

**ON THE PROBABILITY OF THE OCCURRENCE OF AT LEAST  $m$   
EVENTS AMONG  $n$  ARBITRARY EVENTS**

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**Introduction.** Let  $E_1, \dots, E_n$ , denote  $n$  arbitrary events. Let  $p_{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_j}$ , where  $0 \leq i \leq j \leq n$  and  $(\nu_1, \dots, \nu_j)$  is a combination of the integers  $(1, \dots, n)$ , denote the probability of the non-occurrence of  $E_{\nu_1}, \dots, E_{\nu_i}$  and the occurrence of  $E_{\nu_{i+1}}, \dots, E_{\nu_j}$ . Let  $p_{[\nu_1 \dots \nu_i]}$  denote the probability of the occurrence of  $E_{\nu_1}, \dots, E_{\nu_i}$  and no others among the  $n$  events. Let  $S_j = \sum p_{\nu_1 \dots \nu_j}$ , where the summation extends to all combinations of  $j$  of the  $n$  integers  $(1, \dots, n)$ . Let  $p_m(\nu_1, \dots, \nu_k)$ , ( $1 \leq m \leq k \leq n$ ), denote the probability of the occurrence of at least  $m$  events among the  $k$  events  $E_{\nu_1}, \dots, E_{\nu_k}$ .

By the set  $(x_1, \dots, x_b, \dots, x_a) - (x_1, \dots, x_b)$  (where  $b \leq a$ ) we mean the set  $(x_{b+1}, \dots, x_a)$ . And by a  $\binom{a}{b}$ -combination out of  $(x_1, \dots, x_a)$  we mean a combination of  $b$  integers out of the  $a$  integers  $(x_1, \dots, x_a)$ .

We often use summation signs with their meaning understood, thus for a fixed  $k$ ,  $1 \leq k \leq n$ , the summations in  $\sum p_{\nu_1 \dots \nu_k}$ , or  $\sum p_m(\nu_1, \dots, \nu_k)$ , extend to all the  $\binom{n}{k}$ -combinations out of  $(1, \dots, n)$ .

The following conventions concerning the binomial coefficients are made:

$$\binom{0}{0} = 1, \quad \binom{a}{b} = 0 \quad \text{if} \quad a < b \quad \text{or if} \quad b < 0.$$

It is a fundamental theorem in the theory of probability that, if  $E_1, \dots, E_n$  are incompatible (or "mutually exclusive"), then

$$p_1(1, \dots, n) = p_1 + \dots + p_n.$$

When the events are arbitrary, we have Boole's inequality

$$p_1(1, \dots, n) \leq p_1 + \dots + p_n.$$

Gumbel<sup>1</sup> has generalized this inequality to the following:

$$p_1(1, \dots, n) \leq \frac{\sum p_1(\nu_1, \dots, \nu_k)}{\binom{n-1}{k-1}},$$

<sup>1</sup> *C. R. Acad. Sc.* Vol. 205(1937), p. 774.

for  $k = 1, \dots, n$ . The case  $k = 1$  gives Boole's inequality. Fréchet<sup>2</sup> has announced that Gumbel's result can be sharpened to the following

$$(1) \quad A_{k+1} = \frac{\sum p_i(\nu_1, \dots, \nu_{k+1})}{\binom{n-1}{k}} \leq \frac{\sum p_i(\nu_1, \dots, \nu_k)}{\binom{n-1}{k-1}} = A_k,$$

for  $k = 1, \dots, n - 1$ . Thus,  $A_k$  is non-increasing for  $k$  increasing. On the other hand, Poincaré has obtained the following formula which expresses  $p_1(1, \dots, n)$  in terms of the  $S_j$ 's,

$$(2) \quad p_1(1, \dots, n) = \sum p_{\nu_1} - \sum p_{\nu_1\nu_2} + \sum p_{\nu_1\nu_2\nu_3} - \dots + (-1)^n p_{1\dots n} = \sum_{j=1}^n (-1)^{j-1} S_j.$$

In the present paper we shall study the more general function  $p_m(\nu_1, \dots, \nu_k)$  as defined above. First we generalize Poincaré's formula and Fréchet's inequalities. In Theorem 1 we establish (for  $1 \leq m \leq n$ )

$$(3) \quad \begin{aligned} p_m(1, \dots, n) &= \sum p_{\nu_1 \dots \nu_m} - \binom{m}{1} \sum p_{\nu_1 \dots \nu_{m+1}} \\ &+ \binom{m+1}{2} \sum p_{\nu_1 \dots \nu_{m+2}} + \dots + (-1)^{n-m} \binom{n-1}{m-1} p_{1\dots n} \\ &= \sum_{i=0}^{n-m} (-1)^i \binom{m+i-1}{i} S_{m+i}. \end{aligned}$$

Although this result is well known, we prove it in preparation for Theorem 2. Theorem 3 establishes

$$(4) \quad A_{k+1}^{(m)} = \frac{\sum p_m(\nu_1, \dots, \nu_{k+1})}{\binom{n-m}{k+1-m}} \leq \frac{\sum p_m(\nu_1, \dots, \nu_k)}{\binom{n-m}{k-m}} = A_k^{(m)},$$

for  $k = 1, \dots, n - 1$  and  $1 \leq m \leq k$ .

