Komatsu, T.
Osaka J. Math.
14 (1977), 125–137

ON THE PROBLEM OF OPTIMAL CONTROL FOR A STOCHASTIC LINEAR DYNAMIC SYSTEM

Takashi KOMATSU

(Received February 25, 1976)

0. Introduction

The object of this paper is to solve a problem of optimal control for a stochastic linear dynamic system which consists of an unknown process and an observable process. It is shown by Joseph-Tou [5] and Wonham [7] that the control problem of this type can be solved through a reduction to a filtering problem and a stochastic control problem of Markov type. But in their papers it is an essential assumption that the initial distribution of the unknown process is a normal one. Our main aim of this paper is to show the fact that if the cost functional is quadratic, then the control problem can be solved without the normality condition for the initial distribution. The last section is devoted to show the fact that, in the class of initial distributions with given second moment, the maximum of the risk corresponding to the optimal control is attained by a normal distribution.

1. Statement of the problem and the main theorem

Let Ξ (respectively Z) be the space of continuous functions on $[0, T]$ taking values in R^m (resp. in R^n, $n \geqq m$, and taking value 0 at time 0). ξ_t (resp. ζ_t) denotes the projection $\Xi \times Z \ni (\xi, \zeta) \leadsto \xi(t)$ (resp. $\leadsto \zeta(t)$). Let $\mathcal{F}_t^\xi$, $\mathcal{F}_t^\zeta$ and $\mathcal{F}_t$ be the σ-fields on $\Xi \times Z$ generated by $\{\xi_s; s \leqq t\}$, $\{\zeta_s; s \leqq t\}$ and $\{(\xi_s, \zeta_s); s \leqq t\}$ respectively. We shall say that an R^l-valued process $u \equiv u_t(\zeta)$ is an admissible control and write $u \in \mathcal{U}$ if it is non-anticipating with respect to the σ-fields $(\mathcal{F}_t^\zeta)$ and if

$$(1.1) \qquad \sup_{t,\zeta} |u_t(\zeta)|^2/(1+\|\zeta\|_t^2) < \infty ,$$

where $\|\zeta\|_t = \sup_{s \leqq t} |\zeta(s)|$. Let $p(d\theta)$ be a probability measure on R^m such that $\int |\theta|^2 p(d\theta) < \infty$. Give some continuous functions taking matrices-values

a) $F(t)$: $m \times l$— matrix,
b) $G(t)$: $m \times m$— matrix, positive definite,
c) $H(t)$: $n \times m$— matrix, continuously differentiable and $H^*H(t)$ is positive definite (H^* denotes the transposed matrix of H),

d) $L(t)$: $m\times m$— matrix, non-negative definite.

Lemma 1. *For each* $u\in\mathcal{U}$, *there is a unique probability* P *on the space* $(\Xi\times Z, \mathcal{F}_T)$ *such that*

$$P[\xi_0\in d\theta] = p(d\theta)\,, \tag{1.2}$$

$$\xi_t = \xi_0+\int_0^t F(s)u_s ds+\int_0^t \sqrt{G(s)}\,d\beta_s\,, \tag{1.3}$$

$$\zeta_t = \int_0^t H(s)\xi_s ds+ W_t\,,$$

where $\{(\beta_t, W_t), P\}$ *is an* $n+m$*-dimensional Brownian motion independent of* $\{\xi_0, P\}$.

This lemma is proved in section 2. For a moment, let P_u denote the probability in the lemma corresponding to $u\in\mathcal{U}$. The control problem is to find $u\in\mathcal{U}$ which minimizes the risk

$$R(u) = \int\left\{\int_0^T (|u_t|^2+\xi_t\cdot L(t)\xi_t)dt\right\}dP_u\,. \tag{1.4}$$

In order to state the main theorem, we shall introduce some functions. Let $C(t)$ and $S(t)$ be functions of $m\times m$-matrices satisfying the following ordinary differential equation

$$\frac{d}{dt}C = S\,H^*H,\quad \frac{d}{dt}S = C\,G\,, \tag{1.5}$$

with the initial condition $C(0)=I$ (unit matrix) and $S(0)=0$ (null matrix). It is known (see Buchy-Joseph [1]) that the matrix $C(t)$ is invertible, the matrix $D(t) =C^{-1}S(t)$ is positive definite for any $t>0$ and it satisfies the Riccati equation

$$\frac{d}{dt}D = G-D(H^*H)D,\quad D(0) = 0\,. \tag{1.6}$$

Let $A(t)$ be the solution of another Riccati equation

$$-\frac{d}{dt}A = L-A(FF^*)A,\quad A(T) = 0\,. \tag{1.7}$$

Put

$$B_s^t = \int_s^t (HC^{-1})^*(HC^{-1})\,(\sigma)\,d\sigma\,,$$

and let $\Delta(t, x)$ and $\Omega(s, t, x)$ denote the functions

$$\Delta(t, x) = \int p(d\theta)\, e^{x\cdot\theta-\theta\cdot B_0^t\theta/2}\,, \tag{1.8}$$

$$\Omega(s, t, x) = [(2\pi)^m \det B_s^t]^{-1/2} e^{-x\cdot(B_s^t)^{-1}x/2}\,.$$

We have the following result.

