ON THE PROBABILITY THEORY OF ARBITRARILY LINKED EVENTS

By Hilda Geiringer

1. **Introduction.** The classical Poisson problem can be stated as follows: Let $p_1, p_2, \cdots p_n$ be the probabilities of n *independent* events $E_1, E_2, \cdots E_n$ respectively; i.e. the probability of the simultaneous occurrence of E_i and E_j is equal to $p_i p_j$, that of E_i, E_j, E_k is equal to $p_i p_j p_k$ and so on. We seek the probability $P_n(x)$ that x of the events shall occur. If, $p_1 = p_2 = \cdots = p_n$ the problem is known as the Bernoulli problem.

More generally the n events may be regarded as *dependent*. Let p_{ij} be the probability of the simultaneous occurrence of E_i and E_j; p_{ijk} that of E_i, E_j, E_k and finally $p_{12\cdots n}$ that of $E_1, E_2, \cdots E_n$. There shall arise again the problem of determining the probability $P_n(x)$ that x of the n events will take place.[1] Furthermore the asymptotic behaviour of $P_n(x)$ for large n can be studied; and we shall especially be interested in the problem of the convergence of $P_n(x)$ towards a normal distribution or a Poisson distribution.

Even in the general case which we just explained, the sums

$$S_1 = \sum_{i=1}^{n} p_i, \qquad S_2 = \sum_{i,j=1}^{n} p_{ij}, \cdots S_n = p_{12\cdots n}$$

of our probabilities differ only by constant factors from the *factorial moments* $M_n^{(1)}, M_n^{(2)}, \cdots M_n^{(n)}$ of $P_n(x)$. For we have

$$S_\nu = \frac{1}{\nu!} M_n^{(\nu)} = \frac{1}{\nu!} \sum_{x=\nu}^{n} x(x-1) \cdots (x-\nu+1) P_n(x).$$

Starting from this remark the author has, in earlier papers, [8, 9, 10] established a theory of the asymptotic behaviour of $P_n(x)$, making use of the theory of moments. The criterion for the convergence of $P_n(x)$ towards the normal—or the Poisson—distribution consists of certain conditions[2] which the S_ν must satisfy.

In the following section a concise statement of the whole problem will be given, independently of the author's earlier publications. For the convergence towards the normal distribution we shall be able to establish a theorem under wider conditions in a manner which seems to be simpler. Finally, some applications of the theory will be considered.

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[1] See, for instance, references [1]–[7] at end of paper.

[2] Using the "theorem of the continuity of moments," Professor v. Mises [11] established sufficient conditions for the convergence of $P_n(x)$ towards a Poisson distribution in the case of the problem of "iterations." However, his reasoning can be applied to the general case without much difficulty.

2. **Formulation of the problem.** Let us consider the n-dimensional *collective* (Kollectiv) consisting of a sequence of any n trials. In the simplest case these trials will be *alternatives*, i.e. for every trial there will exist only two results, which we may denote by "occurrence," "non-occurrence" or by "1," "0." The single trial may eventually be composed in various manners. For instance we may draw $m > n$ times from an urn, which contains counters, bearing in arbitrary proportions numbers from 0 to 9. The first "event" E_1 may consist of the fact that the first three extracted counters bear even numbers; the second trial E_2 will be regarded as successful, if the sum of the counters extracted at the second, third and fourth drawings is greater than five, etc. In every case the result of the n trials will be expressed by n numbers, each of them equal to 0 or 1. The result (1, 1, 0, 0, 0, $\cdots$ 1), for instance, means that the first, the second, and the last trial were successful, the third, fourth, $\cdots$ unsuccessful, and we have an arithmetical probability distribution $v(x_1, x_2, \cdots x_n)$ ($x_k = 0, 1$; $k = 1, 2, \cdots n$), where

$$(1) \qquad \sum_{x_1} \cdots \sum_{x_n} v(x_1, x_2, \cdots x_n) = 1.$$

Instead of the $2^n - 1$ values of v we will deal with certain groups of *partial sums* of them; the first is

$$\sum \cdots \sum v(x_1, x_2, \cdots x_{i-1}, 1, x_{i+1} \cdots x_n) = p_i \qquad (i = 1, 2, \cdots n)$$

