[4] K. Ramachandra, *Application of Baker's theory to two problems considered by Erdös and Selfridge*, J. Indian Math. Soc. 37 (1973), pp. 25–34.
[5] — *Lectures on Transcendental Numbers*, Ramanujan Institute, Madras 1969.
[6] — *Contributions to the theory of transcendental numbers (II)*, Acta Arith. 14 (1968), pp. 73–88.
[7] T. N. Shorey, *On gaps between numbers with a large prime factor II*, ibid. 25 (1974), pp. 365–373.
[8] H. M. Stark, *Further advances in the theory of linear forms in logarithms*, Diophantine Approximations and its Applications, New York 1973, pp. 255–294.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Bombay, India

Received on 18. 9. 1974 (619)

ACTA ARITHMETICA
XXX (1976)

On the product of the conjugates outside the unit circle of an algebraic integer

by

A. BAZYLEWICZ (Warszawa)

The aim of this paper is to extend some results of A. Schinzel [4] and to make them more precise.

Let K be a number field of degree $|K|$, let

$$P(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_n$$

be a polynomial over K with the content $C(P) = (p_0, \ldots, p_n)$, let G be the set of all isomorphic injections of K into the complex field C and, for $\sigma \in G$, let

$$\sigma P(z) = \sigma p_0 z^n + \ldots + \sigma p_n = \sigma p_0 \prod_{i=1}^{n} (z - a_{\sigma i}).$$

Generalizing an argument of Smyth [5] concerning the fundamental case $K = Q$, Schinzel proved that if K is totally real, $P(z)$ is non-reciprocal, p_i are integers, $p_0 = 1$ and $p_n \neq 0$, then

$$\max_{\sigma \in G} \prod_{|a_{\sigma j}|>1} |a_{\sigma j}| \geqslant \theta_0, \tag{1}$$

where θ_0 is the real root of the equation $\theta^3 - \theta - 1 = 0$. We extend this in the following manner.

THEOREM 1. *Let K be a totally complex quadratic extension of a totally real field and $\sqrt{-3} \notin K$.*

If $P(z) \in K[z]$ is a monic polynomial with integer coefficients, $P(0) \neq 0$, $z^n \overline{P}(z^{-1}) \neq \text{const} P(z)$, then (1) holds.

If $\sqrt{-3} \in K$ (1) needs not be satisfied, but $\Lambda \geqslant |\theta_1|$ where θ_1 is that root of the equation

$$\theta^2 + \frac{-1+\sqrt{-3}}{2}\,\theta - 1 = 0$$

which is greater in absolute value.

For K being a totally complex quadratic extension of a totally real field Schinzel considered the product

$$\Pi = \prod_{\sigma \in G} \prod_{|\alpha_{\sigma j}|>1} |\alpha_{\sigma j}|$$

and proved that if $p_n \neq 0$ and $|p_n| \neq |p_0|$, then

$$\Pi \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{|K|/2} \left(N_{K/Q} \frac{C(P)}{(p_0)}\right)^{\frac{1}{2}+\frac{1}{\sqrt{5}}} \left(N_{K/Q} \frac{(p_n)}{C(P)}\right)^{\frac{1}{2}-\frac{1}{\sqrt{5}}} \tag{2}$$

with the equality possible only if $\sqrt{5} \in K$, $C(P) = (p_0)$,

$$\left|\frac{p_n}{p_0}\right| = \frac{\pm 1+\sqrt{5}}{2}.$$

He made a conjecture about the possible form of the polynomial P for which the equality in (2) is attained.

We prove this conjecture as:

THEOREM 2. *The equality in* (2) *is attained if and only if*

$$P(z) = p_0 \left(z^k + \varepsilon_1 \frac{1 \pm \sqrt{5}}{2}\right) \prod_{i=2}^{n-k+1} (z - \varepsilon_i) \tag{3}$$

where ε_i are roots of unity.

This theorem is an easy consequence of the following:

THEOREM 3. *Let K be a totally complex quadratic extension of a totally real field.*

If $P(z) \in K[z]$ is a monic polynomial with integer coefficients, $P(0) = \varepsilon \frac{1 \pm \sqrt{5}}{2}$ and

$$|\alpha_{\sigma j}| \geqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \pm\sqrt{5} \quad (j = 1, 2, \ldots, n),$$
$$|\alpha_{\sigma j}| \leqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \mp\sqrt{5} \quad (j = 1, 2, \ldots, n),$$

then (3) *holds.*

The proofs are based on several lemmata.

LEMMA 1. *Let $f(z) = \sum_i e_i z^i$ be a function holomorphic in an open disc containing $|z| \leqslant 1$ and satisfying $|f(z)| \leqslant 1$ for $|z| = 1$. Then*

$$|e_i| \leqslant 1 - |e_0|^2 \quad (i = 1, 2, \ldots), \tag{4}$$

$$\left|e_{2i} + \frac{e_i^2 \bar{e}_0}{1-|e_0|^2}\right| \leqslant 1 - |e_0|^2 - \frac{|e_i|^2}{1-|e_0|^2} \quad (i = 1, 2, \ldots). \tag{5}$$

Proof (due to A. Schinzel). Inequality (4) is proved in [4]. In order to prove (5) let us observe, following Smyth [5], that for all β_0, β_1

$$\int_{|z|=1} |f(z)|^2 |\beta_0 + \beta_1 z^i + z^{2i}|^2 dz \leqslant \int_{|z|=1} |\beta_0 + \beta_1 z^i + z^{2i}| dz;$$

thus

$$|e_0\beta_0|^2 + |e_0\beta_1 + e_1\beta_0|^2 + |e_0 + e_i\beta_1 + e_{2i}\beta_0|^2 \leqslant 1 + |\beta_0|^2 + |\beta_1|^2,$$

and setting

$$\beta_0 = pe_0^{-1}; \quad \beta_1 = \frac{pe_i}{e_0(\delta - e_0)}$$

where $|p| = |\delta| = 1$, we obtain

$$\left|e_0 + \frac{e_i^2 p}{(\delta - e_0)e_0} + \frac{pe_{2i}}{e_0}\right| \leqslant |e_0|^{-1}.$$

Hence

$$\left|e_0^2 p^{-1} + \frac{e_i^2}{\delta - e_0} + e_{2i}\right| \leqslant 1. \tag{6}$$

For an arbitrary ε with $|\varepsilon| = 1$ the number

$$\delta = \frac{e_0\varepsilon + 1}{\varepsilon + \bar{e}_0}$$

has absolute value 1.

