ON THE PROJECTIVE DIMENSION OF TENSOR PRODUCTS OF MODULES

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In memory of Nicholas Ryan Baeth

ABSTRACT. In this paper we consider a question of Roger Wiegand, which is about tensor products of finitely generated modules that have finite projective dimension over commutative Noetherian rings. We construct modules of infinite projective dimension (and of infinite Gorenstein dimension) whose tensor products have finite projective dimension. Furthermore we determine nontrivial conditions under which such examples cannot occur. For example we prove that, if the tensor product of two nonzero modules, at least one of which is totally reflexive (or equivalently Gorenstein-projective), has finite projective dimension, then both modules in question have finite projective dimension.

1. INTRODUCTION

Throughout *R* denotes a commutative Noetherian ring, and all *R*-modules are assumed to be finitely generated. If *R* is assumed to be local, then m and *k* denote the unique maximal ideal and the residue field of *R*, respectively. We refer the reader to [5, 10, 17] for basic unexplained terminology and follow the convention that depth(0) = ∞ [26] and pd(0) = $-\infty$ [1].

This paper was initiated by our discussions with Roger Wiegand, who informed us that the following question was raised at a commutative algebra meeting:

Question 1.1. Let *R* be a commutative ring. If *M* and *N* are *R*-modules such that $pd_R(M) < \infty$ and $pd_R(N) < \infty$, then must $pd_R(M \otimes_R N) < \infty$? What if M = N?

It is not difficult to find counterexamples to Question 1.1, but there are some special affirmative cases that are interesting for us. For example we observed that, if *R* is a *d*-dimensional Cohen-Macaulay local ring and *M* is an *R*-module which is locally free on the punctured spectrum of *R* such that $pd_R(M) \le d/2$, then $pd_R(M \otimes_R M) < \infty$; see A.7 and A.8 in the appendix for the details.

Wiegand [37] showed that, unless the ring considered is regular or has depth zero, Question 1.1 cannot have an affirmative answer. More precisely, he proved:

Theorem 1.2. (Wiegand [37]) Let R be a local ring. Then the following conditions are equivalent:

- (i) If M is an R-module such that $pd_R(M) < \infty$, then $pd_R(M \otimes_R M) < \infty$.
- (ii) If M and N are cyclic R-modules such that $pd_R(M) < \infty$ and $pd_R(N) < \infty$, then $pd_R(M \otimes_R N) < \infty$.

 \square

(iii) depth(R) = 0 or R is regular.

Wiegand's proof of Theorem 1.2 is inspiring to us and it is provided in the appendix. Wiegand, motivated by Question 1.1 and Theorem 1.2, raised the following natural question:

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Question 1.3. (Wiegand [37]) Let *R* be a commutative ring. If *M* and *N* are *R*-modules such that $pd_R(M \otimes_R N) < \infty$, then must $pd_R(M) < \infty$ or $pd_R(N) < \infty$?

If $M \otimes_R N$ is nonzero and free, then M and N must be both projective; see A.1. It is also known that Question 1.3 is true if the derived tensor product $M \otimes_R^L N$ of M and N is considered [6, 1.5.3(a)]. On the other hand Question 1.3 is not true in general, and one of our aims in this paper is to construct examples that give a negative answer to the question. In fact, we observe that the finiteness of the projective dimension of a nonzero tensor product $M \otimes_R N$ over a local ring does not necessarily imply the finiteness of the Gorenstein dimension (or that of the complexity) of M or N; see Examples 2.5 and 2.7. In passing we also observe in Remark 2.8 that Question 1.3 has a negative answer even if projective dimension is replaced with injective dimension.

In addition to providing counterexamples, we point out some conditions under which Question 1.3 is true; see the appendix. Most of these conditions are immediate consequences of some known results from the literature, but they provide motivations for us to further investigate the projective dimension of tensor products, and seek new and nontrivial conditions that imply Question 1.3 to be true. One such result we prove in this direction is:

Theorem 1.4. Let *R* be a ring and let *M* and *N* be *R*-modules with $pd_R(M \otimes_R N) \leq n$ for some $n \geq 0$.