where p_i is the probability that the i-th trial will be successful. In an analogous manner let p_{ij} be the probability that the i-th and the j-th trial are both sucessful, p_{ijk} the probability that the i-th, j-th and k-th trials are simultaneously successful. Let us provisionally denote by $\Sigma^{(i)}$ an $(n - 1)$-tuple sum over all variables, except x_i, by $\Sigma^{(i,j)}$ an $(n - 2)$-tuple sum over all variables except x_i and x_j etc. We shall then have:

$$(2) \qquad \begin{aligned} p_i &= \sum{}^{(i)} v(x_1, \cdots x_{i-1}, 1, x_{i+1}, \cdots x_n) \\ p_{ij} &= \sum{}^{(i,j)} v(x_1, \cdots x_{i-1}, 1, x_{i+1}, \cdots x_{j-1}, 1, x_{j+1}, \cdots x_n) \\ &\cdots\cdots\cdots\cdots\cdots \\ p_{12\cdots n} &= v(1, 1, \cdots 1). \end{aligned}$$

In the following these probabilities $p_i, p_{ij}, p_{ijk} \cdots$ will be assumed as *directly given*. There are

$$\binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n-1} + \binom{n}{n} = 2^n - 1$$

values of this kind and it is easily seen, that the partial sums (2) are *linearly independent*.

If, especially, the probability $v(x_1, x_2, \cdots x_n)$ *depends only on the number* of zeros amongst $x_1, x_2, \cdots x_n$, i.e. if

$$v(1, 0, \cdots 0) = v(0, 1, 0, \cdots 0) = \cdots = v(0, 0, \cdots 1)$$
$$v(1, 1, \cdots 0) = v(1, 0, 1, \cdots 0) = \cdots = v(0, 0, \cdots 0, 1, 1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

the value of p_i is independent of i, the value of p_{ij} independent of i and j, and so on:

$$p_1 = p_2 = \cdots = p_n$$
$$p_{12} = p_{23} = \cdots = p_{n-1,n}$$

In the particular case of *independent* events we have only to deal with n probabilities, namely $p_1, p_2, \cdots p_n$. We have indeed $p_{ij} = p_i p_j$; $p_{ijk} = p_i p_j p_k \cdots p_{12\cdots n} = p_1 p_2 \cdots p_n$.

In the case of *chains* however, we need only know $(2n - 1)$ values, namely $p_1, p_2, \cdots p_n$; $p_{12}, p_{23}, \cdots p_{n-1,n}$. The other p_{ij}, and the $p_{ijk}, \cdots p_{12\cdots n}$ can be expressed in terms of the above probabilities.

Returning now to the general case it is easily seen that in the expression for $P_n(x)$ the $p_i, p_{ij} \cdots$ will appear only in the following combinations

$$(3) \quad S_n(0) = 1, \qquad S_n(1) = \sum_{i}^{1\cdots n} p_i, \qquad S_n(2) = \sum_{i,j}^{1\cdots n} p_{ij}, \cdots S_n(n) = p_{12\cdots n}.$$

Indeed, at the basis of the solution of the "problem of sums," there are the following relations [11] between the $S_n(z)$ and the $P_n(x)$.

$$(4) \qquad S_n(z) = \sum_{x=z}^{n} \binom{x}{z} P_n(x) \qquad \begin{pmatrix} x = 0, \cdots n \\ z = 0, \cdots n \end{pmatrix}$$

The linear equations (4) may be solved (by recurrence) for the $P_n(x)$ and we find the important result that

$$(5) \qquad P_n(x) = \sum_{z=x}^{n} (-1)^{z+x} \binom{z}{x} S_n(z)$$

Let $M_n^{(z)}$ be the z-th *factorial moment* of $P_n(x)$, i.e.

$$(6) \qquad M_n^{(z)} = \sum_{x=z}^{n} x(x-1) \cdots (x-z+1) P_n(x).$$

Making use of (4) and (6) we obtain

$$(7) \qquad M_n^{(z)} = z! S_n(z).$$

Our aim is to obtain information concerning the asymptotic behaviour of $P_n(x)$ by studying that of the moments of $P_n(x)$. The moments however are easily seen to be given in terms of the $S_n(z)$.

3. The asymptotic behavior of $P_n(x)$. Convergence towards the normal distribution.

a. THE PRINCIPAL THEOREM. According as the mean value

$$M_n^{(1)} = S_n(1) = a_n = \sum_{x=1}^{n} xP_n(x) \tag{8}$$

remains bounded or not for indefinitely increasing n, there are two types of passage to a limit. In the first case the distribution will converge (under certain conditions) towards a *Poisson* distribution; in the second case it will approach (under certain conditions) a normal distribution. As regards the convergence towards the *Poisson* distribution the author has published [9] a sufficient condition which seems to be quite simple and general. We shall, however, not resume this problem in the present paper.