We set

$$\xi = e_{2i} + \frac{\bar{e}_0 e_i^2}{1-|e_0|^2}, \quad p = \frac{|\xi|}{\xi}\frac{e_0^2}{|e_0|^2}, \quad \varepsilon = \frac{\xi}{|\xi|}\frac{|e_i|^2}{e_i^2}.$$

Then

$$e_0^2 p^{-1} + \frac{e_i^2}{\delta - e_0} + e_{2i} = \frac{\xi}{|\xi|}|e_0|^2 + e_i^2 \frac{\bar{e}_0 + \varepsilon}{1-|e_0|^2} + e_{2i}$$
$$= \frac{\xi}{|\xi|}\left(|e_0|^2 + \frac{|e_i|^2}{1-|e_0|^2} + |\xi|\right).$$

Hence by (6)

$$\left|\frac{\xi}{|\xi|}\left(|\xi| + |e_0|^2 + \frac{|e_i|^2}{1-|e_0|^2}\right)\right| \leqslant 1 \quad \text{and} \quad |\xi| \leqslant 1 - |e_0|^2 - \frac{|e_i|^2}{1-|e_0|^2},$$

which proves the lemma.

LEMMA 2. *Let $P(z) = z^n + p_1 z^{n-1} + \ldots + p_n$, $|p_n| = 1$, $Q(z) = z^n \bar{P}(z^{-1}) \neq$ const $P(z)$. Then*

$$\frac{\overline{P(0)}P(z)}{Q(z)} = \frac{f(z)}{g(z)},$$

where f and g are holomorphic in an open disc containing $|z| \leqslant 1$,

$$f(0) = g(0) = \prod_{|a_j|>1} |a_j|^{-1}, \quad |f(z)| = |g(z)| = 1 \quad \text{for } |z| = 1,$$

and if the coefficients of P are real, the coefficients of f and g are also real.

Proof (cf. [4], Lemma 2). We set

$$f(z) = \prod_{|a_j|<1} \frac{|a_j|}{(-a_j)} \frac{z-a_j}{1-\bar{a}_j z}, \quad g(z) = \prod_{|a_j|>1} \frac{(-a_j)}{|a_j|} \frac{1-\bar{a}_j z}{z-a_j},$$

and using the equalities

$$\prod_{i=1}^{n} (-a_j) = P(0), \quad |P(0)| = 1,$$

we easily verify all the assertions of the lemma. Note that if $|a_j| = 1$, the factor $z-a_j$ occurs both in $P(z)$ and in $Q(z)$. Also if $P(z)$ has real coefficients,

$$\overline{f(z)} = f(\bar{z}), \quad \overline{g(z)} = g(\bar{z}).$$

LEMMA 3 (Kronecker [2]). *If $a \neq 0$ is an algebraic integer with $\overline{|a|} \leqslant 1$, then a is a root of unity.*

If a is a totally real algebraic integer with $\overline{|a|} \leqslant 2$, then $a = 2\cos w\pi$, where w is rational.

LEMMA 4. *If $a \neq 0$ is an algebraic integer of a field K satisfying the assumptions of Theorem 1, then either a is a root of unity or $\overline{|a|} \geqslant \sqrt{2}$.*

Proof. By the first part of Lemma 3 we can assume that $\overline{|a|} \geqslant 1$. For all $\sigma \in G$ we have

$$\sigma(|a|^2) = \sigma a \cdot \sigma \bar{a} = \sigma a \overline{\sigma a} = |\sigma a|^2;$$

thus $|a|^2$ is totally real and totally positive, $|a|$ is totally real and $\overline{|a|} = \overline{||a||}$.

On the other hand, by the second part of Lemma 3,

$$\overline{||a||} \geqslant 2\cos\frac{2\pi}{4} = \sqrt{2}.$$

Proof of Theorem 1. Let

$$A = \max_{\sigma \in G} \prod_{|a_{\sigma j}|>1} |a_{\sigma j}|.$$

Since $\sqrt{2} > \theta_0$, we can assume that $A \leqslant \sqrt{2}$. It follows that $\overline{|P(0)|} = 1$ since otherwise by Lemma 4

$$A \geqslant \overline{|P(0)|} \geqslant \sqrt{2}.$$

The assumption $P(z)/Q(z) \neq \text{const}$ implies that

$$R(z) = \frac{\overline{P(0)}P(z)}{Q(z)} = 1 + a_k z^k + a_l z^l + \ldots, \tag{7}$$

where on the right-hand side we have infinitely many non-zero coefficients, a_k, a_l being the first two of them. Since a_i are integers of K, $\overline{|a_i|} \geqslant 1$ for $i = k, l$.

Using Lemma 4, we distinguish two cases.

The case $|a_k| \geqslant \sqrt{2}$. Since $A(P) = A(\sigma P)$, we can assume $|a_k| \geqslant \sqrt{2}$ replacing if necessary P by a suitable σP. Applying Lemma 2 to P, we get

$$\frac{\overline{P(0)}P(z)}{Q(z)} = \frac{f(z)}{g(z)} = \frac{c + c_1 z + c_2 z^2 + \ldots}{d + d_1 z + d_2 z^2 + \ldots}; \tag{8}$$

$$|f(z)| = |g(z)| = 1 \quad \text{for} \quad |z| = 1, \quad f, g \text{ holomorphic for } |z| \leqslant 1;$$

$$f(0) = g(0) = \prod_{|a_j|>1} |a_j|^{-1} = c = d.$$

Comparing (7) with (8), we get

$$\begin{cases} c_i = d_i \ (i = 1, 2, \ldots, k-1), \ c_k = d_k + a_k c, \\ c_{k+i} = d_{k+i} + a_k d_i \ (i = 1, 2, \ldots, l-k-1), \ c_l = d_l + a_l c + a_k d_{l-k}. \end{cases} \tag{9}$$

It follows from $c_k = d_k + a_k c$, by Lemma 1, that

$$\sqrt{2}c \leqslant |a_k||c| \leqslant |c_k| + |d_k| \leqslant 2 - 2|c|^2, \quad \sqrt{2} \leqslant c^{-1} \leqslant A.$$

The case $\overline{|a_k|} = 1$, a_k is a root of unity. Let η be a root of unity, $P_\eta(z) = \eta^n P(\eta^{-1}z)$. We have $A(P_\eta) = A(P)$, P_η and $K(\eta)$ satisfying the assumptions of the theorem.

Setting

$$R_\eta(z) = \frac{\overline{P_\eta(0)}P_\eta(z)}{z^n \bar{P}_\eta(z^{-1})} = 1 + \sum_{i=1}^{\infty} a_i' z^i,$$

we get

$$R_\eta(z) = \frac{\eta^{-n}\overline{P(0)}\eta^n P(\eta^{-1}z)}{z^n \eta^{-n} \bar{P}(\eta z^{-1})} = \frac{\overline{P(0)}P(\eta^{-1}z)}{Q(\eta^{-1}z)} = R(\eta^{-1}z);$$

hence for all i

$$a_i' = \bar{\eta}^i a_i, \quad |a_i| = |a_i'|. \tag{9a}$$

Taking $\eta = \sqrt[k]{a_k}$, we get

$$a_k' = 1, \quad a_i' = 0 \ (0 < i < k), \quad a_{2k}' \in K.$$

Therefore, without loss of generality we assume that $a_k = 1$ if $l < 2k$ and $a_k = \pm 1$, $a_{2k} \in K$ if $l \geqslant 2k$ (we admit both signs here for the sake of symmetry).