Theorem 1. *If there is a constant* $\varepsilon>0$ *such that*

$$\int e^{\varepsilon|\theta|^2/2}p(d\theta)<\infty\,, \tag{1.9}$$

then there exists an optimal control $\hat{u}\in\mathcal{U}$ *and*

$$\begin{aligned} R(\hat{u}) &= \int_0^T \text{trace}\,[LD+ADH^*HD]dt \\ &\quad+\int_0^T dt\int p(d\theta)\,(C^{-1}\theta)\cdot(L+2ADH^*H)\,(C^{-1}\theta) \\ &\quad-\int_0^T dt\int |F^*AC^{-1}\Delta_x|^2\Delta^{-2}\Big(\int p(d\theta)\Omega(0,t,x-B_0^t\theta)\Big)dx\,, \end{aligned} \tag{1.10}$$

where $\Delta_x=(\partial\Delta/\partial x_j)_{1\leq j\leq m}$.

2. Filtering and reduced control problem

In this section, $\mathcal{G}_t$ denotes the σ-field on Ξ generated by the mappings ξ_s': $\Xi\in\xi\rightsquigarrow\xi(s)$, $s\leqq t$. And the symbol $E_{\tilde{P}}[\cdot]$ denotes the expectation with respect to the probability $\tilde{P}$.

Proof of Lemma 1. Without loss of generality, we can suppose that the initial distribution p is the unit measure at $\theta\in R^m$. Let $\{\Omega', \mathcal{F}', P'; \beta_t'\}$ be an m-dimensional Brownian motion and put

$$X_t^{\theta,\zeta}=\theta+\int_0^t F(s)u_s(\zeta)ds+\int_0^t\sqrt{G(s)}d\beta_s'\,.$$

Let Q_ζ^θ denote the probability on the space $(\Xi, \mathcal{G}_T)$ indiced by the process $\{\Omega', \mathcal{F}', P'; X_t^{\theta,\zeta}\}$ and $\bar{Q}$ the Wiener measure on the space $(\Xi\times Z, \mathcal{F}_T^\zeta)$. Define a probability Q^θ on the space $(\Xi\times Z, \mathcal{F}_T)$ as follows:

$$Q^\theta((A\times Z)\cap B)=\int_B Q_\zeta^\theta(A)d\bar{Q} \qquad \text{for each } A\in\mathcal{G}_T \text{ and } B\in\mathcal{F}_T^\zeta\,.$$

Then the following property is satisfied.

$$\xi_t=\theta+\int_0^t F(s)u_sds+\int_0^t\sqrt{G(s)}d\beta_s\,, \tag{2.1}$$

$\{(\beta_t, \zeta_t); Q^\theta\}$: $m+n$-dimensional Brownian motion.

Let us introduce a positive process

$$\phi_t=\exp\Big[\int_0^t H(s)\xi_s\cdot d\zeta_s-\frac{1}{2}\int_0^t|H(s)\xi_s|^2ds\Big]\,. \tag{2.2}$$

On the basis of Girzanov's theorem [4], if $E_{Q^\theta}[\phi_T]=1$, then the probability P^θ defined by $dP^\theta/dQ^\theta=\phi_T$ has property (1.3) (where P^θ plays the role of P). Let $T_\nu=\inf\{t;|\xi_t|>\nu\}$ and $dP^{\theta,\nu}/dQ^\theta=\phi_{T_\nu}$. Since $E_{Q^\theta}[\phi_{T_\nu}]=1$, from the Girzanov theorem, the probability $P^{\theta,\nu}$ has property (1.3) on the time interval $[0, T_\nu]$ for each ν. The fact

$$\sup_{\nu} E_{P^{\theta,\nu}}[\max_{s\leq T_\nu}|\xi(s)|^2]<\infty$$

follows immediately from (1.1). On the other hand, we have

$$\begin{aligned}\int_{\phi_{T_\nu}>e^N}\phi_{T_\nu}dQ^\theta &= P^{\theta,\nu}[\log\phi_{T_\nu}>N]\\ &\leq P^{\theta,\nu}\left[\int_0^{T_\nu}|H(s)\xi_s|^2ds\leq N,\ \left|\int_0^{T_\nu}H(s)\xi_s\cdot d\zeta_s\right|>\frac{N}{2}\right]\\ &+P^{\theta,\nu}\left[\int_0^{T_\nu}|H(s)\xi_s|^2ds>N\right]\\ &\leq\frac{4}{N}+\frac{1}{N}E_{P^{\theta,\nu}}\left[\int_0^{T_\nu}|H(s)\xi_s|^2ds\right]\ \text{(cf. Lipcer-Shirjaev [6]).}\end{aligned}$$

Therefore $(\phi_{T_\nu})_\nu$ is uniformly integrable. Since $\phi_{T_\nu}\to\phi_T$ a.e. as $\nu\to\infty$, we know that $E_{Q^\theta}[\phi_T]=1$.

We shall show the uniqueness of the probability P^θ having property (1.3) and $P^\theta[\xi_0=\theta]=1$. The fact that $E_{P^\theta}[\phi_T^{-1}]=1$ can be proved by a similar method to the preceeding one. On the basis of the Girzanov theorem, the probability Q^θ associated with P^θ by $dQ^\theta/dP^\theta=\phi_T^{-1}$ has property (2.1). Since the probability Q^θ having property (2.1) is uniquely determined, so is the probability P^θ.