Next, we extend the inequalities (4), and in Theorem 4 we show that

$$(5) \quad A_k^{(m)} \leq \frac{1}{2}(A_{k-1}^{(m)} + A_{k+1}^{(m)});$$

which states that the differences  $A_k - A_{k+1}$  ( $k = 1, \dots, n - 1$ ) are non-decreasing for increasing  $k$ . From this and a simple result we can deduce (4). Also Theorem 2 establishes that

$$(6) \quad \sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leq p_m(1, \dots, n) \leq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i},$$

<sup>2</sup> Loc. cit., Vol. 208(1939), p. 1703.

for  $2l + 1 \leq n - m$  and  $2l \leq n - m$  respectively. These inequalities throw light on formula (3) and are sharper than the following analogue of Boole's inequality for  $p_m(1, \dots, n)$ , which is a special case of (4):

$$(7) \quad p_m(1, \dots, n) \leq \sum p_{\nu_1 \dots \nu_m}.$$

The last statement will be evident in the proof.

In Theorem 5 we give an "inversion" of the formula (3), i.e. we express  $p_{1 \dots n}$  in terms of the  $p_m(\nu_1, \dots, \nu_k)$ 's, as follows:

$$(8) \quad \begin{aligned} \binom{n-1}{m-1} p_{1 \dots n} &= \sum p_m(\nu_1, \dots, \nu_m) - \sum p_m(\nu_1, \dots, \nu_{m+1}) + \dots \\ &+ (-1)^{n-m} p_m(1, \dots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum p_m(\nu_1, \dots, \nu_{m+i}). \end{aligned}$$

This of course implies the following more general formula for  $p_{\alpha_1 \dots \alpha_r}$ ,

$$\binom{r-1}{m-1} p_{\alpha_1 \dots \alpha_r} = \sum_{i=0}^{r-m} (-1)^i \sum p_m(\nu_1, \dots, \nu_{m+i})$$

where  $(\alpha_1, \dots, \alpha_r)$  is a combination of the integers  $(1, \dots, n)$  and where the second summation extends to all the  $\binom{r}{m+i}$ -combinations of  $(\alpha_1, \dots, \alpha_r)$ . Since it is known<sup>3</sup> that we can express other functions such as  $S_r$ ,  $p_{[\mu_1 \dots \mu_r]}$  in terms of the  $p_{\mu_1 \dots \mu_r}$ 's, we can also express them in terms of the  $p_m(\nu_1, \dots, \nu_k)$ 's, provided  $r \geq m$ .

Finally, for the case  $m = 1$ , we give in Theorem 6 an explicit formula for  $p_{[1 \dots r]}$  in terms of the  $p_1(\nu_1, \dots, \nu_k)$ 's, as shown in (9),

$$(9) \quad \begin{aligned} p_{[1 \dots r]} &= -p_1(r+1, \dots, n) + \sum_{\nu_1} p_1(\nu_1, r+1, \dots, n) \\ &- \sum_{\nu_1, \nu_2} p_1(\nu_1, \nu_2, r+1, \dots, n) + \dots \\ &+ (-1)^{r-1} \sum p_1(1, \dots, r, r+1, \dots, n), \\ &= \sum_{i=1}^r (-1)^{i-1} \sum_{(\nu_1, \dots, \nu_i)} p_1(\nu_1, \dots, \nu_i, r+1, \dots, n), \end{aligned}$$

where  $(\nu_1, \dots, \nu_i)$  runs through all the  $\binom{r}{i}$ -combinations from  $(1, \dots, r)$ .