The case $l < 2k$, $a_k = 1$. Applying to P a suitable $\sigma \in G$, we can obtain $|a_l| \geqslant 1$. We shall exploit the following inequality, due to Smyth ([5], pp. 172, 173):

$$(10)\qquad E = \frac{5}{4}|c|^2 + |c_{l-k} + \gamma c|^2 + \left|\frac{a_l c + c_{l-k}}{2} + \frac{\gamma c}{2} - c_{l-k} + \beta c\right|^2 \leqslant 2 + |\beta|^2 + |\gamma|^2,$$

where β and γ are arbitrary complex numbers.

Put $F(\beta, \gamma, c_{l-k}) = E - |\beta|^2 - |\gamma|^2$. $|t|^2 F\left(\frac{\beta}{t}, \frac{\gamma}{t}, \frac{c_{l-k}}{t}\right)$ is a hermitian form with the matrix

$$M = \begin{bmatrix} |c|^2-1 & \frac{|c|^2}{2} & -\frac{c}{2} & \bar{a}_l \frac{|c|^2}{2} \\ \frac{|c|^2}{2} & \frac{5}{4}|c|^2-1 & \frac{3}{4}c & \frac{1}{4}\bar{a}_l|c|^2 \\ -\frac{\bar{c}}{2} & \frac{3}{4}\bar{c} & \frac{5}{4} & -\frac{1}{4}\bar{a}_l\bar{c} \\ a_l\frac{|c|^2}{2} & \frac{1}{4}a_l|c|^2 & \frac{1}{4}a_l c & \frac{5}{4}|c|^2 + \frac{|a_l|^2|c|^2}{4} \end{bmatrix}$$

with diagonal minors

$$(11)\qquad \begin{aligned} M_1 &= |c|^2 - 1 < 0, \qquad M_2 = |c|^4 - \tfrac{9}{4}|c|^2 - 1 > 0, \\ M_3 &= \tfrac{5}{4} - 2|c|^2 > 0, \qquad M_4 = \tfrac{25}{16}|c|^2 - \tfrac{5}{2}|c|^4 + \frac{|c|^2|a_l|^2}{4} \geqslant \tfrac{29}{16}|c|^2 - \tfrac{5}{2}|c|^4. \end{aligned}$$

In order to justify the second inequality we notice that the equation $c_k = a_k + c$ implies by Lemma 1

$$2 - 2c^2 \geqslant c, \qquad c \leqslant \frac{\sqrt{17}-1}{4},$$

and since c is an algebraic integer,

$$c < \frac{\sqrt{17}-1}{4}, \qquad |c|^4 - \frac{9}{4}|c|^2 + 1 > 0.$$

Now

$$F(\beta, \gamma, c_{l-k}) = M_1|\beta + \ldots|^2 + \frac{M_2}{M_1}|\gamma + \ldots|^2 + \frac{M_3}{M_2}|c_{l-k} + \ldots|^2 + \frac{M_4}{M_3}$$

(see, e.g. [3], p. 461), from (10) we get

$$\frac{M_4}{M_3} = \min_{c_{l-k}} \max_{\beta,\gamma} F(\beta, \gamma, c_{l-k}) \leqslant 2$$

and from (11) we get

$$40|c|^4 - 93|c|^2 + 40 \geqslant 16(2M_3 - M_4) \geqslant 0.$$

As proved by Smyth, the latter inequality implies $\Lambda = c^{-1} > \theta_0$.

The case $l \geqslant 2k$, $a_k = \pm 1$, $a_{2k} \in K$. By (9)

$$c_{2k} = d_{2k} + a_k d_k + a_{2k} c.$$

On applying (5) to c_{2k} and d_{2k} and adding the resulting inequalities, we get

$$(12)\qquad \left|c_{2k} - d_{2k} + \frac{c_k^2 - d_k^2}{1-|c|^2}c\right| = \left|a_{2k}c + a_k d_k + \frac{c_k^2 - d_k^2}{1-c^2}c\right| \leqslant 2 - 2c^2 - \frac{|c_k|^2}{1-c^2} - \frac{|d_k|^2}{1-c^2}$$

(c is real). We now set

$$c_k = c_k^{(1)} + ic_k^{(2)}, \qquad d_k = d_k^{(1)} + id_k^{(2)}, \qquad a_{2k} = a_{2k}^{(1)} + ia_{2k}^{(2)},$$

where $c_k^{(i)}$, $d_k^{(i)}$, $a_{2k}^{(i)}$ are real for $i = 1, 2$; and we get from $c_k = d_k + a_k c$ the equations

$$c_k^{(2)} = d_k^{(2)}, \qquad c_k^2 - d_k^2 = c_k^{(1)2} - d_k^{(1)2} + 2ia_k c d_k^{(2)}.$$

The inequality $|x| \geqslant |\mathrm{Re}\, x|$ applied to (12) gives

$$(13)\qquad \left|a_{2k}^{(1)}c + a_k d_k^{(1)} + \frac{c_k^{(1)2} - d_k^{(1)2}}{1-c^2}c\right| \leqslant 2 - 2c^2 - \frac{c_k^{(1)2}}{1-c^2} - \frac{d_k^{(1)2}}{1-c^2}.$$

The left-hand side of (13) is greater than or equal to

$$|a_{2k}^{(1)}c + a_k d_k^{(1)}| - \left|\frac{c_k^{(1)2} - d_k^{(1)2}}{1-c^2}\right|c = |a_{2k}^{(1)}c + a_k d_k^{(1)}| \pm \left(\frac{c_k^{(1)2} - d_k^{(1)2}}{1-c^2}c\right).$$

Hence

$$(14)\qquad |a_{2k}^{(1)}c + a_k d_k^{(1)}| \leqslant 2 - 2c^2 - \min\left(\frac{c_k^{(1)2}}{1+c} + \frac{d_k^{(1)2}}{1-c}, \frac{c_k^{(1)2}}{1-c} + \frac{d_k^{(1)2}}{1+c}\right).$$

Since $c_k^{(1)} = \pm c + d_k^{(1)}$, we have

$$(15)\qquad |c_k^{(1)}| + |d_k^{(1)}| = c,$$

for otherwise by Lemma 1

$$1 - c^2 \geqslant c, \qquad \Lambda = c^{-1} \geqslant \sqrt{2}.$$

Again by Lemma 1

$$(16)\qquad |c_k^{(1)}| \leqslant |c_k| \leqslant 1-c^2, \quad |d_k^{(1)}| \leqslant |d_k| \leqslant 1-c^2.$$

By (15) and (16)

$$(17)\qquad c^2+c-1 \leqslant |c_k^{(1)}| \leqslant 1-c^2, \quad c^2+c-1 \leqslant |d_k^{(1)}| \leqslant 1-c^2.$$

The further argument depends on $a_{2k}^{(1)}$.

