n} = 0$. Thus, $x^n \in N_M$ and hence $(x^n)^k = 0$ for some $k \geq 0$. Therefore, $x \in N_M$ and so $\overline{x} = 0$. □

Theorem 3.13. *Let N be a submodule of a multiplication R-module M. Then M-$\text{rad}(N) = \{m \in M \mid m^k \subseteq N$ for some $k \geq 0\}$.*

Proof. Let

$$B = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}. \tag{3.12}$$

First, we show that B is a submodule of M. Let $x, y \in B$, and let I and J be presentation ideals of x and y, respectively. Then, $x^n = I^n$ and $y^m = JM \subseteq N$ for some positive integers m and n, and presentation ideals I, J of x and y, respectively. Let $k = \max\{m,n\}$. Then

$$\begin{aligned}(x+y)^k &= (IM+JM)^k = \big((I+J)M\big)^k \\ &= (I+J)^kM = \sum_{i=0}^{k}\binom{k}{i}(IM)^i(JM)^{k-i},\end{aligned} \tag{3.13}$$

that is, $x+y \in B$. Also, for $x \in B$ and $r \in R$, we have $(rx)^n \subseteq N$ since $x^n \subseteq N$. Thus, B is a submodule of M. Suppose that $m \in B$ and A is a presentation of m. Then, $m^k = A^kM \subseteq N$ for some $n \geq 1$ and hence by Theorem 2.11, we have

$$M\text{-rad}\left(m^k\right) = \sqrt{A^kM} = \sqrt{A}M \subseteq M\text{-rad}(N). \tag{3.14}$$

Thus, M-$\text{rad}(Rm) = M$-$\text{rad}(AM) \subseteq M$-$\text{rad}(N)$ and this implies that $B \subseteq M$-$\text{rad}(N)$.

Conversely, let $m \in M$-$\text{rad}(N) = \sqrt{I}M$, where $I = \text{ann}(M/N)$. Then, $m = \sum_{i=1}^{n} r_i m_i$ for $r_i \in \sqrt{I}$ and $m_i \in M$. Thus, $r_i^{n_i} \in I$ for some $n_i \geq 1$. Thus, for a sufficiently large n, we have $m^k \subseteq IM = N$ and hence M-$\text{rad}(N) \subseteq B$. Therefore, $B = M$-$\text{rad}(N)$. □

Corollary 3.14. *Let M be a multiplication R-module. Then N_M is the intersection of all prime submodules of M.*

Proof. By Theorem 2.11, we have M-$\text{rad}(0) = \sqrt{A}M$, where $A = \text{ann}(M)$, and by Theorem 3.13, M-$\text{rad}(N) = N_M$. □

Corollary 3.15. *Let M be a faithful multiplication R-module. Then $N_M = \mathcal{N}M$, where $\mathcal{N}$ is the nilradical of R.*

Theorem 3.16. *Let P be a proper submodule of a multiplication module M. Then P is prime if and only if*

$$UV \subseteq P \Longrightarrow U \subseteq P \quad \text{or} \quad V \subseteq P \tag{3.15}$$

for each submodule U and V of M.

Proof. Let P be prime and $UV \subseteq P$, but $U \nsubseteq P$ and $V \nsubseteq P$ for some submodules U and V of M. Suppose that I and J are presentations of U and V, respectively. Then $UV = IJM \subseteq P$. Thus, there are $ry \in U - P$ and $sx \in U - P$ for some $r \in I$ and $s \in J$. Thus, $rsx \in P$ and hence $rM \subseteq P$, that is, $ry \in P$, which is a contradiction.

Conversely, suppose that condition (3.15) is true. Let $rx \in P$ for some $r \in R$ and $x \in M - P$, but $rM \nsubseteq P$; then, $rm \notin P$ for some $m \in M$. Let I and J be presentation ideals of rx and m, respectively. Then

$$R(rx) \cdot (Rm) = (Rx) \cdot (Rrm) = IM \cdot JM = IJM \subseteq P. \tag{3.16}$$

Now, by hypothesis, we must have $Rx \subseteq P$ or $Rrm \subseteq P$, which implies that $x \in P$ or $rm \in P$, which is a contradiction. Therefore, P is prime. □

Corollary 3.17. *Let P be a proper submodule of M. Then P is prime if and only if*

$$m \cdot m' \subseteq P \Longrightarrow m \in P \quad \text{or} \quad m' \in P \tag{3.17}$$

for every $m, m' \in M$.