This of course implies the following more general formula:

$$p_{[\alpha_1 \dots \alpha_r]} = \sum_{i=1}^r (-1)^{i-1} \sum_{(\nu_1, \dots, \nu_i)} p_1(\nu_1, \dots, \nu_i, \alpha_{r+1}, \dots, \alpha_n),$$

<sup>3,4</sup> Fréchet, "Condition d'existence de systemes d'événements associés à certaines probabilités," *Jour. de Math.*, (1940), p. 51-62.

where  $(\alpha_1, \dots, \alpha_r, \dots, \alpha_n)$  is a permutation of  $(1, \dots, n)$  and where  $(\nu_1, \dots, \nu_i)$  runs through all the  $\binom{r}{i}$ -combinations out of  $(\alpha_1, \dots, \alpha_r)$ . From Theorem 6 and two lemmas we deduce a condition of existence of systems of events associated with the probabilities  $p_i(\nu_1, \dots, \nu_m)$ . The author has not been able to obtain similar elegant results for the general  $m$ . Probably they do not exist.

**2. Generalization of Poincaré's formula; Generalization and sharpening of Boole's inequality.**

THEOREM 1:

$$(3) \quad p_m(1, \dots, n) = \sum p_{\nu_1 \dots \nu_m} - \binom{m}{1} \sum p_{\nu_1 \dots \nu_{m+1}} + \binom{m+1}{2} \sum p_{\nu_1 \dots \nu_{m+2}} - \dots + (-1)^{n-m} \binom{n-1}{n-m} p_{1 \dots n}.$$

PROOF: We have

$$(10) \quad p_m(1, \dots, n) = \sum_{b=0}^{n-m} \sum p_{\{\mu_1 \dots \mu_{m+b}\}},$$

where the second summation extends, for a fixed  $b$ , to all the  $\binom{n}{m+b}$ -combinations of  $(1, \dots, n)$ . Further we have

$$(11) \quad p_{\nu_1 \dots \nu_{m+c}} = \sum_{d=0}^{n-m-c} \sum p_{\{\nu_1 \dots \nu_{m+c} \dots \nu_{m+c+d}\}}$$

where the second summation extends, for a fixed  $d$ , to all the  $\binom{n-m-c}{d}$ -combinations of  $(1, \dots, n) - (\nu_1, \dots, \nu_{m+c})$ . The formulas (10) and (11) are evident by observing that the probabilities in the summations are all additive. Now we count the number of times a fixed  $p_{\{\mu_1 \dots \mu_{m+b}\}}$  appears in (3). By (11) this is equal to the sum

$$\binom{m+b}{m} - \binom{m}{1} \binom{m+b}{m+1} + \binom{m+1}{2} \binom{m+b}{m+2} - \dots + (-1)^{n-m} \binom{n-1}{n-m} \binom{m+b}{m+b} = 1,$$

since this number is the coefficient of  $(-1)^m x^m$  in the expansion of

$$(1-x)^{m+b} \left(1 - \frac{1}{x}\right)^{-m} = (-1)^{-m} x^m (1-x)^b.$$

Thus by (10) we have (3).

**THEOREM 2:** For  $2l \leq n - m$  and  $2l \leq n - m$  respectively, we have

$$(6) \quad \sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leq p_m(1, \dots, n) \leq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i}.$$

**PROOF:** By the reasoning in the previous proof, it is sufficient (in fact also necessary) to show that

$$\sum_{i=0}^{2l} \binom{m-1+i}{i} \binom{m+b}{m+i} \geq 1, \quad \sum_{i=0}^{2l+1} \binom{m-1+i}{i} \binom{m+b}{m+i} < 1.$$

Since

$$\binom{m-1+i}{i} \binom{m+b}{m+i} = \frac{(m+b)!}{(m-1)! b!} \binom{b}{i} \frac{1}{m+i}$$

is an integer, it is sufficient to show that

$$(12) \quad \sum_{i=0}^{2l} (-1)^i \binom{b}{i} \frac{1}{m+i} > 0, \quad \sum_{i=0}^{2l+1} (-1)^i \binom{b}{i} \frac{1}{m+i} \leq 0.$$

Suppose  $b > 0$  is even. For  $i \leq b/2 - 1$ , we have  $\frac{b-i}{i+1} > 1$  so that  $\frac{b-i}{i+1} \geq \frac{i+2}{i+1}$ . Also  $\frac{m+i}{m+i+1} \geq \frac{i+1}{i+2}$  for  $m \geq 1$ . Hence

$$\begin{aligned} \binom{b}{i+1} \frac{1}{m+i+1} &= \frac{b-i}{i+1} \frac{m+i}{m+i+1} \binom{b}{i} \frac{1}{m+i} \\ &\geq \frac{i+2}{i+1} \frac{i+1}{i+2} \binom{b}{i} \frac{1}{m+i} = \binom{b}{i} \frac{1}{m+i}. \end{aligned}$$

For  $i \geq b/2$  we have  $\frac{b-i}{i+1} < 1$  so that  $\frac{b-i}{i+1} \frac{m+i}{m+i+1} < 1$  and

$$\binom{b}{i+1} \frac{1}{m+i+1} < \binom{b}{i} \frac{1}{m+i}.$$

Thus the absolute values of the terms of the alternating series

$$\sum_{i=0}^b (-1)^i \binom{b}{i} \frac{1}{m+i} = \frac{b!}{(m+b)!(m-1)!}$$

are monotone increasing as long as  $i \leq \frac{b}{2} - 1$ , reaching maximum at  $i = \frac{b}{2}$  and then become monotone decreasing.