We distinquish three cases: X. $\overline{|a_{2k}^{(1)}|} \geqslant 1$, Y. $a_{2k}^{(1)} = 0$, Z. $0 < \overline{|a_{2k}^{(1)}|} < 1$.

X. Applying to P a suitable $\sigma \in G$, we can obtain $|a_{2k}^{(1)}| \geqslant 1$. By (17)

$$(18)\qquad |a_{2k}^{(1)}c \pm d_k^{(1)}| \geqslant |a_{2k}^{(1)}|c - |d_k^{(1)}| \geqslant c^2+c-1.$$

By (14) and (18)

$$c^2+c-1 \leqslant M = \max_{c^2+c-1\leqslant x\leqslant 1-c^2}\left(2-2c^2-\frac{x^2}{1+c}-\frac{(c-x)^2}{1-c}\right).$$

As proved by Smyth ([5], p. 175), the latter inequality implies $\Lambda = c^{-1} \geqslant \theta_0$.

Y. By (17)

$$|a_{2k}^{(1)}c \pm d_k^{(1)}| = |d_k^{(1)}| \geqslant c^2+c-1.$$

By (14)

$$c^2+c-1 \leqslant M \quad\text{and}\quad \Lambda \geqslant \theta_0$$

as in X.

Z. Since $\overline{|2a_{2k}^{(1)}|} < 2$ and $2a_{2k}^{(1)} = a_{2k}+\bar{a}_{2k}$ is a totally real algebraic integer, we have by Lemma 3

$$a_{2k}^{(1)} = \cos 2w\pi, \quad w \text{ rational}.$$

Since $c_k^{(1)} = \pm c + d_k^{(1)}$, we have by (14)

$$|c_k^{(1)} \pm (a_{2k}^{(1)}-1)c| \leqslant 2-2c^2-\min\left(\frac{c_k^{(1)2}}{1+c}+\frac{d_k^{(1)2}}{1-c}, \frac{c_k^{(1)2}}{1-c}+\frac{d_k^{(1)2}}{1+c}\right).$$

If $\overline{|a_{2k}^{(1)}-1|} \geqslant 1$ or $a_{2k}^{(1)}-1 = 0$, then the situation differs from that occurring in the case X or Y only by the permutation of $c_k^{(1)}$ and $d_k^{(1)}$. Let $\overline{|a_{2k}^{(1)}-1|} < 1$.

We have

$$a_{2k}^{(1)}-1 = \cos 2w\pi - 1 = -2\sin^2 w\pi;$$

hence

$$\overline{|\sin^2 w\pi|} < \frac{1}{2}, \quad \overline{|\sin w\pi|} < \frac{1}{\sqrt{2}}, \quad \overline{|2\sin w\pi|} < \sqrt{2}.$$

$2\sin w\pi$ is a totally real algebraic integer, and hence, by Lemma 4, $2\sin w\pi = \pm 1$, $a_{2k}^{(1)} = 1-2\sin^2 w\pi = \frac{1}{2}$, $a_{2k} = \frac{1}{2}+ia_{2k}^{(2)}$. If $\sqrt{-3} \notin K$, it is impossible to have $a_{2k}^{(2)} = \pm\sqrt{3}/2$, and thus a_{2k} is not a root of unity. Hence by Lemma 4 $\overline{|a_{2k}|} \geqslant \sqrt{2}$ and applying to P a suitable $\sigma \in G$, we can obtain

$$|a_{2k}| \geqslant \sqrt{2}, \quad |a_{2k}^{(2)}| \geqslant \tfrac{1}{2}\sqrt{7}.$$

We now replace P by $P_\eta(z)$, where $\eta^k = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. By (9a) we get

$$(19)\qquad a_k' = \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)a_k, \quad a_{2k}' = ia_{2k}, \quad a_i' = 0 \quad\text{for}\quad i \leqslant 2k,\ i \neq k, 2k.$$

$\tilde{P}(z) = P(z)\bar{P}(z)$ is a polynomial with totally real coefficients.

$$(20)\qquad \frac{\tilde{P}(z)}{\tilde{Q}(z)} = \frac{\overline{P(0)}P(z)}{Q(z)}\,\frac{P(0)\bar{P}(z)}{\bar{Q}(z)}$$
$$= (1+a_k'z^k+a_{2k}'z^{2k}+\ldots)(1+\bar{a}_k'z^k+\bar{a}_{2k}'z^{2k}+\ldots)$$
$$= 1+\sqrt{2}a_kz^k+(1+a_{2k}'+\bar{a}_{2k}')z^{2k}+\ldots = 1+b_kz^k+b_{2k}z^{2k}+\ldots$$

By Lemma 2

$$(21)\qquad \frac{P(z)}{Q(z)} = \frac{e_0+e_1z+\ldots}{f_0+f_1z+\ldots},$$

where $f_0 = e_0 = c^2$ and e_i, f_i are real.

The series occurring on the right side of (21) are convergent in an open disc containing $|z| \leqslant 1$ and have absolute value 1 on the circle $|z| = 1$.

By the inequality (14)

$$(22)\qquad |b_{2k}e_0+a_k\sqrt{2}f_k| \leqslant 2-2e_0^2-\min\left(\frac{e_k^2}{1+e_0}+\frac{f_k^2}{1-e_0}, \frac{e_k^2}{1-e_0}+\frac{f_k^2}{1+e_0}\right).$$

By (20) and (21)

$$(23)\qquad e_i = f_i,\ i<k; \quad e_k = a_k\sqrt{2}e_0+f_k, \quad e_{2k} = b_{2k}e_0+f_{2k}+a_k\sqrt{2}f_k.$$

The equality $e_k = f_k+a_k\sqrt{2}e_0$ implies

$$(24)\qquad |e_k|+|f_k| = \sqrt{2}e_0,$$

for otherwise, by Lemma 1, $\sqrt{2}e_0 \leqslant 1-e_0^2$, and

$$e_0^{-1} \geqslant \frac{\sqrt{2}+\sqrt{6}}{2} > 1.9 > \theta_0^2, \quad \Lambda = c^{-1} > \theta_0.$$