- (i) If $\operatorname{Tor}_{i}^{R}(M,N) = 0$ for all i = 1, ..., n, then $\operatorname{pd}_{R}(M) \leq n$ and $\operatorname{pd}_{R}(N) \leq n$.
- (ii) If $\operatorname{Ext}^{i}_{R}(M,R) = 0$ for all i = 1, ..., n, then M is projective and $\operatorname{pd}_{R}(N) < \infty$.

The proof of Theorem 1.4 is given in section 3, where the proofs of the first and the second part are entirely distinct. One can view the conclusion of Theorem 1.4(i) as an extension of the following fact: If *M* and *N* are nonzero *R*-modules such that $pd_R(M \otimes_R N) < \infty$ and $Tor_i^R(M, N) = 0$ for all $i \ge 1$, then $pd_R(M) < \infty$ and $pd_R(N) < \infty$; see, for example, [34, 1.1]. The vanishing hypothesis of Theorem 1.4(ii) holds for Gorenstein-projective modules (or equivalently for totally reflexive modules since modules are assumed to be finitely generated [17, 4.2.6]). So the theorem yields the result advertised in the abstract:

Corollary 1.5. Let *R* be a ring and let *M* and *N* be nonzero *R*-modules. Assume *M* is totally reflexive. If $pd_R(M \otimes_R N) < \infty$, then *M* is projective and $pd_R(N) < \infty$.

Note that Corollary 1.5 may fail if M has finite Gorenstein dimension, but M is not totally reflexive; see Example 2.5. We also have a result similar to Corollary 1.5 for Ulrich modules over Cohen-Macaulay local rings; see Corollary 3.13 and the paragraph preceding it.

The second main result of this paper concerning Question 1.3 is the following:

Theorem 1.6. Let *R* be a Cohen-Macaulay local ring and let $M = \Omega_R L$ for some nonfree maximal Cohen-Macaulay *R*-module *L*. If *N* is an *R*-module, then $pd_R(M \otimes_R N) < \infty$ if and only if N = 0.

A point worth mentioning here is that the conclusion of Theorem 1.6 also holds if projective dimension is replaced with injective dimension; see Theorem 3.8 and Corollary 3.10. Let us also note that the theorem may fail if M is a syzygy of a module that is not maximal Cohen-Macaulay: for example, if $R = k[[x, y, z]]/(xy - z^2)$, L = R/(x, y), and $M = N = \Omega_R L$, then $N \neq 0$, but $pd_R(M \otimes_R N) < \infty$ and $id_R(M \otimes_R N) < \infty$; see [15, 2.7].

2. Counterexamples to Questions 1.1 and 1.3

Some examples about Question 1.1. In this subsection we construct examples that corroborate Theorem 1.2 and give a negative answer to Question 1.3.

Example 2.1 is due to Wiegand [37]. The first part of the example is included here to highlight the fact that one cannot replace "or" with "and" in Question 1.3; see also Proposition A.7 concerning the second part of the example.

Example 2.1. Let R = k[[x, y]]/(xy) and let M = R/(x+y).

- (i) Let $N = R/(x^2)$. Then it follows $M \otimes_R N \cong M$ so that $pd_R(M \otimes_R N) = pd_R(M) = 1$ since x + y is a non zero-divisor on R. On the other hand, since x^2 is a zero-divisor on R, we see that $pd_R(N) = \infty$.
- (ii) Let N = R/(x y). Then, since x + y and x y are both non zero-divisors on R, it follows that $pd_R(M) = 1 = pd_R(N)$. Furthermore we have that $pd_R(M \otimes_R N) = \infty$.

We make use of the next lemma to obtain Example 2.3 which gives a negative answer to Question 1.3 over a ring that is not Cohen-Macaulay.

Lemma 2.2. Let *R* be a local ring that has depth one. If *R* is not regular, then there is an *R*-module *M* such that $pd_R(M) = 1$ and $pd_R(M^{\otimes n}) = \infty$ for each $n \ge 2$.

Proof. Assume *R* is not regular. Note that \mathfrak{m} can be minimally generated by $\{x_1, \ldots, x_s\}$ for some elements x_i of *R*, each of which is a non zero-divisor on *R*; see 2.6. Set $M = R/(x_1) \oplus \cdots \oplus R/(x_s)$. Then it follows $pd_R(M) = 1$.