We propose, indeed, to prove in the following pages a new theorem concerning the convergence of

$$V_n(x) = \sum_{t \leq x} P_n(t)$$

towards a *normal distribution.*

For this purpose we introduce the following function of the discontinuous variable $z = 0, 1, 2, \cdots n$

$$g_n(z) = \frac{z+1}{a_n} \frac{S_n(z+1)}{S_n(z)} \tag{9}$$

or, more concisely written $g_z = \frac{z+1}{a} \frac{S_{z+1}}{S_z}$, where $S_n(z)$ is defined by (3). Putting $z = a_n u$, let us consider

$$g_n(a_n u) = h_n(u) \tag{10}$$

where u is regarded as a *continuous* variable in the interval from 0 to ϵ. ($\epsilon > 0$.)

Denoting the variance of x for $V_n(x)$ by $M_2 = s_n^2$ we shall prove the

THEOREM: *Let the function $h_n(u)$, defined by* (10) *satisfy the following conditions*:

(i) *If n is sufficiently large, $h_n(u)$ admits derivatives of every order in the interval* $(0, \epsilon)$

(ii) *At $u = 0$, the first derivative of $h_n(u)$ has a limit, for $n \to \infty$, which is different from* -1.

(iii) *If u is in the interval* $(0, \epsilon)$ *the k-th derivative of $h_n(u)$ remains, for every k, inferior to a bound N_k which is independent of n.*

Then

$$\lim_{n\to\infty} V_n(a_n + ys_n\sqrt{2}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{y} e^{-x^2} dx \tag{11}$$

We shall see that in many applications these conditions may reasonably be assumed as satisfied.

b. DEMONSTRATION OF THE THEOREM.

In order to prove the principal theorem, stated above, we shall at first deduce some properties of the (finite) *differences* of $g_n(z)$ ($z = 0, 1, \cdots$) from the assumptions (i), (ii), (iii) which deal with the *derivatives* of $h_n(u)$. Indeed, the κ-th difference of $g_n(z)$ with respect to z, (which contains the values of $g_n(z)$ for $z = 0, 1, \cdots \kappa$), differs only by the factor a_n^κ from the κ-th divided difference of $h_n(u)$, with respect to u (which is formed by the values of $h_n(u)$ for $u = 0, \frac{1}{\alpha_n}, \cdots \frac{\kappa}{\alpha_n}$. Let $n > \kappa$ and so large that $\kappa/a_n < \epsilon$; then all u-values used in the formation of the κ-th divided difference of $h_n(u)$ will be in the interval $(0, \epsilon)$. Now, as it is well known, the absolute value of any divided difference of order κ can not be larger than the largest derivative in an interval which contains all the abscissae, used in the formation of the divided difference. But according to hypothesis (iii) the κ-th derivatives of $h_n(u)$ in $(0, \epsilon)$ are all inferior to N_κ. Therefore[3] we have

$$(12) \qquad | a_n^\kappa \Delta^\kappa g_n(z) | < N_\kappa$$

and for every $\gamma > 0$

$$(13) \qquad \lim_{n\to\infty} \alpha_n^{\kappa-\gamma} \underset{z=0}{\Delta^\kappa} g_n(z) = 0.$$

On the other hand from condition (ii) it follows, as is easily seen, that

$$(14) \qquad \lim_{n\to\infty} \alpha_n \underset{z=0}{\Delta} g_n(z) = a_n[g_n(1) - g_n(0)] = c \neq -1.$$

The equations (13) and (14) imply but *finite differences of* $g_n(z)$.

