203. On the Product of M-Spaces. I

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1. Introduction. In the present paper all spaces are assumed to be Hausdorff. In his previous paper [2], K. Morita has introduced the notion of M-spaces. A space X is called an M-space if there exists a normal sequence $\{\mathfrak{U}_i \mid i=1, 2, \cdots\}$ of open coverings of X satisfying the condition (M) below:

(M) If $\{K_i\}$ is a sequence of non-empty subsets of X such that $K_{i+1} \subset K_i$, $K_i \subset \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for some fixed point x_0 of X, then $\cap \bar{K}_i \neq \phi$.

As is easily verified, Condition (M) is equivalent to the condition (M_0^*) below:

(M_0) If $\{x_i\}$ is a sequence of points of X such that $x_i \in \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for some fixed point x_0 of X, then $\{x_i\}$ has an accumulation point.

Hereafter we use Condition (M_0) in place of Condition (M).

As for the product $X \times Y$ of two M-spaces X and Y, it seems to be unknown whether $X \times Y$ is also an M-space or not. We can give an affirmative answer for this problem in the following cases:

(a) X satisfies the first axiom of countability.

(b) X is locally compact.

(c) X is paracompact.

The purpose of our papers I and II is to introduce the notion of the spaces belonging to the class $\mathfrak{C}$ and to prove a more general theorem (cf. Theorem 1.1 in II) as follows: If a space X belongs to the class $\mathfrak{C}$, then the product $X \times Y$ is also an M-space for any M-space Y. We denote by $\mathfrak{C}$ the class of all spaces X such that there exists a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying the condition (*) below:

(*) If $\{x_i\}$ is a sequence of points of X such that $x_i \in \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for some fixed point x_0 of X, then there exist a subsequence $\{x_{i(n)} \mid n=1, 2, \cdots\}$ of $\{x_i\}$ which has the compact closure.

The class $\mathfrak{C}$ contains all M-spaces satisfying one of conditions (a), (b), and (c), and further the spaces belonging to $\mathfrak{C}$ have the following properties.

(i) If $f: X \to Y$ is a quasi-perfect map (i.e., a continuous closed surjective map such that $f^{-1}(y)$ is countably compact) of a space X belonging to $\mathfrak{C}$ onto a normal space Y, then Y belongs to $\mathfrak{C}$ (cf. Theorem 2.4).

(ii) If X_i, $i=1, 2, \cdots$, belong to $\mathfrak{C}$, then the product $\prod_{i=1}^{\infty} X_i$ belongs to $\mathfrak{C}$ (cf. Theorem 1.3 in II).

2. The spaces belonging to $\mathfrak{C}$.

Theorem 2.1. *If a space X belongs to $\mathfrak{C}$, then X is an M-space.*

Proof. Since X belongs to $\mathfrak{C}$, there exists a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying Condition $(*)$. Let $\{x_i\}$ be a sequence of points of X such that $x_i \in \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for some fixed point x_0 of X. We shall prove that $\{x_i\}$ has an accumulation point. For this purpose we can assume without loss of generality that $\{x_i\}$ contains a subsequence $\{x_{i(n)}\}$ consisting of distinct points. Since $x_{i(n)} \in \mathrm{St}(x_0, \mathfrak{U}_n)$ for every n, by Condition $(*)$ there exists a subsequence $\{x'_n\}$ of $\{x_{i(n)}\}$ which has the compact closure. Consequently, $\{x'_n\}$ has an accumulation point. If otherwise, $\{x'_n\}$ must be discrete and closed. Since $\{x'_n\}$ consists of distinct points, it cannot be compact. Therefore $\{\mathfrak{U}_i\}$ satisfies Condition (M_0). This completes the proof.

Remark. In the proof of Theorem 2.1, it is sufficient to assume that the closure of the subsequence $\{x'_n\}$ of $\{x_{i(n)}\}$ is countably compact.

The converse of Theorem 2.1 is not valid in general, as is shown in Theorem 2.3.

Theorem 2.2. *If an M-space X satisfies one of the following conditions, then X belongs to $\mathfrak{C}$:*

(a) *X satisfies the first axiom of countability.*

(b) *X is locally compact.*

(c) *X is paracompact.*

Proof. Since X is an M-space, there exists a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying Condition (M_0). Let $\{x_i\}$ be a sequence of points of X such that $x_i \in \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for some fixed point x_0 of X. Then, by Condition (M_0), $\{x_i\}$ has an accumulation point x'.

(i) If X satisfies the first axiom of countability, then $\{x_i\}$ contains a subsequence $\{x_{i(n)}\}$ which converges to x'. Clearly the closure of $\{x_{i(n)}\}$ is compact. Thus X belongs to $\mathfrak{C}$.

(ii) If X is locally compact, then there exists a neighborhood $U(x')$ of x' which has the compact closure. Since $U(x')$ contains infinite number of elements of $\{x_i\}$, we denote them by $\{x_{i(n)}\}$. Then $\{x_{i(n)}\}$ has clearly the compact closure. Thus X belongs to $\mathfrak{C}$.

(iii) If X is paracompact, any countably compact subset is compact. Since $\{x_i\}$ has accumulation points in $\cap \mathrm{St}(x_0, \mathfrak{U}_i)$ and nowhere else, the closure of $\{x_i\}$ is countably compact, and hence it is compact. Thus X belongs to $\mathfrak{C}$.