Therefore (12) evidently holds for  $2l \leq b/2$  and  $2l + 1 \leq b/2$  respectively.

For  $t \geq \frac{b}{2} + 1$  we write

$$\begin{aligned} \sum_{i=0}^t (-1)^i \binom{b}{i} \frac{1}{m+i} &= \frac{b!}{(m+b)!(m-1)!} - \sum_{i=t+1}^b (-1)^i \binom{b}{i} \frac{1}{m+i} \\ &= \frac{b!}{(m+b)!(m-1)!} - \sum_{j=0}^{b-t-1} (-1)^j \binom{b}{j} \frac{1}{m+b-j}. \end{aligned}$$

From the above and the fact that  $\frac{b!}{(m+b)!(m-1)!} \leq \frac{1}{m+b}$  we see that the righthand side is an alternating series whose terms are non-decreasing in absolute values. Hence (12) is true.

If  $b$  is odd, the case is similar.

**3. Generalization of Fréchet's inequalities and related inequalities.** Before proving our remaining theorems, we shall give a more detailed account of the general method which will be used. In the foregoing work we have already given two different expressions for the function  $p_m(1, \dots, n)$ , namely, formulas (3) and (10), but they are not convenient for our later purposes. Formula (3) is inconvenient because it is not additive and because the  $p_{\nu_1 \dots \nu_i}$ 's are related in magnitudes; while formula (10) has gone so far in the separation of the additive constituents that its application raises algebraical difficulties. Let us therefore take an intermediate course.

Let each  $\binom{n}{m}$ -combination  $(\nu_1, \dots, \nu_m)$  out of  $(1, \dots, n)$  be written so that  $\nu_1 < \nu_2 < \dots < \nu_m$ . Then we arrange them in an ordered sequence in the following way: the combination  $(\nu_1, \dots, \nu_m)$  is to precede the combination  $(\mu_1, \dots, \mu_m)$  if, for the first  $\nu_i \neq \mu_i$ , we have  $\nu_i > \mu_i$ . After such an arrangement we symbolically denote these combinations by

$$I, II, \dots, \left[ \binom{n}{m} \right].$$

Further, all the  $\binom{k}{m}$ -combinations out of  $(\nu_1, \dots, \nu_k)$  where the latter is a combination out of  $(1, \dots, n)$  are arranged in the order in which they appear in the sequence just written. For example, all the  $\binom{4}{2}$ -combinations out of  $(1, 2, 3, 4)$  are ordered thus:

$$(12) \quad (13) \quad (14) \quad (23) \quad (24) \quad (34).$$

Let  $U$  denote a typical combination  $(\mu_1, \dots, \mu_m)$ . By  $E_U$  we mean the combination of events  $E_{\mu_1}, \dots, E_{\mu_m}$  so that  $p_U = p_{\mu_1 \dots \mu_m}$ . In general, let the combinations  $U_1, \dots, U_{b-1}, U_b$  be given, then  $p_{U_1 \dots U_{b-1} U_b}$  denotes the probability of the non-occurrence of  $U_1, \dots, U_{b-1}$  and the occurrence of  $U_b$ .

Now let  $I, II, \dots, \left[ \binom{k}{m} - 1 \right] = Y, \left[ \binom{k}{m} \right] = Z$  denote all the  $\binom{k}{m}$ -combinations out of  $(\nu_1, \dots, \nu_k)$  in their assigned order. We have

$$(13) \quad p_m(\nu_1, \dots, \nu_k) = p_I + p_{I'II} + p_{I'I'III} + \dots + p_{I' \dots I'Z}.$$

This fundamental formula is evident. Of course it is possible to identify the  $p$ 's on the right-hand side with the ordinary  $p_{\nu_1 \dots \nu_j}$ 's, but we shall refrain from so doing and be content with the following example:

$$p_2(1, 2, 3, 4) = p_{12} + p_{12'3} + p_{12'3'4} + p_{1'23} + p_{1'23'4} + p_{1'2'34}.$$

