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ON THE PROBABILISTIC DOMAIN INVARIANCE

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Probabilistic version of the invariance of domain for contractive field and Schauder invertibility theorem are proved. As an application, the stability of probabilistic open embedding is established.

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1. Introduction

Menger [7] was the first who introduced the notion of probabilistic metric space which is a generalization of the metric space. The study of this space was expanded rapidly with the pioneering work of Scherwood [9], Schweizer and Sklar [10] and many others. Recently Sehgal and Bharucha-Reid [8], Hadzic [5], Cho [3] and Beg, Rehman and Shahzad [1] have studied fixed point theorems for probabilistic metric spaces. This paper deals with the problem of showing that certain mappings on probabilistic normed spaces are homeomorphisms and gives the sufficient conditions under which addition of open mappings results in open mapping. We also study domain invariance and its applications in probabilistic normed spaces.

2. Preliminaries

Let $\mathcal{D}$ be the set of all left continuous functions defined on $\mathcal{R}$ such that for all $f \in \mathcal{D}$,

$$f(x) = 1, \text{ for all } x \leq 0; \tag{1}$$

$$f(\infty) = 0; \tag{2}$$

and f is a nonincreasing function. On $\mathcal{D}$, we consider the natural ordering, that is, $f \leq g$ $(f, g \in \mathcal{D})$ if and only if $f(x) \leq g(x)$ for all $x \in \mathcal{R}$ and $f < g$ if $f \leq g$ and there exists an x_0

such that $f(x_0) < g(x_0)$. We denote by I the function in $\mathcal{D}$ with the property that $I(x) = 0$ for $x > 0$.

Let $\mathcal{E}$ be a subset of $\mathcal{D}$ containing I. A *triangular function* μ on $\mathcal{E}$ with values in $\mathcal{E}$ is any associative and commutative composition law having the following properties: for a, a_1, a_2, b in $\mathcal{E}$,

$$\mu(a, I) = a; \tag{3}$$

$$\mu(a_1, a) \leq \mu(a_2, a) \text{ whenever } a_1 \leq a_2; \tag{4}$$

$$\mu(a, b) \leq \mu_1(a, b), \tag{5}$$

where $\mu_1(a,b)(x) = \inf_t \min\{a(tx) + b[(1-t)x], 1\}$ {here the infimum is taken over all $t \in [0,1]$}.

Note that among a number of possible universal choices for μ, $\mu(a,b) = \max(a,b)$ is the strongest possible universal μ.

A linear space L over $\mathcal{R}$ (real field) is called a *probabilistic normed space* if there exists a mapping $\| \cdot ; \cdot \| : L \to \mathcal{E}$ such that the following properties hold:

$$\| \phi; \cdot \| = I \text{ if and only if } \phi = 0; \tag{6}$$

$$\| a\phi; x \| = \| \phi; |a|^{-1}x \| \text{ for any } x \in \mathcal{R} \text{ and } a \in \mathcal{R},\ a \neq 0; \tag{7}$$

$$\| \phi + \psi; x \| \leq \mu(\| \phi; x \|, \| \psi; x \|). \tag{8}$$

The mapping $\phi \to \| \phi; \cdot \|$ is called a *probabilistic norm* or a *random norm* on L. A sequence (ϕ_n) of points in a probabilistic normed space L is said *convergent* to a point ϕ if for any $0 < \epsilon \leq 1$, $0 < \delta < \infty$ there exists a positive integer $N_{\epsilon,\delta}$ such that $\| \phi_n - \phi; \delta \| < \epsilon$ for all $n > N_{\epsilon,\delta}$. The sequence (ϕ_n) is said Cauchy sequence if for any $0 < \epsilon \leq 1$, $0 < x < +\infty$, there exists a positive integer $N_{\epsilon,x}$ such that $\| \phi_{n+p} - \phi_n; x \| < \epsilon$, for all $n > N_{\epsilon,x}, p \in \mathbb{N}$. A complete probabilistic normed space is called *probabilistic Banach space*. Denote $U_{\epsilon,\delta}(\phi) = \{\psi \in L;\ \| \phi - \psi; \delta \| < \epsilon\}$, where $0 < \epsilon \leq 1$ and $0 \leq \delta < \infty$. Whenever there is no confusion, we write $U(\phi)$ for $U_{\epsilon,\delta}(\phi)$.