Let $n \ge 2$ be an integer and suppose $pd_R(M^{\otimes n}) < \infty$. Then $R/(x_1, x_2)$, being a direct summand of $M^{\otimes n}$, has finite projective dimension, namely $pd_R(R/(x_1, x_2)) = 1$ (recall that $s \ge 2$). This implies that the ideal (x_1, x_2) is free, that is, principal, which is not true. Thus we conclude that $pd_R(M^{\otimes n}) = \infty$. \Box

Example 2.3. Let $R = k[[x,y]]/(x^2,xy)$ and let $M = R/(x) \oplus R/(y)$. Then it follows that $pd_R(M) = 1$ and $pd_R(M^{\otimes n}) = \infty$ for each $n \ge 2$.; see 2.2.

In Examples 2.1 and 2.3, the modules considered have projective dimension one. Next, in Example 2.4, we build on [15, 2.5] and obtain an example of a tensor product of infinite projective dimension, where one of the modules in question has projective dimension three.

Example 2.4. Let R = k[[x, y, z, w]]/(xy) and let $\mathfrak{p} = (y, z, w)$. Then *R* is a three-dimensional hypersurface and \mathfrak{p} is a prime ideal of *R*. Set M = R/(z, w, x + y), $N = \operatorname{Tr}(R/\mathfrak{p})$ and $X = M \oplus N$. Then it follows that $\operatorname{pd}_R(M) = 3$, $\operatorname{pd}_R(N) = 1$, and $\operatorname{pd}_R(M \otimes_R N) = \infty$. Moreover we have $\operatorname{pd}_R(X) = 3$ and $\operatorname{pd}_R(X \otimes_R X) = \infty$.

To establish these claims, first note that $\{z, w, x + y\}$ is an *R*-regular sequence. So $pd_R(M) = 3$. Moreover, as *M* is cyclic, we see that $pd_R(M \otimes_R M) = 3$. Also, since *N* is torsion-free, we conclude that $pd_R(N \otimes_R N) = 2$; see, for example, A.5.

There is a short exact sequence of the form:

$$0 \to R \xrightarrow[w]{z} R^{\oplus 3} \to N \to 0.$$

We obtain, by tensoring this short exact sequence with M, the following exact sequence of R-modules:

$$M \xrightarrow{\begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}} M^{\oplus 3} \to M \otimes_R N \to 0.$$

This exact sequence implies that $M \otimes_R N \cong M^{\oplus 2} \oplus (M/yM) \cong M^{\oplus 2} \oplus k$. Therefore we conclude that $pd_R(M \otimes_R N) = \infty$. Now we set $X = M \oplus N$. Then it follows that $pd_R(X) = 3$ and $pd_R(X \otimes_R X) = \infty$ because $M \otimes_R N$ is a direct summand of $X \otimes_R X$.

Some examples about Question 1.3. In this section we construct two examples giving a negative answer to Question 1.3; see also [11, section 5] for some examples similar in flavor examining the finiteness of Gorenstein dimension of tensor products of modules.

Example 2.5. Let $R = k[[x, y, z]]/(x^2)$, M = R/(xy, z), and let N = R/(xz, y). Then it follows that $pd_R(M \otimes_R N) = 2$ and $pd_R(M) = \infty = pd_R(N)$.

To establish these claims, first note that $\{y, z\}$ is an *R*-regular sequence. Hence, as $M \otimes_R N \cong R/(y, z)$, we conclude that $pd_R(M \otimes_R N) = 2$.

We have that $M = T_1 \otimes T_2$, where $T_1 = R/(xy)$ and $T_2 = R/(z)$. As *z* is a non zero-divisor on both *R* and T_1 , we conclude that $\operatorname{Tor}_1^R(T_1, T_2) = 0$. Furthermore, as $\operatorname{pd}_R(T_2) = 1$, it follows that $\operatorname{Tor}_i^R(T_1, T_2) = 0$ for all $i \ge 1$. Therefore, $\operatorname{pd}_R(T_1) + \operatorname{pd}_R(T_2) = \operatorname{pd}_R(T_1 \otimes T_2) = \operatorname{pd}_R(M)$. As $\operatorname{pd}_R(T_2) < \infty$, we conclude that $\operatorname{pd}_R(M) = \infty$ if and only if $\operatorname{pd}_R(T_1) = \infty$. However, if $\operatorname{pd}_R(T_1) < \infty$ if and only *xy* is a non zero-divisor on *R*. As x(xy) = 0, we see that *xy* is a zero-divisor so that $\operatorname{pd}_R(T_1) = \infty$ and $\operatorname{pd}_R(M) = \infty$.