By (24) and (22)

$$|b_{2k}e_0+a_k\sqrt{2}f_k|\leqslant \tilde{M} = \max\varphi(x), \tag{25}$$

where

$$\varphi(x) = 2-2e_0^2-\frac{x^2}{1+e_0}-\frac{(\sqrt{2}e_0-x)^2}{1-e_0}.$$

We have

$$\frac{1}{2}\varphi'(x) = \frac{-x}{1+e_0}-\frac{x-\sqrt{2}e_0}{1-e_0} = \frac{-2x+\sqrt{2}e_0(1+e_0)}{1-e_0^2};$$

thus the maximum of $\varphi(x)$ taken for $x=\frac{\sqrt{2}}{2}e_0(1+e_0)$ equals

$$M = \varphi\left(\frac{1}{\sqrt{2}}e_0(1+e_0)\right) = 2-2e_0^2-\frac{1}{2}e_0^2(1+e_0)-\frac{1}{2}e_0^2(1-e_0) = 2-3e_0^2.$$

From the equality $f_k+a_k\sqrt{2}e_0=e_k$ we get by (25)

$$|b'_{2k}e_0+a_k\sqrt{2}e_k|\leqslant 2-3e_0^2, \quad \text{where} \quad b'_{2k}=b_{2k}-2.$$

Since $a'_{2k}=-a_{2k}^{(2)}+i\frac{1}{2}$, $b_{2k}=1-2a_{2k}^{(2)}$, we have $b'_{2k}=-1-2a_{2k}^{(2)}$. Replacing, if necessary, f_k by e_k and b_{2k} by b'_{2k}, we can thus assume that $|b_{2k}|\geqslant 1+\sqrt{7}$.

Hence

$$|b_{2k}e_0+a_k\sqrt{2}e_k|\geqslant |b_{2k}e_0|-\sqrt{2}|e_k|\geqslant \sqrt{2}e_0^2+(1+\sqrt{7})e_0-\sqrt{2}$$

and by (25)

$$2-3e_0^2\geqslant \sqrt{2}e_0^2+(1+\sqrt{7})e_0-\sqrt{2},$$

$$f(e_0^{-1})\geqslant 0,$$

where $f(x) = (2+\sqrt{2})x^2-(1+\sqrt{7})x-(3+\sqrt{2})$.

Since $f(\frac{16}{9})<0$ and $e_0=c^2$, we have $\Lambda=c^{-1}>\frac{4}{3}>\theta_0$.

Consider now the case

$$a_k=\pm 1, \quad a_{2k}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \quad \sqrt{-3}\in K.$$

It follows from (9a) that

$$c_k^2-d_k^2 = a_kc(2d_k^{(1)}+a_kc)+2a_kcd_k^{(2)}$$

and from (15) that

$$|c_k|^2+|d_k|^2 = |d_k^{(1)}|^2+2|d_k^{(2)}|^2+(c-|d_k^{(1)}|)^2.$$

The inequality (12) implies in virtue of the above identities

$$M_1 = 2-2c^2-\frac{|d_k^{(1)}|^2+(c-|d_k^{(1)}|)^2+2|d_k^{(2)}|^2}{1-c^2} - \left|\left(\frac{\frac{1}{2}c+\frac{1}{2}c^3}{1-c^2}+a_k\frac{1+c^2}{1-c^2}d_k^{(1)}\right)+i\left(\pm\frac{\sqrt{3}}{2}c+a_k\frac{1+c^2}{1-c^2}d_k^{(2)}\right)\right|\geqslant 0.$$

Hence it follows that

$$M_2 = \max_{|d_k^{(2)}|}\max_{|d_k^{(1)}|}\left(2-2c^2-\frac{|d_k^{(1)}|^2+(c-|d_k^{(1)}|)^2+2|d_k^{(2)}|^2}{1-c^2} - \left|\frac{1+c^2}{1-c^2}\left(\frac{1}{2}c-|d_k^{(1)}|\right)+i\left(\frac{\sqrt{3}}{2}c-\frac{1+c^2}{1-c^2}|d_k^{(2)}|\right)\right|\right)\geqslant 0.$$

The inner maximum is attained for $|d_k^{(1)}|=c/2$ since then both

$$|d_k^{(1)}|^2+(c-|d_k^{(1)}|)^2 \quad \text{and} \quad \left|\frac{1+c^2}{1-c^2}\left(\frac{1}{2}c-|d_k^{(1)}|\right)\right|$$

attain the minimal value. Thus

$$M_2 = \max_{|d_k^{(2)}|}\left(2-2c^2-\frac{c^2}{2(1-c^2)}-\frac{2|d_k^{(2)}|^2}{1-c^2}-\left|\frac{\sqrt{3}}{2}c-\frac{1+c^2}{1-c^2}|d_k^{(2)}|\right|\right)\geqslant 0.$$

We set

$$g(x) = 2-2c^2-\frac{c^2}{2(1-c^2)}-\frac{2x^2}{1-c^2}-\left|\frac{\sqrt{3}}{2}c-\frac{1+c^2}{1-c^2}x\right|.$$

In the interval $\left(0, \frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right)$ the function $g(x)$ is increasing. Indeed we have in this interval

$$\frac{\sqrt{3}}{2}c-\frac{1+c^2}{1-c^2}x>0, \quad g'(x) = -\frac{4x}{1-c^2}+\frac{1+c^2}{1-c^2}\geqslant\frac{-2\sqrt{3}c}{1+c^2}+\frac{1+c^2}{1-c^2}.$$

On the other hand, by the assumption $\Lambda<\sqrt{2}$ we have

$$c^2>\tfrac{1}{2}, \quad (1+c^2)^2>\tfrac{9}{4},$$

$$2\sqrt{3}c(1-c^2)<\sqrt{3}c<\sqrt{3}<\tfrac{9}{4}<(1+c^2)^2.$$

For $x>\frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}$ we have

$$\frac{\sqrt{3}}{2}c-\frac{1+c^2}{1-c^2}x<0, \quad g'(x) = -\frac{4x}{1-c^2}-\frac{1+c^2}{1-c^2}<0;$$

thus the function $g(x)$ is decreasing. Since it is continuous, we have

$$M_2 = g\left(\frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right),$$

$$g\left(\frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right) = 2-c^2-\frac{c^2}{2(1-c^2)}-\frac{3}{2}c^2\frac{(1-c^2)^2}{(1+c^2)^2(1-c^2)} \geqslant 0,$$

and on simplification

$$1-c^2-c^4-c^6+c^8 \geqslant 0,$$

whence $A^8-A^6-A^4-A^2+1 \geqslant 0$. The equation $x^4-x^3-x^2-x+1=0$ has only one real root greater than 1, namely

$$\tfrac{1}{4}\left(1+\sqrt{13}+\sqrt{2\sqrt{13}-2}\right).$$

It follows that

$$A \geqslant \tfrac{1}{2}\sqrt{1+\sqrt{13}+\sqrt{2\sqrt{13}-2}}.$$

On the other hand, the polynomial $z^2+\varrho z-1$, $\varrho = \dfrac{-1+\sqrt{-3}}{2}$ has two zeros given by the formula

$$\frac{-\varrho\pm\sqrt{\varrho^2+4}}{2} = \frac{1}{4}-i\frac{\sqrt{3}}{2}\pm\frac{1}{2}\left(\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}-i\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right)$$

$$= \left(\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}\right)-i\left(\frac{\sqrt{3}}{4}\pm\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right).$$