Let f be a mapping from a probabilistic normed space L_1 into a probabilistic normed space L_2, then f is said to be *continuous* at $\phi_0 \in L_1$ if $\phi \in U(\phi_0)$ implies $f(\phi) \in U(f(\phi_0))$. Let f be a linear operator from a probabilistic normed space L_1 into a probabilistic normed space L_2, then f is called *bounded* if there exists some $m \geq 0$ such that $\| f(\phi); \cdot \| \leq \| m\phi; \cdot \|$ for all $\phi \in L_1$, and the *probabilistic norm* of f is defined as $\| f; x \| = \inf_{0 \neq \phi \in L_1} \| \, \| \phi; x \| f(\phi); x \|$. If f is a bounded linear operator from a probabilistic normed space L_1 into a probabilistic normed space L_2, then $\| f(\phi); x \| \leq \| \, \| f; x \| \phi; x \|$. Let L_1 and L_2 be two probabilistic normed spaces and $F\colon L_1 \to L_2$ be a mapping satisfying $\| F(\phi) - F(\psi); \cdot \| \leq \| \alpha(\phi - \psi); \cdot \|$ then the constant α, is called Lipschitzian. The smallest such α is called *Lipschitz constant* of F and denote it by $\mathcal{L}(F)$. If $\mathcal{L}(F) < 1$ then the map F is called *contraction*. If $\mathcal{L}(F) = 1$ then the map F is said to be *nonexpansive*.

Let $H(x)$ denote the distribution function defined as follows:

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

A *probabilistic metric space* (abbreviated as *PM-space*) is an ordered pair (S,E) where S is a nonempty set and E is a function defined on $S\times S$ into the set

$$\mathcal{F}=\left\{\begin{array}{c}F\colon\mathcal{R}\to[0,1];\,F\text{ nondecreasing and}\\ \text{left continuous on }\mathcal{R},\;F(-\infty)=0,F(+\infty)=1\end{array}\right\},$$

such that if p,q are points of S, then

$$E_{pq}(0)=0\ (E_{pq}\text{ denotes }E(P,q)); \tag{9}$$

$$E_{pq}(x)=H(x)\text{ for all }x\in\mathcal{R}\text{ if and only if }p=q; \tag{10}$$

$$E_{pq}=E_{qp}; \tag{11}$$

$$\text{if }E_{pq}(x)=1\text{ and }E_{qr}(y)=1\text{, then }E_{pr}(x+y)=1. \tag{12}$$

A mapping $K\colon[0,1]\times[0,1]\to[0,1]$ is called *T-norm* if it satisfies:

$$K(x,1)=x, K(0,0)=0; \tag{13}$$

$$K(x,y)\geq K(u,v)\text{ for }x\geq u, y\geq v; \tag{14}$$

$$K(x,y)=K(y,x); \tag{15}$$

$$K(K(x,y),u)=K(x,K(y,u)); \tag{16}$$

for all x,y,u,v in $[0,1]$.

A *Menger PM-space* is a triplet (S,E,K) where (S,E) is a PM-space and the T-norm K is such that the inequality

$$E_{pq}(x+q)\geq K(E_{pr}(x),E_{rq}(y)); \tag{12*}$$

holds for all p,q,r in S and all $x\geq 0$, $y\geq 0$.

A sequence (ϕ_n) in the PM-space (S,E,K) is said Cauchy sequence if $\lim_{n\to\infty}E_{\phi_{n+p}\phi_n}(x)=1$, for any $x>0$ and $p\in\mathbb{N}$.