Let us now introduce certain new moments F_ν which we could call "factorial moments about the mean." They are indeed related to the factorial moments $M^{(\nu)}$ in exactly the same way as the moments M_ν about the mean are related to the moments M_ν^0 about the origin. Writing, S_z, a and g_z instead of $S_n(z)$, a_n and $g_n(z)$, we set

$$(15) \qquad \begin{aligned} F_\nu &= \underset{z=0}{\Delta^\nu} (M^{(\nu)} a^{\nu-z}) = M^{(\nu)} - \nu M^{(\nu-1)} a + \binom{\nu}{2} M^{(\nu-2)} a^2 - \cdots \pm a^\nu \\ &= \nu ! S_\nu - \nu ! S_{\nu-1} a + \binom{\nu}{2} (\nu - 2) ! S_{\nu-2} a^2 - \cdots \pm a^\nu \end{aligned}$$

where, particularly,

$$(16) \qquad F_0 = 1, \qquad F_1 = 0.$$

From (15) we have:

$$(17) \qquad \begin{aligned} M^{(\nu)} &= \nu ! S_\nu = \sum_{z=0}^{\nu} F_{\nu-z} \binom{\nu}{z} a^z \\ &= F_\nu + \nu F_{\nu-1} a + \binom{\nu}{2} F_{\nu-2} a^2 + \cdots + \binom{\nu}{\nu-2} F_2 a^{\nu-2} + a^\nu \end{aligned}$$

Let us begin by proving the following

[3] If we only want to deduce (13) it is sufficient to suppose that N_k (without being independent of n) increases more slowly than any power of a_n.

LEMMA I: *It follows from* (13) *and* (14) *that we have for the* F_ν *defined by* (15)

$$\lim_{n\to\infty} \frac{F_\nu}{a^{\frac{1}{2}\nu}} = G_\nu = \begin{cases} 0 & \text{if } \nu \text{ odd} \\ 1\cdot 3 \cdots (\nu - 1)c^{\frac{1}{2}\nu} & \text{if } \nu \text{ even.} \end{cases} \tag{18}$$

First we conclude from (15) and (14) that (18) is true for $\nu = 1$ and $\nu = 2$. In order to prove (18) for *every* ν, we shall point out, that

$$\lim_{n\to\infty} \frac{F_\nu}{a^{\frac{1}{2}\nu}} = (\nu - 1)c \lim_{n\to\infty} \frac{F_{\nu-2}}{a^{\frac{1}{2}(\nu-2)}} \cdots \qquad (\nu = 2, 3, \cdots) \tag{19}$$

Setting

$$f_z = g_z - 1 \quad \text{and} \quad m_z = \frac{S_z z!}{a^z} = \frac{M^{(z)}}{a^z} \tag{20}$$

we get

$$g_z = \frac{m_{z+1}}{m_z} \tag{21}$$

and

$$\begin{aligned} \Delta m_z &= m_z f_z \\ \Delta^\nu m_z &= \Delta^{\nu-1}(m_z f_z) \end{aligned} \qquad (z = 0, 1, 2, \cdots) \tag{22}$$

But according to (15) we have

$$\underset{z=0}{\Delta^\nu} m_z = \frac{1}{a^\nu} F_\nu \tag{23}$$

and therefore

$$\frac{F_\nu}{a^{\frac{1}{2}\nu}} = a^{\frac{1}{2}\nu} \underset{z=0}{\Delta^{\nu-1}}(m_z f_z) = a^{\frac{1}{2}\nu} \sum f_{\alpha\beta} \Delta^\alpha m_z \Delta^\beta f_z = \sum f_{\alpha\beta} \frac{F_\alpha}{a^{\frac{1}{2}\alpha}} a^{\frac{1}{2}(\nu-\alpha)} \Delta^\beta f_z \tag{24}$$

$$(\alpha + \beta \geqq \nu - 1;\ \alpha \leqq \nu - 1,\ \beta \leqq \nu - 1).$$

Here we have made use of the fact that the κ-th difference of a product uv can be transformed in a finite sum $\Sigma S_{\alpha\beta}\Delta^\alpha u \Delta^\beta v$ where α and β are non-negative integers and $\alpha \leqq \kappa$, $\beta \leqq \kappa$. $\left(\text{If we concern ourselves with derivatives and not with finite differences, we have, } \alpha + \beta = \kappa \text{ and } S_{\alpha\beta} = \binom{\kappa}{\alpha}\right)$. Suppose

$$\alpha + \beta > \nu - 1.$$

Then $\beta \geqq \nu - \alpha$; therefore, as $\nu > \alpha$ we have $\beta > \frac{\nu - \alpha}{2}$. Since $\Delta^\beta f_z = \Delta^\beta g_z$ the product $a^{\frac{1}{2}(\nu-\alpha)} \underset{z=0}{\Delta^\beta} f_z$ *converges toward* zero, in accordance with (13), whereas the factor $S_{\alpha\beta} \frac{F_\alpha}{a^{\frac{1}{2}\alpha}}$ remains bounded for every $\alpha < \nu$. Now suppose