Hence

$$|\theta_1|^2 = \left(\frac{1}{4}+\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}\right)^2+\left(\frac{\sqrt{3}}{4}+\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right)^2$$

$$= \frac{1}{4}\left(1+\sqrt{13}+\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}+\sqrt{3\left(\frac{\sqrt{13}}{2}-\frac{7}{4}\right)}\right)$$

$$= \frac{1}{4}\left(1+\sqrt{13}+\sqrt{2\sqrt{13}-2}\right)$$

and we get

$$A \geqslant |\theta_1|.$$

It remains to note that the zeros of the polynomial $z^2+\varrho^2 z-1$ are complex conjugates of the zeros of $z^2+\varrho z-1$ and that

$$|\theta_1|^2 < 1.73 < \theta_0^2.$$

LEMMA 5. *$f(z) = a_n z^n+\ldots+a_0$ has all zeros inside the unit circle if and only if $\delta_k(f) > 0$, $k = 0, 1, \ldots, n$ where*

$$(26)\qquad \delta_k(f) = \left|\begin{array}{cccccccc}
a_n & & & & a_0 & a_1 & \cdots & a_{k-1}\\
a_{n-1} & a_n & & & & a_0 & & a_{k-2}\\
\vdots & & \ddots & & & & \ddots & \vdots\\
 & & & & & & & a_1\\
a_{n-k+1} & \cdots & a_{n-1} & a_n & & & & a_0\\
\bar a_0 & & & & \bar a_n & \bar a_{n-1} & \cdots & \bar a_{n-k+1}\\
\bar a_1 & \bar a_0 & & & & \bar a_n & & \\
\vdots & & \ddots & & & & \ddots & \vdots\\
 & & & & & & & \bar a_{n-1}\\
\bar a_{k-1} & \cdots & \bar a_1 & \bar a_0 & & & & \bar a_n
\end{array}\right|.$$

Proof see A. Cohn [1].

LEMMA 6. *If $A = (a_{ij})$, $i, j \leqslant n$, $n \geqslant l > k$, then*

$$(27)\qquad \det A = \sum_{i<j}(-1)^{i+j+k+l}A_{k,l;i,j}\begin{vmatrix} a_{ki} & a_{kj}\\ a_{li} & a_{lj}\end{vmatrix},$$

where $A_{k,l;i,j}$ is the determinant of the matrix obtained from A by crossing out the k-th and l-th rows and the i-th and j-th columns.

The proof follows from Laplace's theorem.

LEMMA 7. *Let K satisfy the assumptions of Theorem 3,*

$$f(z) = a_n z^n+\ldots+a_0,\quad f^*(z) = z^n\bar f(z^{-1}) = \bar a_0 z^n+\bar a_1 z^{n-1}+\ldots+\bar a_n,$$

where a_i are integers of K. If $a_n = 1$, $a_0 = \varepsilon\dfrac{1+\sqrt{5}}{2}$, where ε is a root of unity, then for each $i \leqslant n$ the condition

$$(\mathrm{w}_i)\qquad \begin{array}{lll} \delta_k(\sigma f) > 0 & \text{if} \quad \sigma\sqrt{5} = -\sqrt{5} & (k = 1, 2, \ldots, i),\\ \delta_k(\sigma f^*) > 0 & \text{if} \quad \sigma\sqrt{5} = \sqrt{5} & (k = 1, 2, \ldots, i)\end{array}$$

implies

$$(28)\qquad \delta_k(\sigma f) = \delta_1(\sigma f)^k \quad \text{and} \quad a_{n-k+1}\bar a_0-\bar a_{k-1} = 0$$

for all $k \leqslant i$ and all $\sigma \in G$.

Proof. We shall proceed by induction with respect to i. For $i = 1$

$$\sigma(a_n\bar a_0-\bar a_0) = \sigma 0 = 0,$$

$$\delta_1(\sigma f) = \begin{vmatrix} 1 & \sigma a_0\\ \sigma\bar a_0 & 1\end{vmatrix} = 1-|\sigma a_0|^2 = \begin{cases} -\dfrac{1+\sqrt{5}}{2} & \text{if} \quad \sigma\sqrt{5} = \sqrt{5},\\[2mm] \dfrac{\sqrt{5}-1}{2} & \text{if} \quad \sigma\sqrt{5} = -\sqrt{5}.\end{cases}$$

Let us assume that the lemma is true for $i < i_0$. Then, by applying the formula from Lemma 6 to the rows i_0, $2i_0$ and omitting the terms which involve the coefficient

$$\begin{vmatrix} a_n & 0 \\ \bar{a}_0 & 0 \end{vmatrix} = 0,$$

we get

$$(29)\qquad \delta_{i_0}(f) = \delta_1(f)\,\delta_{i_0-1}(f) + \sum_{j=1}^{i_0-1}(-1)^j \begin{vmatrix} a_{n-i_0+j} & a_n \\ \bar{a}_{i_0-j} & \bar{a}_0 \end{vmatrix} (\delta_{i_0})_{i_0,2i_0;i_0,j}(f).$$

Now we apply (29) to σf and use the inductive assumption. We obtain

$$(30)\qquad \delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0} - \begin{vmatrix} \sigma a_{n-i_0+1} & \sigma a_n \\ \overline{\sigma a_{n-i_0+1}} & \overline{\sigma a_0} \end{vmatrix} (\delta_{i_0})_{i_0,2i_0;1,i_0}(\sigma f),$$

$$\big(\delta_{i_0}(f)\big)_{i_0,2i_0;1,i_0}$$

$$= \begin{vmatrix}
0 & 0 & \cdots & 0 & 0 & a_0 & a_1 & \cdots & a_{i-2} & a_{i-1} \\
a_n & 0 & \cdots & & & 0 & a_0 & a_1 \cdots & a_{i-3} & a_{i-2} \\
a_{n-1} & a_n & \cdots & & & & & & & \\
 & & & & & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\
\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & 0 & 0 & \cdots & a_1 & a_2 \\
a_{n-i+3} & a_{n-i+2} & \cdots & a_{n-1} & a_n & 0 & 0 & \cdots & a_0 & a_1 \\
0 & 0 & \cdots & & 0 & \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_{n-i+2} & \bar{a}_{n-i+1} \\
\bar{a}_0 & 0 & \cdots & & & 0 & \bar{a}_n & \cdots & \bar{a}_{n-i+3} & \bar{a}_{n-i+2} \\
\bar{a}_1 & \bar{a}_0 & \cdots & \cdots & & & 0 & \cdots & & \\
\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\
\bar{a}_{i-3} & \bar{a}_{i-4} & \cdots & \bar{a}_1 & \bar{a}_0 & 0 & 0 & \cdots & \bar{a}_n & \bar{a}_{n-1}
\end{vmatrix}$$