A mapping $F\colon(S,E,K)\to(S,E,K)$ is called a contraction if there exists $\alpha\in(0,1)$ such that

$$E_{F(p)F(q)}(\alpha x)\geq E_{pq}(x),$$

for any $x>0$ and $p,q\in S$.

Lemma 2.1: [4, Theorem 3.4.12]. *Let F be a linear operator from a probabilistic normed space L_1 into a probabilistic normed space L_2. If F is bounded then F is continuous.*

Theorem 2.2: [6, Theorem 11.2.2]. *Every contraction mapping on a PM-space has at most one fixed point.*

Theorem 2.3: [6, Theorem 11.2.4]. *Let (S,E,K) be a complete Menger PM-space and let F be a contraction mapping on S. Then F has a unique fixed point where $K(x,y)=min(x,y)$.*

Remark 2.4: Let $(L,\|\cdot;\cdot\|,\mu)$ be a probabilistic Banach space with $\mu(a,b)=\max\{a,b\}$. If (ϕ_n) is a Cauchy sequence in $(L,\|\cdot;\cdot\|,\mu)$ and F is a contraction on $(L,\|\cdot;\cdot\|,\mu)$, then (L,E,K), with $K(x,y)=\min\{x,y\}$ and $E_{pq}(x)=1-\|p-q;x\|$, becomes a complete Menger PM-space (i.e., (ϕ_n) becomes a Cauchy sequence in

(L, E, K) and the convergence of a sequence in $(L, \parallel \cdot ; \cdot \parallel, \mu)$ is equivalent to the convergence in $(L, E, K))$ and F becomes a contraction on $(L, E, K))$.

For more details, we refer to Serstnev [12], Istratescu [6], Schweizer and Sklar [11] and Chang et al [2].

We will prove the following fixed point theorem in probabilistic Banach spaces for subsequent use in the next sections.

Theorem 2.5: *Let*

$$\mathcal{E} = \left\{ \begin{array}{c} f \in \mathcal{D}; f \textit{ is nonconvex on } [0, +\infty) \textit{ i.e. } af(x) + (1-a)f(y) \\ \leq f(ax + (1-a)y), \forall a \in [0,1], \forall x, y \geq 0 \end{array} \right\}$$

and $\mu: \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ *be a triangular function. Also, let* $(L \parallel \cdot ; \cdot \parallel, \mu)$ *be a probabilistic Banach space* (*where* $\parallel L, \cdot \parallel \subset \mathcal{E}$). *If* $F: (L, \parallel \cdot ; \cdot \parallel, \mu) \to (L, \parallel \cdot ; \cdot \parallel, \mu)$ *is a contraction mapping, then* F *has a unique fixed point.*

Proof: Let $\phi_0 \in L$ be fixed and consider the iterations $\phi_{n+1} = F(\phi_n)$, $n \in \mathbb{N}$. For any $x > 0$ we have

$$\parallel \phi_{n+1} - \phi_n; x \parallel \; = \; \parallel F(\phi_n) - F(\phi_{n-1}); x \parallel \; \leq \; \parallel \alpha(\phi_n - \phi_{n-1}); x \parallel$$

$$\leq \ldots \leq \; \parallel \alpha^n(\phi_1 - \phi_0); x \parallel \; = \; \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel .$$

Then

$$\parallel \phi_{n+2} - \phi_n; x \parallel \; \leq \mu(\parallel \phi_{n+2} - \phi_{n+1}; x \parallel , \parallel \phi_{n+1} - \phi_n; x \parallel)$$

$$\leq \mu(\parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^{n+1}} \parallel , \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel)$$

$$\leq \mu(\parallel \phi_1 - \phi_0 \tfrac{x}{\alpha^n} \parallel , \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel)$$

$$\leq \mu_1(\parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel , \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel)$$

$$= \inf_{t \in [0,1]} \Big\{ \min \Big\{ \parallel \phi_1 - \phi_0; \tfrac{tx}{\alpha^n} \parallel + \parallel \phi_1 - \phi_0; \tfrac{(1-t)x}{\alpha^n} \parallel , 1 \Big\} \Big\}$$