$$\alpha + \beta = \nu - 1.$$

Then $\beta = \nu - 1 - \alpha$. First let $\alpha < \nu - 2$; then $\beta = \nu - 1 - \alpha > \frac{\nu - \alpha}{2}$. Thus $a^{\frac{1}{2}(\nu-\alpha)}\Delta^{\beta}_{z=0} f_z$ converges again towards zero, whereas the other factors are bounded as before. Next, if $\alpha = \nu - 1$, then $\beta = 0$ and $\Delta^0_{z=0} f_z = f_0 = 0$. Thus the corresponding term of our sum is equal to zero. Finally if $\alpha = \nu - 2$, then $\beta = 1$, and $S_{\alpha\beta} = \nu - 1$. The corresponding term of the sum (24) will be

$$(\nu - 1) \lim_{n\to\infty} \frac{F_{\nu-2}}{a^{\frac{1}{2}(\nu-2)}} \cdot \lim_{n\to\infty} a \,\Delta_{z=0} f_z = (\nu - 1)c \lim_{n\to\infty} \frac{F_{\nu-2}}{a^{\frac{1}{2}(\nu-2)}}$$

which completes the proof of Lemma I.

We shall now establish a relation between the *factorial moments* about the mean F_ν and the *ordinary moments* about the mean M_ν. To an expression of the form

(25) $$ca^{\rho} F_{\alpha}$$

(where the constant c is independent of n) let us attribute a "weight" $\rho + \frac{\alpha}{2}$. Then we shall prove the following *lemma*

LEMMA II: *Let* $\nu = 2\mu$(ν even), $\nu = 2\mu + 1$ (ν odd) *and*

(26) $$\alpha_\rho = \frac{\nu!}{(\nu - 2\rho)!\, 2^{\rho} \rho!}.$$

Then

(27) $$M_\nu - \sum_{\rho=0}^{\mu} \alpha_\rho a^{\rho} F_{\nu-2\rho}$$

is equal to a finite sum of terms of the form (25), *each of which has a weight less than* $\nu/2$.

To prove this lemma we begin by expressing the M_ν in terms of the factorial moments $M^{(\rho)}$. We shall then express the $M^{(\rho)}$ by the F_z. Now, let $s_{\kappa z}$ be the "*Stirling* numbers of second kind," i.e., putting

(28) $$x^{(z)} = x(x - 1) \cdots (x - z + 1)$$

we have

(29) $$x^z = \sum_{\kappa=0}^{z} s_{\kappa z} x^{(z-\kappa)} \qquad (z = 0, 1, 2, \cdots)$$

Then by an elementary calculation we obtain

(30) $$M_\nu = \sum_{\rho=0}^{\nu} M^{(\nu-\rho)} \left[s_{\rho\nu} - \nu a s_{\rho-1,\nu-1} + \binom{\nu}{2} a^2 s_{\rho-2,\nu-2} - \cdots \pm \binom{\nu}{\rho} a^{\rho} \right].$$

If we now introduce the F_κ we get

$$
(31) \quad \begin{aligned} M_\nu = \sum_{\rho=0}^{\nu-1} \sum_{r=0}^{\nu-\rho} F_{\nu-r-\rho} a^r \Bigg[\binom{\nu-\rho}{r} s_{\rho\nu} - \binom{\nu}{1}\binom{\nu-\rho-1}{r-1} s_{\rho,\nu-1} \\ + \binom{\nu}{2}\binom{\nu-\rho-2}{r-2} s_{\rho,\nu-2} - \cdots \pm \binom{\nu}{r} s_{\rho,\nu-r} \Bigg]. \end{aligned}
$$

Furthermore we may easily verify that

$$
(32) \quad \begin{aligned} \binom{\nu-\rho-x}{r-x} = \binom{\nu-\rho}{r} - \binom{\nu-\rho-1}{r} x \\ + \frac{1}{2!}\binom{\nu-\rho-2}{r} x^{(2)} + \cdots \pm \frac{1}{(\nu-r-\rho)!} x^{(\nu-r-\rho)}. \end{aligned}
$$

But the $s_{\kappa z}$ for $z = 0, 1, 2, \cdots$ are equal to the values of a polynomial in z, of degree 2κ, the highest term of which is equal to $\frac{z^{2\kappa}}{\kappa!\,2^\kappa}$. The degree of the product

$$
(33) \qquad \binom{\nu-\rho-x}{r-x} s_{\rho,\nu-x} = \varphi(x)
$$

is therefore equal to $(\nu - r - \rho) + 2\rho = \nu - r + \rho$. On the other hand the expression between brackets in the right hand member of (31) is nothing other than the ν-th difference of $\zeta(x)$. (The missing terms of this difference are indeed equal to zero, the corresponding $s_{\kappa\nu}$ being equal to zero.)