Note that $N = T_3 \otimes_R T_4$, where $T_3 = R/(xz)$ and $T_4 = R/(y)$. Furthermore, it follows that *y* is a non zero-divisor on *R* and T_4 , and *xz* is a zero-divisor on *R*. Therefore, by using a similar argument we used for *M*, we conclude that $pd_R(N) = \infty$.

The following fact is used for the argument of Example 2.7.

2.6. Let *R* be a local ring and let $0 \neq x \in \mathfrak{m}$. It follows that $pd_R(R/xR) < \infty$ if and only if *x* is a non zero-divisor on *R*; see [3, 6.3] and also [5, 1.2.7(2)].

The conclusion of Example 2.7, as far as Question 1.3 is considered, is stronger from that of Example 2.5: Example 2.7 points out two modules M and N over a local ring R which is not Gorenstein, where both M and N have infinite Gorenstein dimension and infinite complexity [4] such that $M \otimes_R N$ has finite projective dimension; see also [11, 5.4].

Example 2.7. Let $R = k[[x, y, z, w]]/(x^2, xy, y^2)$, M = R/(xw, z), and let N = R/(xz, w). Then it follows that $pd_R(M \otimes_R N) = 2$, and $G-\dim_R(M) = \infty = G-\dim_R(N)$.

To see these, note that $M \otimes_R N = R/(xw, z, xz, w) = R/(z, w)$; hence $pd_R(M \otimes_R N) = 2$ since $\{z, w\}$ is an *R*-regular sequence.

We can write M = T/zT, where T = R/(xw). Note that z is a non zero-divisor on T. Hence $G-\dim_R(M) < \infty$ if and only if $G-\dim_R(T) < \infty$; see, for example, [17, 1.2.9].

Next we note that *R* is Golod since it is a non-Gorenstein ring which has codimension two; see [5, (5.0.1) and 5.3.4]. So, if G-dim_{*R*}(*T*) < ∞ , then it follows $\operatorname{Ext}_{R}^{i}(T,R) = 0$ for all $i \gg 0$ [17, 1.2.7], which

implies that $pd_R(T) < \infty$ and *xw* is a non zero-divisor on *R*; see 2.6. As *xw* is a zero-divisor on *R*, we conclude that $G-\dim_R(T) = \infty$, that is, $G-\dim_R(M) = \infty$. Similarly we can observe that $G-\dim_R(N) = \infty$.

Finally we note that M and N both have infinite complexity; more precisely, the Betti numbers of both M and N grow exponentially as these modules do not have finite projective dimension and R is a Golod ring; see [5, 5.3.3(2)] for the details.

We finish this section by pointing out that Question 1.3 can also fail if projective dimension is replaced with injective dimension:

Remark 2.8. Let *R* be a Cohen-Macaulay local ring with a canonical module ω . Assume *R* admits a nontrivial semidualizing module *C*, that is, $C \ncong \omega$ and $C \ncong R$; see [35, 2.3.2] for such an example.

It follows that $id_R(C) = \infty$ [35, 2.1.8 and 2.2.13]. Suppose $id_R(C^{\dagger}) < \infty$, where $()^{\dagger} = Hom_R(-, \omega)$. Then $C^{\dagger} \cong \omega^{\oplus n}$ for some $n \ge 0$. This implies that $C \cong C^{\dagger \dagger} \cong R^{\oplus n}$, that is, $C \cong R$. Therefore $id_R(C^{\dagger}) = \infty$. On the other hand, since $C \otimes_R C^{\dagger} \cong \omega$, we have that $id_R(C \otimes_R C^{\dagger}) < \infty$; see [35, 3.1.4 and 3.1.10].

3. PROOFS OF THEOREM 1.4, THEOREM 1.6, AND SOME COROLLARIES

Proof of Theorem 1.4. This subsection is devoted to a proof of Theorem 1.4. We start by proving the first part of the theorem:

A proof of Theorem 1.4(i). We assume $pd_R(M \otimes_R N) \le n$ and $Tor_i^R(M,N) = 0$ for all i = 1, ..., n for some $n \ge 1$, and aim to show that $pd_R(M) \le n$ and $pd_R(N) \le n$. For that, we may assume R is local.