(because $\parallel \phi_1 - \phi_0; \cdot \parallel$ is nonconvex)

$$\leq \inf_{t \in [0,1]} \big\{ \min \big\{ \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel , 1 \big\} \big\} = \; \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel .$$

By induction, it easily follows that in general,

$$\parallel \phi_{n+p} - \phi_n; x \parallel \; \leq \; \parallel \phi_1 - \phi_0; \tfrac{x}{\alpha^n} \parallel , \forall n \in \mathbb{N}, p \in \mathbb{N}.$$

Passing to limit with $n \to +\infty$, we obtain that (ϕ_n) is a Cauchy sequence. Because $(L, \parallel \cdot ; \cdot \parallel, \mu)$ is a probabilistic Banach space, therefore there exists $\phi^* \in L$, such that $\forall x > 0, \lim_{n \to \infty} \parallel \phi_n - \phi^*; x \parallel = 0$. Because

$$\parallel \phi^* - F(\phi^*); x \parallel \; \leq \mu(\parallel \phi^* - \phi_{n+1}; x \parallel , \parallel \phi_{n+1} - F(\phi^*); x \parallel)$$

$$= \mu(\parallel \phi^* - \phi_{n+1}; x \parallel , \parallel F(\phi_n) - F(\phi^*); x \parallel)$$

$$\leq \mu(\parallel \phi^* - \phi_{n+1}; x \parallel , \parallel \alpha(\phi_n - \phi^*); x \parallel)$$

$$\leq \mu(\parallel \phi^* - \phi_{n+1}; x \parallel , \parallel \phi_n - \phi^*; x \parallel).$$

Let $x > 0$ be fixed and $0 < \epsilon < \frac{1}{2}$. Then there exists $N_{x,\epsilon} > 0$ such that for all $n > N_{x,\epsilon}$, we have

$$\parallel \phi^* - \phi_{n+1}; x \parallel < \epsilon \text{ and } \parallel \phi_n - \phi^*; x \parallel < \epsilon.$$

It follows that

$$\parallel \phi^* - F(\phi^*); x \parallel \leq \mu(\epsilon, \epsilon) \leq \mu_1(\epsilon, \epsilon) < 2\epsilon,$$

and ϵ is arbitrary (sufficiently small), we obtain $\parallel \phi^* - F(\phi^*); x \parallel = 0$, $\forall x > 0$, i.e., $\phi^* = F(\phi^*)$.

In order to prove the uniqueness, let us suppose that there exist two fixed points $\phi \neq \psi \in L$, for F. We get

$$\parallel \phi - \psi; x \parallel = \parallel F(\phi) - F(\psi); x \parallel \leq \parallel \phi - \psi; \tfrac{x}{\alpha} \parallel \leq \parallel \phi - \psi; x \parallel ,$$

which implies $\parallel \phi - \psi; x \parallel = \parallel \phi - \psi; \frac{x}{\alpha} \parallel$, $\forall x > 0$, and taking into account that $\parallel \phi - \psi; +\infty \parallel = 0$, we obtain $\parallel \phi - \psi; x \parallel = 0$, $\forall x > 0$, i.e., $\phi - \psi = 0$, which proves the theorem.

Remark 2.6: For reasons of simplicity, throughout the rest of the paper, a probabilistic Banach space $(L, \parallel \cdot ; \cdot \parallel , \mu)$ with $\mu(a,b) = \max(a,b)$ or satisfying the hypothesis in the statement of Theorem 2.5, will be called of *S-type*.