Q.E.D.

Since probabilities P^θ and Q^θ are mutually absolutely continuous, so are the restriction $\bar{P}^\theta=P^\theta|\mathcal{F}_T^\zeta$ of the probability P^θ on the field $\mathcal{F}_T^\zeta$ and the Wiener measure $\bar{Q}\equiv Q^\theta|\mathcal{F}_T^\zeta$. Put

$$m_t^\theta = E_{P^\theta}[\xi_t|\mathcal{F}_t^\zeta]\,, \tag{2.3}$$

$$\bar{\phi}_t(\theta) = \exp\left[\int_0^t H(s)m_s^\theta\cdot d\zeta_s-\frac{1}{2}\int_0^t|H(s)m_s^\theta|^2ds\right]. \tag{2.4}$$

Since the process

$$\nu_t^\theta = \zeta_t-\int_0^t H(s)m_s^\theta ds \tag{2.5}$$

is a Brownian motion with respect to $(\mathcal{F}_t^\zeta, \bar{P}^\theta)$, by Girzanov's theorem, the relation $d\bar{P}^\theta=\bar{\phi}_T(\theta)d\bar{Q}$ must hold. Noting that $dP^\theta=\phi_T dQ^\theta$, we obtain

$$E_{P^\theta}[f|\mathcal{F}_t^\zeta] = \bar{\phi}_t(\theta)^{-1}E_{Q^\theta}[f\phi_t|\mathcal{F}_t^\zeta] \tag{2.6}$$

for each $\mathcal{F}_t$-measurable function $f\geq 0$.

Lemma 2. *The conditional distribution of the random variable (ξ_t, P^θ) given the σ-field $\mathcal{F}_t^\zeta$ is the Gaussian distribution. Its variance matrix $D(t)$ is a solution of equation* (1.6) *and its mean m_t^θ satisfies the following equation.*

$$m_t^\theta = \theta + \int_0^t F(s)u_s ds + \int_0^t D(s)H^*(s)\,(d\zeta_s - H(s)m_s^\theta ds)\,. \tag{2.7}$$

Proof. It seems possible to prove this by a similar way to Wonham [7], but it is essentially assumed in [7] that the control u is Lipshitz continuous. Therefore we must prove this by another way. Let $a(t)$, $b(t, \zeta)$ and $c(t, \zeta)$ be solutions of the equations

$$\frac{d}{dt}a = H^*H - aGa,\quad a(0) = 0\,,$$

$$\frac{d}{dt}b = -aGb + aFu_t - aG\hat{\zeta}_t,\quad b(0, \zeta) = 0\,,$$

$$\frac{d}{dt}c = \mathrm{trace}(aG) - 2(\hat{\zeta}_t + b)\cdot Fu_t + (\hat{\zeta}_t + b)\cdot G(\hat{\zeta}_t + b),\quad c(0, \zeta) = 0\,,$$

where $\hat{\zeta}_t = \int_0^t H^*(s)d\zeta_s$. Then we have

$$\phi_t = \psi_t^\zeta \exp\Big[-\frac{1}{2}\xi_t\cdot a(t)\xi_t + \xi_t\cdot(\hat{\zeta}_t + b(t, \zeta)) + \frac{1}{2}c(t, \zeta)\Big]\,, \tag{2.8}$$

where
$$\psi_t^\zeta = \exp\Big[\int_0^t (a(s)\xi_s - \hat{\zeta}_s - b(s, \zeta))\cdot\sqrt{G(s)}d\beta_s - \frac{1}{2}\int_0^t |\sqrt{G(s)}(a(s)\xi_s - \hat{\zeta}_s - b(s, \zeta))|^2 ds\Big]\,.$$

We see that the process ψ_t^ζ is a martingale on the space $(\Xi, \mathcal{G}_T, (\mathcal{G}_t), Q_\zeta^\theta)$. Therefore the probability $\hat{P}_\zeta^\theta$ on the space $(\Xi, \mathcal{G}_T)$ given by $d\hat{P}_\zeta^\theta = \psi_t^\zeta dQ_\zeta^\theta$ has the property

$$\xi_t = \theta + \int_0^t \{F(s)u_s + G(s)(a(s)\xi_s - \hat{\zeta}_s - b(s, \zeta))\}ds + \int_0^t \sqrt{G(s)}d\beta_s^\zeta\,, \tag{2.9}$$

$\{\Xi, \mathcal{G}_T, (\mathcal{G}_t), \hat{P}_\zeta^\theta;\ \beta_t^\zeta\}$: m-dimensional Brownian motion.

This implies that the process $\{\xi_t, \hat{P}_\zeta^\theta\}$ is a Gaussian Markov process. Let $J_\zeta(t,x)$ denote the density (with respect to dx) of the distribution of the random valiable $\{\xi_t, \hat{P}_\zeta^\theta\}$. From (2.6) and (2.8), we have

$$E_{P^\theta}[f(\xi_t)\,|\,\mathcal{F}_t^\zeta] = \bar{\phi}_t(\theta)^{-1}\int \hat{J}_\zeta(t, x)f(x)dx\,,$$

for each function $f(x) \geqq 0$, where

$$\hat{J}_\zeta(t, x) = J_\zeta(t, x)\exp\Big[-\frac{1}{2}x\cdot a(t)x + x\cdot(\hat{\zeta}_t + b(t, \zeta)) + \frac{1}{2}c(t, \zeta)\Big]\,.$$

This implies that the conditional distribution of the random variable (ξ_t, P^θ)

given the σ-field $\mathcal{F}_t^\xi$ is Gaussian. Equation (1.6) and (2.7) follow immediately from the filtering equation (see Fujisaki-Kallianpur-Kunita [2]). Q.E.D.