This ν-th difference will certainly vanish if

$$
\nu - r + \rho < \nu \text{ i.e. } r > \rho.
$$

Now, let $r = \rho$. Then the ν-th difference, i.e. the coefficient α_ρ of $F_{\nu-r-\rho}a^r = F_{\nu-2\rho}a^\rho$ in (31), is equal to $\nu!$ multiplied by the coefficient of $x^{\nu-2\rho}$ in $\varphi(x)$:

$$
\alpha_\rho = \nu! \frac{1}{(\nu-2\rho)!} \frac{1}{2^\rho \rho!}.
$$

Finally, let $r < \rho$. Then the weight of $F_{\nu-r-\rho}a^r$ is inferior to $\nu/2$. We have thus established Lemma II.

We have for instance for $\nu = 1, 2, 3, 4, 5$

$$M_1 = F_1 = 0,\ M_2 = F_2 + a,\ M_3 = F_3 + 3F_2 + a$$

$$M_4 = (F_4 + 6\,aF_2 + 3\,a^2) + 6\,F_3 + (7F_2 + a)$$

$$M_5 = (F_5 + 10\,F_3 a) + (10\,F_4 + 40\,F_2 a + 10\,a^2) + 25\,F_3 + (15\,F_2 + a)$$

Inversely in an analogous manner, we can express F_ν by the

$$
M_\rho(\rho = 1, 2, \cdots \nu).
$$

We can now terminate our demonstration by proving the following

LEMMA III: *If the conditions* (18) *are satisfied, then*

$$\lim_{n\to\infty}\frac{M_\nu}{M_2^{\frac{1}{2}\nu}} = H_\nu = \begin{cases} 0 \cdots \nu \text{ odd} \\ 1\cdot 3 \cdots (\nu - 1) \cdots \nu \text{ even}. \end{cases} \tag{34}$$

First the equation (18) for $\nu = 2$ gives

$$\lim_{n\to\infty}\frac{F_2}{a} = \lim_{n\to\infty}\frac{M_2 - a}{a} = c$$

thus

$$\lim_{n\to\infty}\frac{M_2}{a} = 1 + c \qquad (c \neq -1). \tag{35}$$

It is therefore obviously sufficient to prove the relation

$$\lim_{n\to\infty}\frac{M_\nu}{a^{\frac{1}{2}\nu}} = H_\nu(1 + c)^{\frac{1}{2}\nu}. \tag{36}$$

Putting $\nu = 2\mu$ and $\nu = 2\mu + 1$ respectively we obtain however from our lemma

$$\frac{M_\nu}{a^{\frac{1}{2}\nu}} = \sum_{\rho=0}^{\mu} \alpha_\rho a^{\rho-\frac{1}{2}\nu} F_{\nu-2\rho} + Ra^{-\frac{1}{2}\nu}.$$

Here R represents a finite sum of terms of the form (25), of "weight" inferior to $\frac{\nu}{2}$. But by virtue of (18) such a term, divided by $a^{\frac{1}{2}\nu}$ converges towards zero and we obtain

$$\lim_{n\to\infty}\frac{M_\nu}{a^{\frac{1}{2}\nu}} = \sum_{\rho=0}^{\mu}\alpha_\rho \lim_{n\to\infty}\frac{F_{\nu-2\rho}}{a^{\frac{1}{2}\nu-\rho}} = \sum_{\rho=0}^{\mu}\frac{\nu!}{(\nu-2\rho)!\,2^\rho\rho!}G_{\nu-2\rho}. \tag{37}$$

For an odd ν, $G_{\nu-2\rho}$ *is equal to zero*; for an even $\nu(= 2\mu$, say) however, we have

$$G_{2\mu-2\rho} = c^{\mu-\rho}\frac{(2\mu-2\rho)!}{2^{\mu-\rho}(\mu-\rho)!}$$

and we obtain

$$\begin{aligned} \lim_{n\to\infty}\frac{M_{2\mu}}{a^\mu} &= \sum_{\rho=0}^{\mu}\frac{(2\mu)!}{(2\mu-2\rho)!\,2^\rho\rho!}\cdot\frac{(2\mu-2\rho)!}{2^{\mu-\rho}(\mu-\rho)!} \\ &= \frac{(2\mu)!}{2^\mu\mu!}\sum_{\rho=0}^{\mu}\frac{\mu!}{\rho!(\mu-\rho)!}c^{\mu-\rho} = H_{2\mu}(1+c)^\mu \end{aligned} \tag{38}$$

in accordance with (36). Lemma III is therefore proved.