3. Invariance of Domain for Contractive Fields

Lemma 3.1: *Let L be a probabilistic Banach space of S-type and $B[\phi_0, I(x-r)] = \{\phi;\ \parallel \phi - \phi_0; x \parallel < I(x-r)\}$. Let $F: B \to L$ be a contraction with $\mathcal{L}(F) = \alpha < 1$. If $\parallel F(\phi_0) - \phi_0; x \parallel < I(x - (1-\alpha)r)$, then F has a fixed point.*

Proof: Choose $\epsilon < r$ so that

$$\parallel F(\phi_0) - \phi_0; x \parallel \leq I(x - (1-\alpha)\epsilon) < I(x - (1-\alpha)r).$$

The mapping F maps the closed ball $\overline{D} = \{\phi;\ \parallel \phi - \phi_0; x \parallel \leq I(x - \epsilon)\}$ into itself. Indeed, if $\phi \in \overline{D}$, then

$$\parallel F(\phi) - \phi_0; x \parallel = \parallel F(\phi) - F(\phi_0) + F(\phi_0) - \phi_0; x \parallel$$

$$\leq \mu(\parallel F(\phi) - F(\phi_0); x \parallel , \parallel F(\phi_0) - \phi_0; x \parallel)$$

$$\leq \mu(\parallel \phi - \phi_0; \tfrac{x}{\alpha} \parallel , I(x - (1-\alpha)\epsilon)$$

$$\leq \mu(I(\tfrac{x}{\alpha} - \epsilon), I(x - (1-\alpha)\epsilon)$$

$$= \mu(I(x - \alpha\epsilon), I(x - (1-\alpha)\epsilon) = I(x - \epsilon')$$

where $\alpha\epsilon = \epsilon' < r$, $\alpha\epsilon < \alpha r < r$ and $0 < \epsilon < 1$. Thus if $\phi \in \overline{D}$ then $F(\phi) \in \overline{D}$.

Because L is complete, it is sufficient to prove that $\overline{D}$ is closed. In this sense, let (ϕ_n) be a sequence of points in $\overline{D}$, convergent to a point $\phi^* \in L$. We obtain

$$\| \phi^* - \phi_0; x \| \leq \mu(\| \phi^* - \phi_n; x \| , \| \phi_n - \phi_0; x \|)$$

$$\leq \mu_1(\| \phi^* - \phi_n; x \| , \| \phi_n - \phi_0; x \|)$$

$$= \inf_{t \in [0,1]} \{\min\{ \| \phi^* - \phi_n; tx \| + \| \phi_n - \phi_0; (1-t)x \| , 1\}\}.$$

We have two cases:

a) $x \in (-\infty, \epsilon]$; ***b***) $x \in (\epsilon, +\infty)$

Case **a)** obviously $I(x - \epsilon) = 1$ and therefore

$$\| \phi^* - \phi_0; x \| \leq 1 = I(x - \epsilon).$$

b) For each $t \in [0,1]$, fixed, and each $0 < \eta < 1$, there exists $N_{t,\eta} \in \mathbb{N}$, such that for $n > N_{t,\eta}$ we get $\| \phi^* - \phi_n; tx \| < \eta$.

It follows

$$\| \phi^* - \phi_0; x \| \leq \inf_{t \in [0,1]} \{\min\{\eta + I((1-t)x - \epsilon), 1\}\} < \eta,$$

because for $t < 1 - \frac{\epsilon}{x}$ we have $I((1-t)x - \epsilon) = 0$. Since η is arbitrary (independent of x), we get

$$\| \phi^* - \phi_0; x \| = 0 = I(x - \epsilon) \text{ for } x > \epsilon.$$

As a conclusion, $\phi^* \in \overline{D}$. Since $\overline{D}$ is complete, we get that F has a fixed point by Theorem 2.3 and by Theorem 2.5.

Definition 3.2: Let M be a subset of a probabilistic Banach space L. Given a map $F: M \to L$, the map $f: M \to L$ defined by $f(\phi) = \phi - F(\phi)$ is called the *field associative with* F. The field f determined by a contraction F is called a *contractive field*.