Remark. A space X belonging to $\mathfrak{C}$ does not necessarily satisfy the first axiom of countability, because there is a compact space which does not satisfy the first axiom of countability. Further a space belonging to $\mathfrak{C}$ is not necessarily locally compact, because there is a metric space which is not locally compact.

Theorem 2.3. *There exists an M-space which does not belong to $\mathfrak{C}$, and further there exists a space which belongs to $\mathfrak{C}$ but is not a paracompact M-space.*

To prove Theorem 2.3, we mention two examples of the spaces satisfying the required properties.

Example 1. (*An M-space which does not belong to* $\mathfrak{C}$). We show that a countably compact space A_1, which was constructed by J. Novák [4], satisfies the required conditions. Let $\beta(N)$ be the Čech-compactification of the set N of natural numbers. Then by [4] there exist two subsets P and Q of $\beta(N)$ such that $P \cup Q = \beta(N) - N$, $P \cap Q = \phi$ and that $\bar{S} \cap P \neq \phi$ and $\bar{S} \cap Q \neq \phi$ for any countable infinite subset S of $\beta(N)$. Let us put $A_1 = P \cup N$. Then the subspace A_1 of $\beta(N)$ is countably compact, and hence it is an M-space.

Now we shall prove that the space A_1 does not belong to $\mathfrak{C}$. Let S be any countable infinite subset of A_1. Then the set S has no compact closure in A_1. If otherwise, then the closure of S in A_1 is compact, and hence it is compact in $\beta(N)$, too. But, as the construction of the sets P and Q shows, the closure of S in $\beta(N)$ contains a point of the set $Q = \beta(N) - A_1$. This is a contradiction. Let $\{\mathfrak{U}_i\}$ be any normal sequence of open coverings of A_1, and let $x_0 \in P$. Then $\mathrm{St}(x_0, \mathfrak{U}_i)$ contains infinite points of N for each i, and hence we can choose a sequence $\{n_i\}$ of distinct points of N such that $n_i \in \mathrm{St}(x_0, \mathfrak{U}_i)$. As is shown above, the closure of $\{n_i\}$ in A_1 is not compact. Thus the space A_1 does not belong to $\mathfrak{C}$.

Example 2. (*A space X which belongs to $\mathfrak{C}$ but is not a paracompact M-space*). Let X be the space $\{\alpha \mid \alpha < \Omega\}$ of ordinals with the order topology, where Ω is the first uncountable ordinal. Since X is countably compact and satisfies the first axiom of countability, it belongs to $\mathfrak{C}$. But it is not paracompact.

Theorem 2.4. *Let $f: X \to Y$ be a quasi-perfect map. If X belongs to $\mathfrak{C}$ and if X or Y is normal, then Y belongs to $\mathfrak{C}$. If $f: X \to Y$ is perfect and if Y belongs to $\mathfrak{C}$, then X belongs to $\mathfrak{C}$.*

The proof of Theorem 2.4 is performed by the similar way as in the proof of the first part of [3, Theorem 2.2]. For this purpose we introduce a class $\mathfrak{C}^*$ of the spaces. We denote by $\mathfrak{C}^*$ the class of all spaces X such that there exists a sequence $\{\mathfrak{F}_i \mid i=1, 2, \cdots\}$ of locally finite closed coverings of X satisfying Condition (*). As for the spaces belonging to $\mathfrak{C}^*$, the following lemmas are valid.

Lemma 2.5. *Let $f: X \to Y$ be a quasi-perfect map. If X belongs to $\mathfrak{C}^*$, then Y also belongs to $\mathfrak{C}^*$.*

Lemma 2.6. *If X belongs to $\mathfrak{C}$, then X belongs to $\mathfrak{C}^*$ and satisfies the property* (C) *below:*

(C) *For any locally finite collection $\{F_\lambda\}$ of closed sets of X there exists a locally finite collection $\{G_\lambda\}$ of open sets of X such that $F_\lambda \subset G_\lambda$ for each λ.*

In case X is normal, the converse is true.

Since Lemma 2.5 can be proved by the similar way as in the proof of [1, Theorem 2.3] and Lemma 2.6 can be proved by the similar way as in the proof of [3, Theorem 1.1], we omitt the proof.

Proof of Theorem 2.4. If X is normal, so is Y. Further by [3, Lemma 2.1], if X has Property (C), so has Y. Hence the first part follows from Lemmas 2.5 and 2.6. The second part follows from the fact that, if $f: X \to Y$ is perfect, then $f^{-1}(C)$ is compact for every compact set C of Y. Thus the proof is completed.

By the same way as in the proof of [3, Theorem 3.1], it follows from Theorem 2.4 that, if $\{A_\lambda \mid \lambda \in \Lambda\}$ is a locally finite closed covering of a space X and if each A_λ is a normal space belonging to $\mathfrak{C}$, then X is a normal space belonging to $\mathfrak{C}$. In [3], K. Morita shows by an example that a space Y which is the union of closed subspaces C_i, $i=1, 2$, each of which is an M-space, is not an M-space in general. In this example, each C_i is a locally compact M-space, and hence belongs to $\mathfrak{C}$, while Y does not belong to $\mathfrak{C}$.

Finally we note that a space belonging to $\mathfrak{C}^*$ does not belong to $\mathfrak{C}$ in general. This is an immediate consequence of the following result obtained by K. Morita [3]: There is a perfect map $f: X \to Y$ such that X is a locally compact M-space but Y is not an M-space. In fact, Y belongs to $\mathfrak{C}^*$ by Lemma 2.5 but does not belong to $\mathfrak{C}$.

References

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