Our principal theorem is now an obvious consequence of the well known theorem of the continuity of moments. By virtue of this theorem the convergence of $V_n(a_n + ys_n\sqrt{2})$ towards a normal distribution as given by (7) will indeed be assured if the moments of V_n converge towards the moments of the corresponding normal distribution; i.e. if (34) is true. Thus our principal theorem is completely demonstrated.

4. Some applications.

EXAMPLE 1. We shall consider the following play as a very simple application of our theorem: An urn contains $m = 2n$ counters bearing the numbers $1, 2, \cdots m$. We draw them all, one after the other, without returning the counters previously drawn. *We ask for the probability $P_{2n}(x)$ that an even counter will appear at a drawing of even number x times* ($0 \leqq x \leqq n$).
As can be easily found, we have

$$p_2 = p_4 = \cdots = p_{2n} = \tfrac{1}{2}$$

$$p_{2,4} = p_{2,6} = \cdots = p_{2n-2,2n} = \frac{1}{4}\frac{2n-2}{2n-1}$$

Consequently

$$S_1 = \frac{n}{2}, \qquad S_2 = \binom{n}{2}\frac{1}{4}\frac{2n-2}{2n-1}, \qquad S_3 = \binom{n}{3}\frac{1}{8}\frac{(2n-2)(2n-4)}{(2n-1)(2n-2)}, \tag{39}$$
$$\cdots\cdot$$
$$S_z = \frac{1}{2^z}\binom{n}{z}\frac{(2n-2)(2n-4)\cdots(2n-2z+2)}{(2n-1)(2n-2)\cdots(2n-z+1)}.$$

From (39) it follows that

$$g_n(z) = \frac{n-z}{n}\frac{2n-2z}{2n-z}. \tag{40}$$

Setting $z/\frac{1}{2}n = u$, we get

$$h_n(u) = \frac{(2-u)^2}{2\left(2-\dfrac{u}{2}\right)}. \tag{41}$$

The conditions (i), (ii), (iii) of our principal theorem are obviously satisfied if $\epsilon < 4$ and we have

$$h'_n(0) = -\tfrac{3}{4} = c$$

The probability defined above is thus seen to converge (*according to* (11)) *towards a normal distribution, having a mean equal to* $\frac{n}{2}$ *and a variance* $M_2 \sim \frac{n}{8}$

EXAMPLE 2. *Probability of an "occupation."* Let k stones be distributed by chance over n places. Then the probability that any stone will occupy a certain place will be equal to $1/n$. *We ask for the probability $P_n(x)$ that there shall be x places, every one of which is occupied by exactly m stones.*[4]

By certain simple considerations, well known in combinatory calculus, we obtain:

$$a_n = n\frac{k!}{m!(k-m)!}\left(\frac{1}{n}\right)^m\left(1-\frac{1}{n}\right)^{k-m} \tag{42}$$

[4] The problem presents itself for instance if we ask for the probability that in a certain county there will be x villages, everyone of m inhabitants.

$$S_z = \frac{n!}{z!(n-z)!}\,\frac{k!}{(m!)^z(k-mz)!}\left(\frac{1}{n}\right)^{mz}\left(1-\frac{z}{n}\right)^{n-mz}. \tag{43}$$

Let $k/n = \alpha$. From (43) we deduce that

$$\begin{aligned} g_n(z) &= \frac{n-z}{n}\left(\frac{1-\dfrac{z+1}{n}}{1-\dfrac{z}{n}}\right)^{n\alpha}\frac{1}{\left(1-\dfrac{1}{n}\right)^{n\alpha}}\,\frac{\left(1-\dfrac{z}{n}\right)^{mz}\left(1-\dfrac{1}{n}\right)^{m}}{\left(1-\dfrac{z+1}{n}\right)^{mz+m}} \\ &\quad\cdot\frac{\left(\alpha-\dfrac{zm}{n}\right)\left(\alpha-\dfrac{zm+1}{n}\right)\cdots\left(\alpha-\dfrac{zm+m-1}{n}\right)}{\alpha\left(\alpha-\dfrac{1}{n}\right)\cdots\left(\alpha-\dfrac{m-1}{n}\right)} \end{aligned} \tag{44}$$