Theorem 3.3: *Let L be a probabilistic Banach space of S-type, $U \subset L$ open, and $F: U \to L$ be a contraction. Let $f: U \to L$ be the field associative with F. Then,*

(i) *the field f is an open mapping; in particular, $f(U)$ is open in L, and*

(ii) *the mapping $f: U \to f(U)$ is a homeomorphism.*

Proof: (i) To show that f is an open mapping, it is enough to show that for any $\phi \in U$, if $B[\phi, r] \subset U$, then $B[f(\phi), (1-\alpha)r] \subset f(B[\phi, r])$. For this purpose, choose any $\psi_0 \in B[f(\phi), (1-\alpha)r]$ and define $G: B[\phi, r] \to L$ by $G(\psi) = \psi_0 + F(\psi)$. Then G is a contraction and

$$\| G(\phi) - \phi; x \| = \| \psi_0 + F(\phi) - \phi; x \|$$

$$= \| \psi_0 - f(\phi); x \| < I(x - (1-\alpha)r).$$

By Lemma 3.1 there exists $\phi_0 \in B[\phi, r]$ with $\phi_0 = \psi_0 + F(\phi_0)$, therefore $f(\phi_0) = \psi_0 = \phi_0 - F(\phi_0)$. Thus $B[f(\phi), (1-\alpha)r] \subset f(B[\phi, r])$. So f is an open mapping and, in particular, $f(U)$ is open in L.

(ii) If $\phi, \psi \in U$, then

$$\| f(\phi) - f(\psi); x \| = \| (\phi - \psi) - (F(\phi) - F(\psi)); x \|$$

$$\geq \| (1-\alpha)(\phi - \psi); x \| .$$

So that f is injective. Since $f\colon U \to f(U)$ is a continuous open bijection, it is a homeomorphism.

Corollary 3.4: *Let L be a probabilistic Banach space of S-type and $F\colon L \to L$ be a contraction. Then the corresponding field $f = Id - F$ is a homeomorphism.*

Proof: Let $\psi_0 \in L$, define $G\colon L \to L$ by $G(\phi) = \psi_0 + F(\phi)$. Then G is a contraction, so G has a fixed point $\phi_0 = \psi_0 + F(\phi_0)$, therefore $f(\phi_0) = \psi_0$. Thus $f(L) = L$.

4. Domain Invariance and Invertibility of Linear Operators

Proposition 4.1: *Let F be a linear operator on a probabilistic Banach space L of S-type. If $\| Id - F; 1 \| = 0$, then F is invertible.*

Proof: The map $Id - F\colon L \to L$ is a contraction. Indeed,

$$\| (Id - F)(\phi - \psi); x \| \ \le \ \| \ \| (Id - F); x \| \ (\phi - \psi); x \|$$

$$= \ \| \alpha(\phi - \psi); x \| \ , \alpha \in (0, 1).$$

Corollary 3.4 further implies that $Id - (Id - F) = F$ is a homeomorphism. Thus F is invertible.

Lemma 4.2: *Let L_1 and L_2 be two probabilistic normed spaces and $F\colon L_2 \to L_1$ and $G\colon L_1 \to L_2$ be two bounded linear operators. Then $\| FG; x \| \ \le \ \| F \| G; x \| ; x \|$.*

Proof: We have

$$\| FG; x \| = \inf_{0 \ne \phi \in L_1} \| \ \| \phi; x \| \ (FG)(\phi); x \|$$

$$= \inf_{0 \ne \phi \in L_1} \| F(\ \| \phi; x \| \ G(\phi)); x \| \ \le \ \| F \| G; x \| ; x \| \ .$$

Remark 4.3: Let L_1 and L_2 be two probabilistic Banach spaces and $S\colon L_1 \to L_2$ be a linear operator. If there is some $m > 0$ such that $\| S(\phi); x \| \ \ge \ \| m\phi; x \|$ for $\phi \in L_1$, then S is injective; if such an S is also surjective, then it is invertible because $\| S^{-1}(\psi); x \| \ \le \ \| \frac{\psi}{m}; x \|$ for $\psi \in L_2$. Therefore S^{-1} is continuous by Lemma 2.1.