We develop the determinant according to the first row and the i_0-th row. When these rows are left out, the (i_0-1)-th column consists of zeros only, and hence it suffices to compute the minors $M_{1,i_0;i_0-1,i}$. For the elements a of K, we have $\sigma(\bar{a}) = \overline{(\sigma a)}$; thus

$$\begin{vmatrix} \sigma a_0 & \sigma a_i \\ \overline{\sigma a_n} & \overline{\sigma a_{n-i}} \end{vmatrix} = \sigma(a_0)\overline{\sigma(a_{n-i})} - \sigma(a_i)\overline{\sigma(a_n)}$$

$$= \sigma(a_0\bar{a}_{n-i} - a_i\bar{a}_n) = \overline{\sigma(\bar{a}_0 a_{n-i} - \bar{a}_i a_n)}.$$

By the inductive assumption

$$\delta_{i_0}(\sigma f)_{i_0,2i_0;1,i_0} = \delta_{i_0-2}(\sigma f)\begin{vmatrix} \sigma(a_0) & \sigma(a_{i_0-1}) \\ \overline{\sigma(a_n)} & \overline{\sigma(a_{n-i_0+1})} \end{vmatrix}$$

$$= \delta_1(\sigma f)^{i_0-2}\,\overline{\sigma(\bar{a}_0 a_{n-i_0+1} - \bar{a}_{i_0-1} a_n)}.$$

Substituting the computed value of $(\delta_{i_0})_{i_0,2i_0;1,i_0}$ into (30), we get

$$(31)\qquad \delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}\,|\sigma(\bar{a}_0 a_{n-i_0+1} - \bar{a}_{i_0-1} a_n)|^2.$$

Since $\delta_k(f^*)$ is obtained from $\delta_k(f)$ by a transposition of the ith row and the $(k+i)$-th row of the latter, we have

$$\delta_k(f^*) = (-1)^k\delta_k(f) \quad\text{and}\quad \delta_k(\sigma f^*) = (-1)^k\delta_k(\sigma f).$$

Hence the condition (w_{i_0}) is equivalent to the condition

$$\delta_{i_0}(\sigma f) > 0 \quad\text{if}\quad \sigma\sqrt{5} = -\sqrt{5},$$
$$(-1)^{i_0}\delta_{i_0}(\sigma f) > 0 \quad\text{if}\quad \sigma\sqrt{5} = \sqrt{5}.$$

We set

$$x = \bar{a}_0 a_{n-i_0+1} - \bar{a}_{i_0-1} a_n$$

and distinguish two cases:

A. i_0 is even. We have, for all $\sigma \in G$,

$$\delta_{i_0}(\sigma f) = \delta_1^{i_0}(\sigma f) - \delta_1^{i_0-2}(\sigma f)\,|\sigma x|^2 > 0,$$

and since

$$\delta_1(\sigma f)^{i_0} \quad\text{and}\quad \delta_1(\sigma f)^{i_0-2} > 0,$$

we get

$$|\sigma x|^2 < \delta_1^2(\sigma f) = \begin{cases} \dfrac{3+\sqrt{5}}{2} & (\sigma\sqrt{5} = \sqrt{5}), \\[2mm] \dfrac{3-\sqrt{5}}{2} & (\sigma\sqrt{5} = -\sqrt{5}), \end{cases}$$

$$\prod_{\sigma\in G}|\sigma x|^2 < \prod_{\sigma\in G}\delta_1^2(\sigma f).$$

The number $N = \prod_{\sigma\in G}|\sigma x|^2$ is a non-negative rational integer, as the square of the norm of x,

$$\prod_{\sigma\in G}\delta_1^2(\sigma f) = 1.$$

Since $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$ occur equally often as factors in the product, we have $M = 0$ and $x = 0$.

By (31)

$$\delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0},$$

$$\bar{a}_0 a_{n-i_0+1} - \bar{a}_{i_0-1} a_n = 0, \qquad \sigma(\bar{a}_0 a_{n-i_0+1} - a_n\bar{a}_{i_0-1}) = 0$$

and the inductive assertion is proved.

B. i_0 in odd. We have

$$\delta_{i_0}(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = -\sqrt{5},$$
$$\delta_{i_0}(\sigma f) < 0 \quad \text{if} \quad \sigma\sqrt{5} = \sqrt{5}.$$

If

$$\sigma\sqrt{5} = -\sqrt{5}, \quad \delta_1(\sigma f) = \frac{\sqrt{5}-1}{2} > 0,$$

we have

$$\delta_1(\sigma f)^{i_0} > 0, \quad \delta_1(\sigma f)^{i_0-2} > 0$$

and the inequality

$$\delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}|\sigma x|^2 > 0$$

implies

$$\frac{3-\sqrt{5}}{2} > |\sigma x|^2. \tag{32}$$

If

$$\sigma\sqrt{5} = \sqrt{5}, \quad \delta_1(\sigma f) = -\frac{1+\sqrt{5}}{2} < 0,$$

we have

$$\delta_1^{i_0}(\sigma f) < 0, \quad \delta_1(\sigma f)^{i_0-2} < 0$$

and the inequality

$$\delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}|\sigma x|^2 < 0$$

implies

$$\frac{3+\sqrt{5}}{2} > |\sigma x|^2. \tag{33}$$

The inductive assertion follows from (32) and (33) as in the case A.

LEMMA 8. *Let K satisfy the assumptions of Theorem 3,*

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + \varepsilon\frac{1+\sqrt{5}}{2},$$

where a_i are integers of K, ε is a root of unity, and let

$$\sigma f(z) = \prod_{i=1}^{n}(z-\alpha_{\sigma j}), \quad \sigma \in G.$$

If

$$(34) \qquad |\alpha_{\sigma j}| > 1 \quad \textit{if} \quad \sigma\sqrt{5} = \sqrt{5}, \qquad |\alpha_{\sigma j}| < 1 \quad \textit{if} \quad \sigma\sqrt{5} = -\sqrt{5},$$

then

$$f(z) = z^n + \varepsilon\frac{1+\sqrt{5}}{2}.$$

Proof. $f(z)$ has all zeros outside the circle $|z| \geq 1$ if and only if $f^*(z) = z^n\bar{f}(z^{-1})$ has all zeros inside the circle $|z| < 1$. By Lemma 5 the condition (34) is equivalent to the condition

$$(34') \qquad \delta_k(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = -\sqrt{5} \quad (k = 1, 2, \ldots, n), \qquad \delta_k(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = \sqrt{5} \quad (k = 1, 2, \ldots, n).$$

The latter is the same as condition (w_n) considered in Lemma 7 and in virtue of that lemma

$$a_{n-i}\bar{a}_0 - \bar{a}_i = 0 \quad (i = 1, 2, \ldots, n-1), \tag{35}$$

where $a_0 = \varepsilon\frac{1+\sqrt{5}}{2}$.