Now, let n *and* k *tend simultaneously to* ∞, *in such a way that* $\alpha = \dfrac{k}{n}$ *remains bounded.* We get at first

$$\lim_{n\to\infty}\frac{a_n}{n} = \frac{\alpha^m}{m!}e^{-\alpha}. \tag{45}$$

As a_n is seen to be of the order of magnitude of n we introduce the new variables

$$\frac{z}{n} = v \quad\text{and}\quad v = u\,\frac{a_n}{n}.$$

We have then (writing h and $\bar{h}$ instead of h_n and $\bar{h}_n$):

$$g_n(z) = g_n(nv) = \bar{h}(v)$$

$$\bar{h}(v) = \bar{h}\left(u\,\frac{a_n}{n}\right) = h(u).$$

Therefore

$$\begin{aligned} \bar{h}(v) &= (1-v)\left(1-\frac{1}{n(1-v)}\right)^{n\alpha}\frac{1}{\left(1-\dfrac{1}{n}\right)^{n\alpha}}\left(\frac{1-\dfrac{1}{n}}{1-v-\dfrac{1}{n}}\right)^{m} \\ &\quad\left(1-\frac{1}{n(1-v)}\right)^{-nmv}\cdot\frac{(\alpha-mv)\left(\alpha-\dfrac{1}{n}-mv\right)\cdots\left(\alpha-\dfrac{m-1}{n}-mv\right)}{\alpha\left(\alpha-\dfrac{1}{n}\right)\cdots\left(\alpha-\dfrac{m-1}{n}\right)}. \end{aligned} \tag{46}$$

These formulae show that the k-th derivative of $\bar{h}(v)$ with respect to v contains only rational expressions, [in the denominators of which there appear powers of $(1-v)$], and positive powers of $\log\left(1-\dfrac{1}{n(1-v)}\right)$. The conditions (i) and (iii) of our principal theorem are therefore satisfied if $\epsilon < 1$. Furthermore we have

$$\left(\frac{d\bar{h}}{dv}\right)_{v=0} = -1 - \frac{\alpha}{1-\frac{1}{n}} - mn\log\left(1-\frac{1}{n}\right) + \frac{m}{1-\frac{1}{n}}$$

$$(47) \qquad -m\,\frac{\left(\alpha-\frac{1}{n}\right)\left(\alpha-\frac{2}{n}\right)\cdots\left(\alpha-\frac{m-1}{n}\right)+\alpha\left(\alpha-\frac{2}{n}\right)\cdots\left(\alpha-\frac{m-1}{n}\right)+\cdots\alpha\left(\alpha-\frac{1}{n}\right)\cdots\left(\alpha-\frac{m-2}{n}\right)}{\alpha\left(\alpha-\frac{1}{n}\right)\cdots\left(\alpha-\frac{m-1}{n}\right)}$$

and consequently

$$\lim_{n\to\infty}\left(\frac{dh}{du}\right)_{u=0} = \left[-1-\frac{m^2}{\alpha}-\alpha+2m\right]\lim_{n\to\infty}\frac{a_n}{n}$$
$$= -\left(1+\frac{(m-\alpha)^2}{\alpha}\right)\frac{\alpha^m}{m!}e^{-\alpha} = c.$$

We have thus obtained the interesting result that,
The probability $V_n(x)$ that x places at most are occupied, each one by m stones, converges towards a normal distribution if k and n tend simultaneously to ∞ in such a way that $\lim_{n\to\infty}\frac{k}{n} = \alpha$ is bounded. We have then

$$(48) \qquad \lim_{n\to\infty} V_n(a_n + u\sqrt{2}\,s_n) = \phi(u)$$

with

$$(49) \qquad \lim_{n\to\infty}\frac{a_n}{n} = \frac{\alpha^m}{m!}e^{-\alpha}, \qquad \lim_{n\to\infty}\frac{s_n^2}{a_n} = 1 - \frac{\alpha^m e^{-\alpha}}{m!}\cdot\left[1+\frac{(m-\alpha)^2}{\alpha}\right].$$

UNIVERSITY OF ISTANBUL,
ISTANBUL, TURKEY.

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