From (2.7) and the equality $(d/dt)C^{-1}=-DH^*HC^{-1}$, we have

$$m_t^\theta = C(t)^{-1}\theta + m_t^0 . \tag{2.10}$$

Since $E_{\bar{P}^0}[d\bar{P}^\theta/d\bar{P}^0 \mid \mathcal{F}_t^\xi] = \bar{\phi}_t(\theta)/\bar{\phi}_t(0)$ and since

$$\frac{\bar{\phi}_t(\theta)}{\bar{\phi}_t(0)} = \exp\left[\theta\cdot\int_0^t (HC^{-1})^*(s)dv_s^0 - \frac{1}{2}\theta\cdot B_0^t\theta\right],$$

$$\begin{aligned} R(u) &= \int p(d\theta)E_{\bar{P}^\theta}\left[\int_0^T (|u_t|^2+\mathrm{trace}(LD(t))+m_t^\theta\cdot L(t)m_t^\theta)dt\right] \\ &= E_{\bar{P}^0}\left[\int_0^T dt\int p(d\theta)\{|u_t|^2+\mathrm{trace}(LD(t)) \right. \\ &\qquad \left. +|\sqrt{L(t)}(C(t)^{-1}\theta+m_t^0)|^2\}\, e^{\eta\cdot X_t-\theta\cdot B_0^t\theta/2}\right], \end{aligned}$$

where

$$X_t = \int_0^t (HC^{-1}(s))^* dv_s^0 . \tag{2.11}$$

Put $Y_t = m_t^0 - S^*(t)X_t$. Then we have

$$Y_t = \int_0^t (F(s)u_s(\zeta) - GC^*(s)X_s)ds . \tag{2.12}$$

Let $\Delta_x(t, x)$ and $\Delta_{xx}(t, x)$ be the vector $(\partial\Delta/\partial x_j)_j$ and the matrix $(\partial^2\Delta/\partial x_i\partial x_j)_{i,j}$ respectively. The risk $R(u)$ is expressed as follows:

$$\begin{aligned} R(u) = E_{\bar{P}^0}\Big[\int_0^T &\{|u_t|^2\Delta(t, X_t)+\mathrm{trace}(LD)\Delta(t, X_t) \\ &+|\sqrt{L}(Y_t+S^*X_t)|^2\Delta(t, X_t)+2(Y_t+S^*X_t)\cdot LC^{-1}\Delta_x(t, X_t) \\ &+\mathrm{trace}(C^{*-1}LC^{-1}\Delta_{xx}(t, X_t))\}\,dt\Big]. \end{aligned} \tag{2.13}$$

Therefore the original problem of control by incomplete data is reduced to problem (2.11), (2.12) and (2.13) of control by complete data.

3. Proof of the main theorem

The Bellmann equation corresponding to our problem (2.11), (2.12) and (2.13) is given as follows:

$$\begin{aligned} \min_u\Big\{\frac{\partial\Phi}{\partial t}&+\frac{1}{2}\mathrm{trace}(B\Phi_{xx})+(Fu-GC^*x)\cdot\Phi_y+|u|^2\Delta \\ &+\mathrm{trace}(LD)\Delta+|\sqrt{L}(y+S^*x)|^2\Delta+2(y+S^*x)\cdot LC^{-1}\Delta_x \end{aligned} \tag{3.1}$$

$$+\mathrm{trace}(C^{*-1}LC^{-1}\Delta_{xx})\Big\} = 0, \qquad \Phi(T, x, y)\equiv 0\,,$$

where $B(t)=(HC^{-1})^*(HC^{-1})(t)$ and $\Phi_{xx}=(\partial^2\Phi/\partial x_i\partial x_j)$, $\Phi_x=(\partial\Phi/\partial x_j)$, $\Phi_y=(\partial\Phi/\partial y_j)$. Obviously the minimum in (3.1) is attained by

$$u = \Gamma(t, x, y)\equiv -\frac{1}{2}\frac{1}{\Delta(t, x)}F^*(t)\Phi_y(t, x, y)\,. \tag{3.2}$$

Equation (3.1) is non-linear and degenerate, but it can be solved in the concrete. Putting

$$\Phi(t, x, y) = y\cdot U(t, x)y+2y\cdot V(t, x)+W(t, x)\,, \qquad U(t, x)\text{: symmetric matrix,} \tag{3.3}$$

equation (3.1) is reduced to the following three equations.