**THEOREM 3.** For  $k = 1, \dots, n - 1$  and  $1 \leq m \leq k$  we have

$$\binom{n - m}{k - m} \Sigma p_m(\nu_1, \dots, \nu_{k+1}) \leq \binom{n - m}{k + 1 - m} \Sigma p_m(\nu_1, \dots, \nu_k).$$

**PROOF.** Substitute (13) and a similar formula for  $k + 1$  into the two sides respectively. After this substitution we observe that the number of terms is the same on both sides, since

$$\binom{n - m}{k - m} \binom{n}{k + 1} \binom{k + 1}{m} = \binom{n - m}{k + 1 - m} \binom{n}{k} \binom{k}{m}.$$

Also, the number of terms with a given  $U = (\mu_1, \dots, \mu_m)$  unaccented is the same, since

$$\binom{n - m}{k - m} \binom{n - m}{k + 1 - m} = \binom{n - m}{k + 1 - m} \binom{n - m}{k - m}.$$

Let the sum of all the terms with  $U$  unaccented in the two summations be denoted by  $\sigma_{k+1} = \sigma_{k+1}(\mu_1, \dots, \mu_m)$  and  $\sigma_k = \sigma_k(\mu_1, \dots, \mu_m)$  respectively. It is sufficient to prove that

$$(14) \quad \binom{n - m}{k - m} \sigma_{k+1} \leq \binom{n - m}{k + 1 - m} \sigma_k,$$

for any  $U$ .  $\sigma_k$  contains  $\binom{n - m}{k - m}$  terms each of the form  $p_{\nu_1 \dots \nu_l \mu_1 \dots \mu_m}$  where  $0 \leq l \leq \mu_m - m$  and where  $(\nu_1, \dots, \nu_l, \mu_1, \dots, \mu_m)$  is a  $\binom{\mu_m}{m + l}$ -combination out of  $(1, \dots, \mu_m)$ . For fixed  $(\mu_1, \dots, \mu_m)$  and a fixed  $l$  but varying  $\lambda$ 's,  $\sigma_k$  contains  $\binom{n - \mu_m}{k - m - l}$  terms of the form  $p_{\nu_1 \dots \nu_l \mu_1 \dots \mu_m}$ , with exactly  $l$  accented subscripts. Let the sum of all such terms be denoted by  $\sigma_k^{(l)}$ . Evidently  $\sigma_k^{(l)}$  has  $\binom{\mu_m - m}{l}$  terms. As a check we have

$$\begin{aligned} \binom{n - \mu_m}{k - m} \binom{\mu_m - m}{0} + \binom{n - \mu_m}{k - m - 1} \binom{\mu_m - m}{1} + \dots \\ + \binom{n - \mu_m}{k - \mu_m} \binom{\mu_m - m}{\mu_m - m} = \binom{n - m}{k - m}, \end{aligned}$$

which is the total number of terms in  $\sigma_k$ .

We decompose these  $p$ 's partially, as follows:

$$p_{\nu_1 \dots \nu_l \mu_1 \dots \mu_m} = \sum_{b=0}^{\mu_m - m - l} \sum_{\mu_{m+1}, \dots, \mu_{m+b}} p_{\nu_1 \dots \nu_{l+c} \mu_1 \dots \mu_{m+b}},$$

where  $(\nu_1, \dots, \nu_{l+c}, \mu_1, \dots, \mu_{m+b})$  is a permutation of  $(1, \dots, \mu_m)$  and where the second summation extends, for a fixed  $b$ , to all the  $\binom{\mu_m - m - l}{b}$ -combinations out of  $(1, \dots, \mu_m) - (\nu_1, \dots, \nu_l, \mu_1, \dots, \mu_m)$ .

Now consider a given

$$p_{\rho_1 \dots \rho_i \lambda_1 \dots \lambda_s \mu_1 \dots \mu_m}$$

where  $0 \leq t \leq \mu_m - m$  and  $(\rho_1 \dots \rho_i \lambda_1 \dots \lambda_s \mu_1 \dots \mu_m)$  is a permutation of  $(1, \dots, \mu_m)$ . It appears  $\binom{t}{l}$  times in  $\sigma_k^{(l)}$ . Hence it appears

$$\binom{n - \mu_m}{k - m} \binom{t}{0} + \binom{n - \mu_m}{k - m - 1} \binom{t}{1} + \dots + \binom{n - \mu_m}{k - m - t} \binom{t}{t} = \binom{n - \mu_m + t}{k - m}$$

times in  $\sigma_k$ .

Therefore to prove (14) it is sufficient to prove that

$$\binom{n - m}{k - m} \binom{n - \mu_m + t}{k + 1 - m} \leq \binom{n - m}{k + 1 - m} \binom{n - \mu_m + t}{k - m}.$$

By an easy reduction we have

$$(n - \mu_m + t - k + m) \leq n - k$$

or

$$- \mu_m + t + m \leq 0;$$

since  $t \leq \mu_m - m$  this is obvious.