Theorem 4.4: [Schauder Invertibility Theorem] *Let L_1 and L_2 be two probabilistic Banach spaces of S-type and $S, T\colon L_1 \to L_2$ be two linear operators, with S invertible. Assume that there is an $m > 0$ such that, for each $0 \le t \le 1$, the operator $E_t = (1 - t)S + tT$ satisfies $\| E_t(\phi); x \| \ \ge \ \| m\phi; x \|$ for all $\phi \in L_1$. Then E_t is invertible for all $0 \le t \le 1$, and in particular, T is invertible.*

Proof: If an operator E_s is invertible, then for each t in the open interval $J_s = \{t\colon \mid t - s \mid \ < \frac{m}{\|T - S; x\|}\}$, the operator E_t is invertible or equivalently, $E_s^{-1}E_t\colon L_1 \to L_1$ is invertible for each $t \in J_s$. For this purpose, note that

$$E_t = S + s(T - S) + (t - s)(T - S)$$

$$= E_s + (t - s)(T - S).$$

So that,

$$E_s^{-1}E_t = I + (t - s)E_s^{-1}(T - S).$$

Because

$$\| E_s\phi; x \| \ \geq \ \| m\phi; x \| \ ,$$

so for $t \in J_s$, we have

$$\| -(t-s)E_s^{-1}(T-S); x \| \ = \ \| E_s^{-1}(T-S); \tfrac{x}{-(t-s)} \|$$

$$\leq \ \| E_s^{-1} \| T-S; x \| \ ; \tfrac{x}{-(t-s)} \|$$

$$\leq \ \| E_s^{-1} \tfrac{m}{\|T-S;x\|} \| T-S; x \| \ ; x \| \ \leq I(x-1).$$

Therefore

$$\| Id - E_s^{-1}E_t; x \| \ \leq I(x-1).$$

Proposition 4.1 further implies that $E_s^{-1}E_t$ is invertible. Thus E_t is invertible. Now assume that $\mathcal{F} = \{t \in [0,1]: E_t \text{ is invertible}\}$, then $\mathcal{F}$ is an open set. If $t \notin \mathcal{F}$, then the operator E_s with $s \in J_t$ is not invertible. So that $[0,1] - \mathcal{F}$ is also an open set. Because $[0,1]$ is connected and $\mathcal{F}$ is nonempty, we get $\mathcal{F} = [0,1]$ and the proof is complete.

5. Stability of Open Embeddings

Theorem 5.1: *Let L be a probabilistic normed space, M be a probabilistic Banach space of S-type and $F: L \to M$ be an open embedding of L onto an open subset $U \subset M$. Let $G: L \to M$ be a map such that $G \circ F^{-1}: U \to L$ is a contraction. Then $\phi \to F(\phi) + G(\phi)$ is also an open embedding of L into M.*

Proof: Consider $h = (F+G) \circ F^{-1} = Id + GF^{-1}: U \to M$. By the domain invariance, this h maps U homeomorphically onto an open $h(U) \subset M$. Since $F + G = h \circ F = (F+G) \circ F^{-1} \circ F$, it follows that $h \circ F$ is an open embedding of L into M.

Theorem 5.2: *Let L be a probabilistic metric space, M be a probabilistic Banach space of S-type and $F: L \to M$ be an open embedding such that F^{-1} is Lipschitzian. Let $G: L \to M$ be a Lipschitzian map such that $\mathcal{L}(G)\mathcal{L}(F^{-1}) < 1$. Then $\phi \to F(\phi) + G(\phi)$ is also an open embedding of L into M.*

Proof: Since $\mathcal{L}(G \circ F^{-1}) \leq \mathcal{L}(G)\mathcal{L}(F^{-1}) < 1$, we get that $G \circ F^{-1}$ is a contraction. By Theorem 5.1, $F + G$ is also an open embedding of L into M.

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