$$\frac{\partial}{\partial t}U+\frac{1}{2}\mathrm{trace}(BU_{xx})-\frac{1}{\Delta}UFF^*U+\Delta L = 0,\quad U(T, x) = 0\,. \tag{3.4}$$

$$\frac{\partial}{\partial t}V+\frac{1}{2}\mathrm{trace}(BV_{xx})-\frac{1}{\Delta}UFF^*V -UGC^*x+\Delta LS^*x+LC^{-1}\Delta_x = 0, \qquad V(T, x) = 0\,. \tag{3.5}$$

$$\frac{\partial}{\partial t}W+\frac{1}{2}\mathrm{trace}(BW_{xx})-2x\cdot CGV-\frac{1}{\Delta}V\cdot FF^*V+\mathrm{trace}(LD)\Delta +x\cdot SLS^*x\Delta+2x\cdot SLC^{-1}\Delta_x+\mathrm{trace}(C^{*-1}LC^{-1}\Delta_{xx}) = 0\,, \qquad W(T, x) = 0\,. \tag{3.6}$$

Since $\partial\Delta/\partial t+1/2\,\mathrm{trace}(B\Delta_{xx})=0$, the solution of equation (3.4) is given as follows:

$$U(t, x) = \Delta(t, x)A(t)\,,$$

where $A(t)$ is the solution of (1.7), and the solution of equation (3.5) is given as follows:

$$V(t, x) = \Delta(t, x)A(t)S^*(t)x+A(t)C(t)^{-1}\Delta_x(t, x)\,.$$

It is easy to show that equation (3.6) turns into the equation

$$\frac{\partial W}{\partial s}+\frac{1}{2}\mathrm{trace}(BW_{xx})-2x\cdot\left[\frac{d}{ds}(SA)\right]C^{-1}\Delta_x+\mathrm{trace}(C^{*-1}LC^{-1}\Delta_{xx}) -x\cdot\left[\frac{d}{ds}(SAS^*)\right]x\Delta+\mathrm{trace}(LD)\Delta-|F^*AC^{-1}\Delta_x|^2\Delta^{-1} = 0\,, \qquad W(T, x) = 0\,. \tag{3.7}$$

Lemma 3. *Condition* (1.9) *implies that, for* $1\leqq i, j\leqq m$,

$$\sup_{t,x}\frac{|(\partial/\partial x_i)\Delta(t, x)|}{(1+|x|)\Delta(t, x)}<\infty,\ \sup_{t,x}\frac{|(\partial^2/\partial x_i\partial x_j)\Delta(t, x)|}{(1+|x|^2)\Delta(t, x)}<\infty\,. \tag{3.8}$$

Proof. Put

$$q_t(d\theta)=\exp\left(\frac{\varepsilon}{2}|\theta|^2-\frac{1}{2}\theta\cdot B_0^t\theta\right)p(d\theta).$$

Then $q_t(d\theta)$ is a finite measure. It is obvious that there is a constant $d>0$ such that

$$\int_{|\theta|\leqq d}q_t(d\theta)\geqq\frac{1}{2}\int q_t(d\theta)\qquad\text{for all } t.$$

For a certain constant $c>0$,

$$\frac{\int p(d\theta)|\theta|e^{x\cdot\theta-\theta\cdot B_0^t\theta/2}}{\Delta(t,x)}=\frac{\int q_t(d\theta)|\theta|e^{-\varepsilon|\theta-\varepsilon^{-1}x|^2/2}}{\int q_t(d\theta)e^{-\varepsilon|\theta-\varepsilon^{-1}x|^2/2}}$$

$$\leqq\varepsilon^{-1}|x|+\frac{\int q_t(d\eta+\varepsilon^{-1}x)|\eta|e^{-\varepsilon|\eta|^2/2}}{\int q_t(d\eta+\varepsilon^{-1}x)e^{-\varepsilon|\eta|^2/2}}$$

$$\leqq\text{const.}(1+|x|)+\frac{\int q_t(d\eta)}{\int_{|\eta|\leqq d}q_t(d\eta)}\times\frac{\sup\{|\eta|e^{-\varepsilon|\eta|^2/2};|\eta|>c(1+|x|)\}}{\inf\{e^{-\varepsilon|\eta|^2/2};|\eta|\leqq d+\varepsilon^{-1}|x|\}}.$$

Since there exists a constant c such that

$$\sup_{|\eta|>c(1+|x|)}|\eta|e^{-\varepsilon|\eta|^2/2}\leqq e^{-\varepsilon(d+\varepsilon^{-1}|x|)^2/2}\qquad\text{for all } x,$$

the former of (3.8) follows immediately. The latter of (3.8) can be proved similarly. Q.E.D.

Put

$$\text{(3.9)}\qquad \tilde{W}(s,x)=-\int_s^T dt\int\Omega(s,t,y-x)\left|F^*AC^{-1}\frac{\Delta_y}{\Delta}(t,y)\right|^2\Delta(t,y)dy$$

$$\equiv-\int_s^T dt\int p(d\theta)\Big\{e^{x\cdot\theta-\theta\cdot B_0^s\theta/2}$$

$$\times\int\left|F^*AC^{-1}\frac{\Delta_y}{\Delta}(t,y)\right|^2\Omega(s,t,y-x-B_s^t\theta)dy\Big\}.$$