**THEOREM 4:** For  $2 \leq k \leq n - 1$  and  $1 \leq m \leq k$  we have

$$(5) \quad \frac{\Sigma p_m(\nu_1, \dots, \nu_k)}{\binom{n - m}{k - m}} \leq \frac{1}{2} \frac{\Sigma p_m(\nu_1, \dots, \nu_{k-1})}{\binom{n - m}{k - 1 - m}} + \frac{1}{2} \frac{\Sigma p_m(\nu_1, \dots, \nu_{k+1})}{\binom{n - m}{k + 1 - m}}.$$

**PROOF:** By the reasoning in the previous proof, it is sufficient to show that

$$2 \binom{n - m}{k - 1 - m} \binom{n - m}{k + 1 - m} \binom{n - \mu_m + t}{k - m} \leq \binom{n - m}{k - m} \binom{n - m}{k + 1 - m} \binom{n - \mu_m + t}{k - 1 - m} + \binom{n - m}{k - m} \binom{n - m}{k - 1 - m} \binom{n - \mu_m + t}{k + 1 - m},$$

for  $0 \leq t \leq \mu_m - m$ . By an easy reduction this is equivalent to

$$2(n - k)(n - \mu_m + t - k + m + 1) \leq (n - k + 1)(n - k) + (n - \mu_m + t - k + m + 1)(n - \mu_m + t - k + m)$$

or

$$(n - \mu_m + t - k + m + 1)(\mu_m - t - m) \leq (n - k)(\mu_m - t - m).$$

For  $t = \mu_m - m$  we have equality, otherwise we have

$$- \mu_m + t + m + 1 \leq 0.$$



We can deduce Theorem 3 from Theorem 4 and the following result (a case of generalized Gumbel inequalities):

$$(15) \quad \binom{n-1}{n-2} p_m(1, \dots, n) \leq \Sigma p_m(\nu_1, \dots, \nu_{n-1}).$$

PROOF OF (15): Substitute from (13). Consider the  $p$ 's with  $U$  unaccented. The number of such terms is the same on both sides. But on the left-hand side they are all the same  $p_{U, U, \dots, (U-1), U}$ , while those on the right-hand side, being of the form  $p_{U_1, \dots, U_\lambda, U}$  where  $0 \leq \lambda \leq U - 1$  and  $(U_1, \dots, U_\lambda)$  is a combination out of  $(1, \dots, U - 1)$ , are greater than or equal to it. Hence the result.

**4. The  $p_{\alpha_1 \dots \alpha_i}$ 's in terms of the  $p_m(\nu_1, \dots, \nu_k)$ 's and the  $p_{[\alpha_1 \dots \alpha_i]}$ 's in terms of the  $p_i(\nu_1, \dots, \nu_k)$ 's.**

THEOREM 5: For  $1 \leq m \leq n$  we have

$$(8) \quad \begin{aligned} \binom{n-1}{m-1} p_{1 \dots n} &= \sum p_m(\nu_1, \dots, \nu_m) - \sum p_m(\nu_1, \dots, \nu_{m+1}) + \dots \\ &\quad + (-1)^{n-m} p_m(1, \dots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum_{\nu_1, \dots, \nu_{m+i}} p_m(\nu_1, \dots, \nu_{m+i}). \end{aligned}$$

PROOF: As in the proof of Theorem 3, consider  $\sigma_k(\mu_1, \dots, \mu_m)$ . Here  $m \leq k \leq n$ . Since a given

$$(16) \quad p_{\rho_1 \dots \rho_t \lambda_1 \dots \lambda_s \mu_1 \dots \mu_m},$$

appears  $\binom{n - \mu_m + t}{k - m}$  times in  $\sigma_k$ , it appears

$$\begin{aligned} \sum_{k=m}^n (-1)^{k-m} \binom{n - \mu_m + t}{k - m} &= \sum_{j=0}^{n-m} (-1)^j \binom{n - \mu_m + t}{j} \\ &= \sum_{j=0}^{n-\mu_m+t} (-1)^j \binom{n - \mu_m + t}{j} = \begin{cases} 0, & \text{if } n - \mu_m + t \geq 1, \\ 1, & \text{if } n - \mu_m + t = 0. \end{cases} \end{aligned}$$

times on the right hand side of (8). Hence for fixed  $(\mu_1, \dots, \mu_m)$ , the only  $p$ 's of the form (16) which actually appears are those with  $t = \mu_m - n$ . But  $\mu_m \leq n$ , thus  $t = 0$ ,  $\mu_m = n$ , and  $(\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_m)$  is a permutation of  $(1, \dots, n)$ . The term in question is therefore  $p_{1 \dots n}$ . Since the number of  $\binom{n}{m}$ -combinations of  $(1, \dots, n)$  with  $\mu_m = n$  is  $\binom{n-1}{m-1}$ , we have the theorem.