(35) gives $a_i\bar{a}_0 - \bar{a}_{n-i} = 0$ and on passing to complex conjugates we get

$$-a_{n-i} + a_0\bar{a}_i = 0 \quad (i = 1, 2, \ldots, n-1). \tag{35'}$$

Since

$$\begin{vmatrix} \bar{a}_0 & -1 \\ -1 & a_0 \end{vmatrix} = |a_0|^2 - 1 = \frac{1+\sqrt{5}}{2} \neq 0,$$

(35) and (35′) imply $a_i = a_{n-i} = 0$ for $i = 1, 2, \ldots, n-1$. Hence $f(z) = z^n + a_0$.

Proof of Theorem 3. Assume first that

$$P(0) = \varepsilon\frac{1+\sqrt{5}}{2}.$$

Let

$$P(z) = f(z)\prod_{i=2}^{n-k+1}(z-\varepsilon_i),$$

where ε_i are roots of unity, but no zero of $f(z)$ is a root of unity. The product $\prod(z-\varepsilon_i)$ divides $(z^m-1)^m$ for a suitable m; hence

$$\prod_{i=2}^{n-k+1}(z-\varepsilon_i) = (P(z), (z^m-1)^m) \in K[z]$$

and $f(z) \in K[z]$, f is monic with integer coefficients and $f(0) = \varepsilon_1\frac{1+\sqrt{5}}{2}$.

For $\sigma \in G$ let

$$\sigma f(z) = \prod_{j=1}^{k}(z-\alpha_{\sigma j}). \tag{36}$$

By the assumption about $P(z)$, we have

$$(37)\qquad \begin{aligned} |\alpha_{\sigma j}| &\geqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \sqrt{5},\\ |\alpha_{\sigma j}| &\leqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = -\sqrt{5}.\end{aligned}$$

Suppose that for a $\sigma_0 \in G$ and a $j_0 \leqslant k$ we have

$$(38)\qquad |\alpha_{\sigma_0 j_0}| = 1.$$

Consider the field $L = K^{\sigma_0}(\alpha_{\sigma_0 j_0})$ and any isomorphic injection τ of L into C. We have $\sigma_0\sqrt{5} = \overline{\sigma_0\sqrt{5}}$; thus $\tau\sigma_0\sqrt{5} = \tau\overline{\sigma_0\sqrt{5}}$.

If $\tau\sigma_0\sqrt{5} = \sqrt{5}$, then, since $\tau\sigma_0 f(\alpha_{\sigma_0 j_0}) = \tau\sigma_0 f(\tau\alpha_{\sigma_0 j_0}) = 0$, we have by (36) and (37)

$$|\tau\alpha_{\sigma_0 j_0}| \geqslant 1 \quad \text{and} \quad |\tau\overline{\alpha_{\sigma_0 j_0}}| \geqslant 1.$$

If $\tau\sigma_0\sqrt{5} = \tau\overline{\sigma_0\sqrt{5}} = -\sqrt{5}$ we have similarly

$$|\tau\alpha_{\sigma_0 j_0}| \leqslant 1 \quad \text{and} \quad |\tau\overline{\alpha_{\sigma_0 j_0}}| \leqslant 1.$$

On the other hand, by (38)

$$\tau\alpha_{\sigma_0 j_0}\cdot\tau\overline{\alpha_{\sigma_0 j_0}} = \tau(\alpha_{\sigma_0 j_0}\cdot\overline{\alpha_{\sigma_0 j_0}}) = \tau|\alpha_{\sigma_0 j_0}|^2 = 1;$$

thus $|\tau\alpha_{\sigma_0 j_0}| = 1$ for all τ. By Lemma 3, $\alpha_{\sigma_0 j_0}$ is a root of unity and by (36) a certain conjugate of it is a zero of $f(z)$, contrary to the definition of f. The contradiction obtained above shows that f satisfies all the assumptions of Lemma 8 and in virtue of that lemma

$$f(z) = z^k + \varepsilon_1\frac{1+\sqrt{5}}{2}.$$

Assume now that $P(0) = \varepsilon\frac{1-\sqrt{5}}{2}$. Then, for a σ with $\sigma\sqrt{5} = -\sqrt{5}$, σP satisfies the assumptions of the theorem and $\sigma P(0) = \frac{\sqrt{5}+1}{2}\sigma(\varepsilon)$. Thus, by the already proved case of the theorem, formula (3) holds.

Proof of Theorem 2. If (3) holds, we clearly have equality in (2). Suppose that for a polynomial $P \in K[z]$ with the leading coefficient p_0 the equality in (2) is obtained. By the equality $C(P) = (p_0)$ quoted in the introduction, $P_0(z) = p_0^{-1}P(z)$ has integral coefficients. Moreover $\sqrt{5} \in K$ and $|P_0(0)| = \frac{\pm 1+\sqrt{5}}{2}$. Since for $a \in K$, $\sigma \in G$

$$\sigma(|a|^2) = |\sigma a|^2,$$

we have for all $\sigma \in G$

$$\left|\sigma\left(\frac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}}\right)\right| = \left|\sigma\left(\frac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}\right)\right|^2 = \sigma\left(\left|\frac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}\right|^2\right) = 1.$$

$\frac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}} = P_0(0)^2\frac{3\pm\sqrt{5}}{2}$ is an integer, and hence by Lemma 4 $\frac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}}$ is a root of unity and $\frac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}$ is also one.

Thus

$$P_0(z) = z^n + p_1 z^{n-1} + \ldots + \varepsilon\frac{1\pm\sqrt{5}}{2},$$

where p_i are integers of K. On the other hand,

$$(39)\qquad \prod_{\sigma\in G}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}| = \prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\prod_{|\alpha_{\sigma j}|<1}|\alpha_{\sigma j}|^{-1}\right)\prod_{\sigma\sqrt{5}=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}|$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{|K|}{2}}\prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|<1}|\alpha_{\sigma j}|^{-1}\prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}|,$$

and the equality in (2) implies that both double products on the right-hand side of (39) are empty. Thus $P_0(z)$ satisfies the assumptions of Theorem 3 and in virtue of that Theorem 3 holds.

References

[1] A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Zeitschr. 14 (1922), pp. 110–148.

[2] L. Kronecker, *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, J. Reine Angew. Math. 53 (1857), pp. 173–175.

[3] A. Mostowski and M. Stark, *Introduction to Higher Algebra*, Warszawa 1968.

[4] A. Schinzel, *On the product of the conjugates of an algebraic number outside the unit circle*, Acta Arith. 24 (1973), pp. 385–399.

[5] C. J. Smyth, *On the product of the conjugates of an algebraic integer outside the unit circle*, Bull. London Math. Soc. 3 (1971), pp. 169–175.

Received on 11. 11. 1974 (640)