From Lemma 3, we see that there is a continuous function $\rho_1(t)\geqq 0$ such that $\rho_1(T)=0$ and $|\tilde{W}(s,x)|\leqq\rho_1(s)\Delta(s,x)(1+|x|^2)$. It is a routine work to show that the function $\tilde{W}(s,x)$ is continuously differentiable in s, continuously differentiable up to second order in $x_i(1\leqq i\leqq m)$ and it is a solution of the equation

$$\frac{\partial\tilde{W}}{\partial s}+\frac{1}{2}\operatorname{trace}(B\tilde{W}_{xx})-\left|F^*AC^{-1}\frac{\Delta_x}{\Delta}\right|^2\Delta=0,\ \ \tilde{W}(T,x)=0.$$

Using these properties, it is easy to show that the solution of equation (3.7) is given as follows:

$$(3.10)\qquad \begin{aligned} W(s, x) &= \tilde{W}(s, x)+x\cdot SAS^*x\Delta(s, x)+2x\cdot SAC^{-1}\Delta_x(s, x)\\ &\quad+\int_s^T \mathrm{trace}(LD+ADH^*HD)dt\cdot\Delta(s, x)\\ &\quad+\mathrm{trace}\left[\left(\int_s^T C^{*-1}(L+2ADH^*H)C^{-1}dt\right)\Delta_{xx}(s, x)\right] \end{aligned}$$

and that there exists a continuous function $\rho_2(t)\geqq 0$ such that

$$(3.11)\qquad |W(s, x)|\leqq\rho_2(s)\Delta(s, x)(1+|x|^2),\quad \rho_2(T)=0\,.$$

Lemma 4. *For each $u\in\mathcal{U}$, $R(u)\geqq\Phi(0, 0, 0)$.*

Proof. Let $T_N=\inf\{t;|X_t|^2+|Y_t|^2>N\}$, where (X_t, Y_t) is the solution of (2.11) and (2.12). By Ito's formula for stochastic integrals (see Gikhman-Skorokhod [3]),

$$\begin{aligned} &E_{\bar{P}^0}[\Phi(T_N, X_{T_N}, Y_{T_N})]-\Phi(0, 0, 0)\\ &= E_{\bar{P}^0}\left[\int_0^{T_N}\left\{\frac{1}{2}\mathrm{trace}(B(t)\Phi_{xx}(t, X_t, Y_t))+\frac{\partial}{\partial t}\Phi(t, X_t, Y_t)\right.\right.\\ &\qquad\left.\left.+(F(t)u_t(\zeta)-G(t)C^*(t)Y_t)\cdot\Phi_y(t, X_t, Y_t)\right\}dt\right]\\ &\geqq -R(u)\,. \end{aligned}$$

It follows from (3.11) that there is a continuous function $\rho(t)\geqq 0$ such that $\rho(T)=0$ and $|\Phi(t, x, y)|\leqq\rho(t)(1+|x|^2+|y|^2)\Delta(t, x)$. Therefore

$$\begin{aligned} &E_{\bar{P}^0}[|\Phi(T_N, X_{T_N}, Y_{T_N})|]\leqq E_{\bar{P}^0}[\rho(T_N)\Delta(T_N, X_{T_N})(1+|X_{T_N}|^2+|Y_{T_N}|^2)]\\ &=\int p(d\theta)E_{\bar{P}^\theta}[\rho(T_N)(1+|X_{T_N}|^2+|Y_{T_N}|^2)]\\ &=E_P[\rho(T_N)(1+|X_{T_N}|^2+|Y_{T_N}|^2)]\,. \end{aligned}$$

Since $d\nu_t^0=d\nu_t^\theta+HC^{-1}\theta dt$,

$$\begin{aligned} E_{\bar{P}^\theta}[\sup_t|X_t|^2]&\leqq 2E_{\bar{P}^\theta}\left[\sup_t\left|\int_0^t(HC^{-1})^*d\nu_s^\theta\right|^2+\sup_t|B_0^t\theta|^2\right]\\ &\leqq \mathrm{const.}(1+|\theta|^2)\,. \end{aligned}$$

Therefore $E_P[\sup_t|X_t|^2]<\infty$. And since $E_P[\sup_t|\zeta(t)|^2]<\infty$,

$$E_P[\sup_t|Y_t|^2]\leqq\mathrm{const.}\ E_P[\sup_t|u_t(\zeta)|^2+\sup_t|X_t|^2]<\infty\,.$$

This implies that $P[T_N<T]\to 0$. Thus

$$E_P[\rho(T_N)(1+|X_{T_N}|^2+|Y_{T_N}|^2)]\to 0\quad\text{as } N\to\infty\,.$$

Q.E.D.

We shall show that there is a control $\hat{u}\in\mathcal{U}$ such that $R(\hat{u})=\Phi(0, 0, 0)$. Since the function

$$\Gamma(t, x, y) = -F^*A\Big(y+S^*x+C^{-1}\frac{\Delta_x}{\Delta}\Big) \tag{3.12}$$

is locally Lipshitz continuous in (x, y) and since

$$\sup_{t,x,y} \frac{\Gamma(t, x, y)}{1+|x|+|y|}<\infty\,, \tag{3.13}$$

the equation

$$\begin{aligned} dX_t &= (HC^{-1})^*(d\zeta_t-H(Y_t+S^*X_t)dt)\,, \\ dY_t &= (F\Gamma(t, X_t, Y_t)-GC^*X_t)dt, \quad (X_0, Y_0) = (0, 0) \end{aligned} \tag{3.14}$$