THEOREM 6: For  $1 \leq r \leq n - 1$ , we have

$$(9) \quad \begin{aligned} p_{[1 \dots r]} &= - p_1(r + 1, \dots, n) + \sum_{\nu_1} p_1(\nu_1, r + 1, \dots, n) \\ &\quad - \sum_{\nu_1, \nu_2} p(\nu_1, \nu_2, r + 1, \dots, n) + \dots + (-1)^{r-1} \sum p_1(1, \dots, n) \\ &= \sum_{i=1}^r (-1)^{i-1} \sum_{\nu_1, \dots, \nu_i} p_1(\nu_1, \dots, \nu_i, r + 1, \dots, n), \end{aligned}$$

where  $(\nu_1, \dots, \nu_i)$  runs through all the  $\binom{r}{i}$ -combinations out of  $(1, \dots, r)$ .

PROOF: We rewrite (14) for the special case  $m = 1$ ,

$$(17) \quad p_1(\mu_1, \dots, \mu_k) = p_{\mu_1} + p_{\mu_1\mu_2} + \dots + p_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100}$$

where  $\mu_1 < \mu_2 < \dots < \mu_k$ . Substitute into the right hand side of (9). After the substitution let the sum of all those  $p$ 's with  $\mu$  unaccented be denoted by  $\sigma_\mu$ . The terms in  $\sigma_\mu$  are of the form  $p_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100}$  where  $1 \leq s \leq \mu$  and  $(\mu_1, \dots, \mu_{s-1})$  is a combination out of  $(1, \dots, \mu - 1)$ .

First consider a fixed  $\mu \leq r$ . For a fixed  $p_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100}$  we count the number of times it appears in  $\sigma_\mu$ , that is, on the right hand side of (9). This is evidently equal to

$$\sum_{j=s}^r (-1)^j \binom{r-\mu}{j-s} = \sum_{j=s}^{r-\mu+s} (-1)^j \binom{r-\mu}{j-s} = \begin{cases} 0, & \text{if } r-\mu \geq 1, \\ 1, & \text{if } r-\mu = 0. \end{cases}$$

Thus the only terms that actually appear are those with  $\mu = r$ ; and each of such terms  $p_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100}$  appears exactly once with the sign  $(-1)^s$ . Hence their total contribution is

$$(18) \quad p_r - \sum_{\nu_1} p_{\nu_1 r} + \sum_{\nu_1, \nu_2} p_{\nu_1 \nu_2 r} - \dots + (-1)^{r-1} p_{1\dots(r-1) r} = p_{1\dots r},$$

by an easy modification of Poincaré's formula.

Next consider a fixed  $\mu \geq r + 1$ . Every term with  $\mu$  unaccented in  $\sigma_\mu$  is of the form (with the usual convention for  $\mu = r + 1$ )  $p_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100}$  where  $(\mu_1, \dots, \mu_s)$  is a combination out of  $(1, \dots, r)$ ; and it appears exactly once with the sign  $(-1)^s$ . Their total contribution is therefore

$$- p_{(r+1)\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}\mu_{12}\mu_{13}\mu_{14}\mu_{15}\mu_{16}\mu_{17}\mu_{18}\mu_{19}\mu_{20}\mu_{21}\mu_{22}\mu_{23}\mu_{24}\mu_{25}\mu_{26}\mu_{27}\mu_{28}\mu_{29}\mu_{30}\mu_{31}\mu_{32}\mu_{33}\mu_{34}\mu_{35}\mu_{36}\mu_{37}\mu_{38}\mu_{39}\mu_{40}\mu_{41}\mu_{42}\mu_{43}\mu_{44}\mu_{45}\mu_{46}\mu_{47}\mu_{48}\mu_{49}\mu_{50}\mu_{51}\mu_{52}\mu_{53}\mu_{54}\mu_{55}\mu_{56}\mu_{57}\mu_{58}\mu_{59}\mu_{60}\mu_{61}\mu_{62}\mu_{63}\mu_{64}\mu_{65}\mu_{66}\mu_{67}\mu_{68}\mu_{69}\mu_{70}\mu_{71}\mu_{72}\mu_{73}\mu_{74}\mu_{75}\mu_{76}\mu_{77}\mu_{78}\mu_{79}\mu_{80}\mu_{81}\mu_{82}\mu_{83}\mu_{84}\mu_{85}\mu_{86}\mu_{87}\mu_{88}\mu_{89}\mu_{90}\mu_{91}\mu_{92}\mu_{93}\mu_{94}\mu_{95}\mu_{96}\mu_{97}\mu_{98}\mu_{99}\mu_{100} + \dots + (-1)^{r-1} p_{1\dots(r+1)\mu} = - p_{1\dots(r+1)\mu},$$