Proof. If P is prime, then, clearly, (3.17) is true. Conversely, suppose that (3.17) is true, and $UV \subseteq P$ for submodules U and V of M, but $U \nsubseteq P$ and $V \nsubseteq P$. Thus, there are $u \in U - P$ and $v \in V - P$. Then $uv = RuRv \subseteq UV \subseteq P$ and hence by (3.17), we must have $u \in U$ or $v \in V$, which is a contradiction. Therefore, P is prime. □

Definition 3.18. An element u of an R-module M is said to be a *unit* provided that u is not contained in any maximal submodule of M.

Theorem 3.19. *Let M be a multiplication R-module. Then $u \in M$ is a unit if and only if $\langle u \rangle = M$.*

Proof. The sufficiency is clear. For a necessary part, let u be a unit element. Then $\langle u \rangle$ is not contained in any maximal submodule of M. Thus, by Theorem 2.7, we must have $\langle u \rangle = M$. □

Theorem 3.20. *Let M be an R-module (not necessarily multiplicative) such that M has a unit u. Then $m \in \operatorname{rad}(M)$ if and only if $u - rm$ is unit for every $r \in R$.*

Proof. See [7, Theorem 4.8]. □

Theorem 3.21. *Every homomorphic image of a multiplication module is a multiplication module.*

Proof. Let M be a multiplication R-module, $\phi : M \to M'$ an R-module homomorphism, and $K = \phi(M)$. Let $k \in K$, then $k = (\phi m)$ for some $m \in M$. Since M is a multiplication, then by Lemma 2.5, there is an ideal I of R such that $Rm = IM$. Thus,

$$\varphi(IM) = I\varphi(M) = IK = \varphi(Rm) = R\varphi(m) = Rk. \tag{3.18}$$

Therefore, by Lemma 2.5, K is a multiplication R-module. □

Corollary 3.22. *Let M be a multiplication R-module and N a submodule of M. Then, M/N is a multiplication R-module.*

Theorem 3.23 (a version of Nakayama lemma)**.** *Let M be a faithful multiplication R-module such that M has a unit u. Then, for every submodule N, the following conditions are equivalent:*

(i) *N is contained in every maximal submodule of M;*
(ii) *$u - rx$ is a unit for all $r \in R$ and for all $x \in N$;*
(iii) *if M is a finitely generated R-module such that $NM = M$, then $M = 0$;*
(iv) *if M is finitely generated and K is a submodule of M such that $M = NM + K$, then $M = K$.*

Proof. (i)⇒(ii) is an immediate consequence of Theorem 3.19.

(ii)⇒(iii). Since M is finitely generated, there must be a minimal generating set $X = \{m_1, \dots, m_n\}$ of M. If $M \neq 0$, then $m_1 \neq 0$ by minimality. Now, let I be a presentation of N. Then, $NM = M$ implies that $M = IM \cdot M = M$, and since M is faithful, then by Theorem 2.13, we have $N \subseteq \operatorname{rad}(M) = J_1(M)M \subseteq J(R)M$. Thus, $m_1 = j_1m_1 + j_2m_2 + \cdots + j_nm_n$ $(j_i \in J(R))$ whence $j_1m_1 = m_1$ so that $(1 - j_1)m_1 = 0$ if $n = 1$, and

$$(1 - j_1)m_1 = j_2m_2 + \cdots + j_nm_n, \quad n > 1. \tag{3.19}$$

Since $1 - j_1$ is a unit in R, $m_1 = (1 - j_1)^{-1}(1 - j_1)m_1 + \cdots + (1 - j_1)^{-1}j_nm_n$. Thus, if $n = 1$, then $m_1 = 0$, which is a contradiction. If $n > 1$, then m_1 is a linear combination of $m_2, m_3, \dots, m_n$; consequently, $\{m_2, \dots, m_n\}$ generates M, which contradicts the choice of X.

(iii)⇒(iv). Since for every submodule K/N of M/N, we have $K/N = \operatorname{ann}(M/N/K/N)M/N = \operatorname{ann}(M/K)M/N$; then by Corollary 3.22, M/N is a multiplication R-module. Now, it is easy to verify that $\operatorname{rad}(M/N) = M/N$ and hence, by (iii), we must have $M = K$.

(iv)⇒(i). Let K be any maximal submodule of M, then $K \subseteq NM = K$. Consequently, $NM + M = M$ by maximality of K, otherwise $M = K$ by (iv) a contradiction. Therefore, $N = NM \subseteq K$. □

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