(see (2.5), (2.11), (2.12) and (3.2)) has a unique solution (in the path-wise sense). Let $(\hat{X}_t(\zeta), \hat{Y}_t(\zeta))$ denote the solution and put

$$\hat{u}_t(\zeta) = \Gamma(t, \hat{X}_t(\zeta), \hat{Y}_t(\zeta))\,. \tag{3.15}$$

Lemma 5. *The process $\hat{u}_t(\zeta)$ is an admissible control and $R(\hat{u})=\Phi(0,0,0)$.*

Proof. Obviously $\hat{u}_t(\zeta)$ is non-anticipating with respect to the σ-fields $(\mathcal{F}_t^\zeta)$. From (3.13) we see that

$$(1+|\hat{X}_t|+|\hat{Y}_t|)\leqq\Big(1+\Big|\int_0^t(HC^{-1})^*d\zeta_s\Big|\Big)+K\int_0^t(1+|\hat{X}_s|+|\hat{Y}_s|)ds$$

for some constant K. It follows immediately that

$$(1+|\hat{X}_t|+|\hat{Y}_t|)\leqq\sup_{s\leqq t}\Big(1+\Big|\int_0^s(HC^{-1})^*(\sigma)d\zeta_\sigma\Big|\Big)e^{Kt}\,.$$

Since $(HC^{-1})(t)$ is continuously differentiable,

$$\begin{aligned} \Big|\int_0^t(HC^{-1})^*(\sigma)d\zeta_\sigma\Big| &= \Big|(HC^{-1})^*(t)\zeta_t-\int_0^t\Big[\frac{d}{d\sigma}(HC^{-1})^*(\sigma)\Big]\zeta_\sigma d\sigma\Big| \\ &\leqq \text{const.}\sup_{s\leqq t}|\zeta(s)|\,. \end{aligned}$$

Therefore

$$|\hat{u}_t(\zeta)|\leqq\text{const.}\,(1+|\hat{X}_t|+|\hat{Y}_t|)\leqq\text{const.}\,(1+\sup_{s\leqq t}|\zeta(s)|)\,.$$

this means that $\hat{u}_t\in\mathcal{U}$. The proof of the fact $R(\hat{u})=\Phi(0, 0, 0)$ is similar to the proof of Lemma 4. Q.E.D.

Lemma 4 and 5 mean that the control $\hat{u}=\hat{u}_t(\zeta)$ is optimal. Since $\Phi(0, 0, 0)=W(0, 0)$, equality (1.10) follows immediately. Thus Theorem 1 is completely proved. Now, we shall give a remark. It is obvious from the proof of Lemma

4 that if $R(u)=\Phi(0, 0, 0)$, then

$$\int_0^T \bar{P}^0[u_t(\zeta)\mp\hat{u}_t(\zeta)]dt = 0\,.$$

This implies that

$$\int_0^T P[u_t\mp\hat{u}_t]dt = \int p(d\theta)\int_0^T \bar{P}^\theta[u_t\mp\hat{u}_t]dt = 0$$

because $\bar{P}^0$ and $\bar{P}^\theta$ are mutually absolutely continuous. Therefore the optimal control is uniquely determined up to measure zero with rspect to $dt\times dP$.

4. Maximum of the risk corresponding to the optimal control

Let $\mathcal{P}[M]$ be the class of probabilities $p(d\theta)$ on R^m such that

a) there is a constant $\varepsilon>0$ such that $\int e^{\varepsilon|\theta|^2/2}p(d\theta)<\infty$,

b) $\int\theta_i\theta_j p(d\theta)=M_{ij}$, where $M=(M_{ij})$ is a given positive matrix.

The purpose of this section is to show the following fact.

Theorem 2. i) *The maximum of the risk $R(\hat{u})$ under the condition that the initial distribution $p(d\theta)$ belongs to the class $\mathcal{P}[M]$ is attained by the normal distribution with parameter* $(0, M)$:

$$p(d\theta) = [(2\pi)^m \det M]^{-1/2}\exp\left[-\frac{1}{2}\theta\cdot M^{-1}\theta\right]d\theta\,. \tag{4.1}$$

ii) *If there is a time t_0 such that $L(t_0)>0$ and $FF^*(t_0)>0$, then the maximum is attained only by distribution* (4.1).

Proof. Put, for $t>0$,

$$\Lambda(t, x) = \int p(d\theta)\Omega(0, t, x-B_0^t\theta)\,. \tag{4.2}$$

Then we have

$$\int\Lambda(t, x)dx = 1,\ \int x_i x_j\Lambda(t, x)dx = (B_0^t+B_0^t M B_0^t)_{ij}\,, \tag{4.3}$$

$$\Delta_x(t, x)/\Delta(t, x) = \Lambda_x(t, x)/\Lambda(t, x)+(B_0^t)^{-1}x\,, \tag{4.4}$$

where $\Lambda_x=(\partial\Lambda/\partial x_j)$. Using (4.3) and (4.4), we obtain that

$$R(\hat{u}) = \int_0^T\{\text{trace}\,[(L+ADH^*H)D+(L+2ADH^*H)C^{-1}MC^{*-1}$$
$$-\tilde{F}\tilde{F}^*((B_0^t)^{-1}+M)+2\tilde{F}\tilde{F}^*(B_0^t)^{-1}]-\int|\tilde{F}^*\Lambda_x|^2\Lambda^{-1}dx\}\,dt\,,$$

where $\tilde{F}=C^{*-1}AF$. Then assertion i) follows immediately from the following lemma.