by another application of Poincaré's formula. Summing up for  $\mu = r + 1, \dots, n$ , we obtain

$$(19) \quad - (p_{1\dots r(r+1)} + p_{1\dots r(r+1)(r+2)} + \dots + p_{1\dots r(r+1)\dots(n-1)n}).$$

Adding (18) and (19), we obtain as the sum of the right-hand side of (9)

$$p_{1\dots r} - (p_{1\dots r(r+1)} + p_{1\dots r(r+1)(r+2)} + \dots + p_{1\dots r(r+1)\dots(n-1)n}) = p_{1\dots r(r+1)(r+2)\dots n} = p_{[1\dots r]}$$

by an easy modification of (17).

**5. A condition for existence of systems of events associated with the probabilities  $p_1(\nu_1, \dots, \nu_k)$ .**

LEMMA 1: Let any  $2^n - 1$  quantities  $q(\alpha_1, \dots, \alpha_k)$  be given, where  $k =$

1, \dots, n, and for a fixed k, (\alpha\_1, \dots, \alpha\_k) runs through all the \binom{n}{k}-combinations out of (1, \dots, n). Let the quantities Q(\alpha\_1, \dots, \alpha\_k) be formed as follows:

$$Q(0) = 1 - q(1, \dots, n),$$

$$Q(\alpha_1, \dots, \alpha_k) = -q(\alpha_{k+1}, \dots, \alpha_n) + \sum_{\nu_1} q(\nu_1, \alpha_{k+1}, \dots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\nu_1, \nu_2, \alpha_{k+1}, \dots, \alpha_n) + \dots + (-1)^{k-1} q(1, \dots, n),$$

where (\nu\_1, \dots, \nu\_i) runs through all the \binom{k}{i}-combinations out of (1, \dots, n) - (\alpha\_{k+1}, \dots, \alpha\_n). Then the sum of all these Q's is equal to 1.

PROOF: Add all these Q's and count the number of times a fixed q(\mu\_1, \dots, \mu\_k) appears in the sum. For 1 \le k \le n this number is equal to

$$-1 + \binom{k}{1} - \binom{k}{2} + \dots + (-1)^{k-1} \binom{k}{k} = 0.$$

Hence we have the lemma.

LEMMA 2: (Fréchet) Given 2^n quantities Q\_{[\alpha\_1 \dots \alpha\_r]} where (\alpha\_1, \dots, \alpha\_r) runs through all combinations out of (1, \dots, n) including the empty one. The necessary and sufficient condition that there exist systems of events E\_1, \dots, E\_n for which

$$p_{[\alpha_1 \dots \alpha_r]} = Q_{[\alpha_1 \dots \alpha_r]}$$

(where p\_{[0]} denotes the probability for the non-occurrence of E\_1, \dots, E\_n) is that each Q \ge 0 and that their sum is equal to 1.

PROOF: Since the probabilities p\_{[\alpha\_1 \dots \alpha\_r]} are independent, i.e., unrelated in magnitudes except that their sum is equal to 1, the lemma is evident.

THEOREM 7: Given 2^n - 1 quantities q(\alpha\_1, \dots, \alpha\_k) as in Lemma 1, the necessary and sufficient condition that there exist systems of events E\_1, \dots, E\_n for which

$$p_1(\alpha_1, \dots, \alpha_k) = q(\alpha_1, \dots, \alpha_k)$$

is that for any combination (\alpha\_{r+1}, \dots, \alpha\_n), 1 \le r \le n - 1, out of (1, \dots, n) we have

$$-q(\alpha_{r+1}, \dots, \alpha_n) + \sum_{\nu_1} q(\alpha_{\nu_1}, \alpha_{r+1}, \dots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{r+1}, \dots, \alpha_n) + \dots + (-1)^{r-1} q(1, \dots, n) \ge 0,$$

and thus

$$1 - q(1, \dots, n) \ge 0.$$

PROOF: The condition is necessary by Theorem 6. It is sufficient by Lemma 1, 2 and an obvious formula expressing p\_1(\alpha\_1, \dots, \alpha\_r) in terms of the p\_{[\nu\_1 \dots \nu\_i]}'s.