Lemma 6. *If $p\in\mathcal{P}[M]$, then*

$$\int|\tilde{F}^*\Lambda_x|^2\Lambda^{-1}dx\geqq \operatorname{trace}(\tilde{F}\tilde{F}^*(B_0^t+B_0^tMB_0^t)^{-1}). \tag{4.5}$$

And the equality holds if $\Lambda(t, x)dx$ is the normal distribution with parameter $(0, B_0^t+B_0^tMB_0^t)$. Further, if $\tilde{F}\tilde{F}^*$ is invertible, then the equality holds only for the same normal distribution.

Proof of Lemma 6. Put $N(t)=B_0^t+B_0^tMB_0^t$. Without loss of generality, we can suppose that $\operatorname{trace}(\tilde{F}\tilde{F}^*N^{-1})>0$. Using the fact $\int(\partial\Lambda/\partial x_j)x_i dx=\delta_{ij}$, we have

$$\int\tilde{F}^*\Lambda_x\cdot\tilde{F}^*N^{-1}x\,dx=-\operatorname{trace}(\tilde{F}\tilde{F}^*N^{-1}).$$

On the other hand, by the Schwarz inequality,

$$\left(\int\tilde{F}^*\Lambda_x\cdot\tilde{F}^*N^{-1}x\,dx\right)^2=\left(\int\left(\tilde{F}^*\frac{\Lambda_x}{\Lambda}\right)\cdot(\tilde{F}^*N^{-1}x)\Lambda dx\right)^2$$

$$\leqq\left(\int\left|\tilde{F}^*\frac{\Lambda_x}{\Lambda}\right|^2\Lambda dx\right)\left(\int|\tilde{F}^*N^{-1}x|^2\Lambda dx\right)$$

$$=\left(\int|\tilde{F}^*\Lambda_x|^2\Lambda^{-1}dx\right)\operatorname{trace}\left(\tilde{F}\tilde{F}^*N^{-1}\right).$$

And the equality holds only if vector valued functions $\tilde{F}^*\Lambda_x/\Lambda$ and $\tilde{F}^*N^{-1}x$ are linearly dependent. In the case when $\tilde{F}\tilde{F}^*$ is invertible, the equality holds if and only if $\Lambda_x/\Lambda=kN^{-1}x$ for a certain constant k which may depend on t. Since $\int\Lambda dx=1$ and $\int x_ix_j\Lambda dx=N_{ij}$, the equality $\Lambda_x/\Lambda=kN^{-1}x$ implies that

$$\Lambda(t, x)=[(2\pi)^m\det N(t)]^{-1/2}\exp\left[-\frac{1}{2}x\cdot N(t)^{-1}x\right]. \quad \text{Q.E.D.} \tag{4.6}$$

We shall prove ii) of Theorem 2. If there is a time t_0 such that $LFF^*(t_0)>0$, there is an open interval $\mathcal{J}$ such that $LFF^*(t)>0$ for each $t\in\mathcal{J}$. It is obvious from equation (1.7) that $A(t)>0$ for each $t\in\mathcal{J}$. Therefore the matrix $\tilde{F}\tilde{F}^*(t)$ is invertible for all $t\in\mathcal{J}$. Thus, if the maximum of $R(\hat{u})$ is attained by $p\in\mathcal{P}[M]$ and if $\Lambda(t, x)$ is given by (4.2), equality (4.6) must hold for each $t\in\mathcal{J}$. By the Fourier transformation, we have

$$\left(\int e^{i\eta\cdot B_0^t\theta}p(d\theta)\right)e^{-\eta\cdot B_0^t\eta/2},$$

$$=\int e^{i\eta\cdot x}\Lambda(t, x)dx=e^{-(B_0^t\eta)\cdot(1+MB_0^t)\eta/2},$$

for each $t\in\mathcal{J}$. Since $\eta\in R^m$ is arbitrary, we have

$$\int e^{i\eta\cdot\theta}p(d\theta)=e^{-\eta\cdot M\eta/2},$$

which implies (4.1). Q.E.D.

OSAKA CITY UNIVERSITY

References

[1] R.S. Buchy and P.D. Joseph: Filtering for stochastic processes with applications to guidance, Interscience Publishers, 1968.

[2] M. Fujisaki, G. Kallianpur and H. Kunita: *Stochastic differential equations for the non-linear filtering problem,* Osaka J. Math. **9** (1972), 19–40.

[3] I.G. Gikhman and A.V. Skorokhod: Introduction to the theory of random processes, 1965.

[4] I.V. Girzanov: *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures,* Th. Prob. Appl. **5** (1960), 285–301.

[5] P.D. Joseph and J.T.Tou: *On linear control theory,* AIEE Trans. Application and Industry **80** (1961), 193–196.

[6] R.Sh. Lipcer and A.N. Shirjaev: *The absolute continuity with respect to Wiener measures that correspond to processes of diffusion type* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 842–889.

[7] W.M. Wonham: *On the separation theorem of stochastic control,* SIAM J. Control **6** (1968), 312–326.