Let $X_{\bullet} = P_{\bullet} \otimes_R Q_{\bullet}$ be the total tensor product complex with differentials $\partial_{\bullet}^{X_{\bullet}}$, where P_{\bullet} and Q_{\bullet} are the minimal free resolutions of M and N, respectively. Note that $\operatorname{Tor}_{i}^{R}(M,N) = \operatorname{H}_{i}(X_{\bullet})$ for all $i \ge 0$. As $\operatorname{Tor}_{i}^{R}(M,N) = 0$ for all i = 1, ..., n, the following (minimal) complex is exact at $X_{0}, ..., X_{n}$:

(1.4.1)
$$X_{n+1} \xrightarrow{\partial_{n+1}^{X_{\bullet}}} X_n \xrightarrow{\partial_n^{X_{\bullet}}} \dots \to X_1 \xrightarrow{\partial_1^{X_{\bullet}}} X_0 \to M \otimes_R N \to 0.$$

Now, because $\Omega_R^{n+1}(M \otimes_R N) = \operatorname{im}(\partial_{n+1}^{X_{\bullet}})$ and $\operatorname{pd}_R(M \otimes_R N) \leq n$, (1.4.1) implies that $\partial_{n+1}^{X_{\bullet}} = 0$. This implies, by the definition of the differential map $\partial_{\bullet}^{X_{\bullet}}$ of the total complex, that $\operatorname{im}(\partial_i^P) \otimes_R Q_{n+1-i} = 0$ and $P_{n+1-i} \otimes_R \operatorname{im}(\partial_i^Q) = 0$ for all $i = 1, \ldots, n+1$. Therefore $\operatorname{im}(\partial_{n+1}^P) \otimes_R Q_0 = 0$ and $P_0 \otimes_R \operatorname{im}(\partial_{n+1}^Q) = 0$. As M and N are nonzero so are P_0 and Q_0 . Consequently it follows that $\operatorname{im}(\partial_{n+1}^P) = 0$ and $\operatorname{im}(\partial_{n+1}^Q) = 0$, and this implies $\operatorname{pd}_R(M) \leq n$ and $\operatorname{pd}_R(N) \leq n$, as claimed.

We proceed to prove the second part of Theorem 1.4. In fact we give three distinct proofs, each of which seems interesting on its own. Our first proof relies upon the next two lemmas:

Lemma 3.1. Let *R* be a local ring and let *M* be an *R*-module. If $\text{Ext}_R^1(M, \Omega_R(M \otimes_R N)) = 0$ for some nonzero *R*-module *N*, then *M* is free.

Proof. It suffices to observe that $\operatorname{Tor}_{1}^{R}(\operatorname{Tr} M, M \otimes_{R} N) = 0$, where $\operatorname{Tr} M$ is the Auslander transpose of M; see [30, 3.3(1)]. As $\operatorname{Tor}_{1}^{R}(\operatorname{Tr} M, M \otimes_{R} N) \cong \operatorname{Hom}_{R}(M, M \otimes_{R} N)$ [39, 3.9], we apply $\operatorname{Hom}_{R}(M, -)$ to the syzygy short exact sequence $0 \to \Omega_{R}(M \otimes_{R} N) \to F \to M \otimes_{R} N \to 0$, where F is free, and see that the induced map $\operatorname{Hom}_{R}(M, F) \to \operatorname{Hom}_{R}(M, M \otimes_{R} N)$ is surjective. This implies that each R-module homomorphism $M \to M \otimes_{R} N$ factors through F, that is, $\operatorname{Hom}_{R}(M, M \otimes_{R} N) = 0$.

We omit the proof of the following observation which can be proved by induction on r.

Lemma 3.2. Let *R* be a ring and let *A* and *B* be *R*-modules. Assume there exists an exact sequence $0 \to A^{\oplus n_r} \to \cdots \to A^{\oplus n_0} \to B \to 0$, where $n_i \ge 0$ for all $i = 0, 1, \dots, r$. If *M* is an *R*-module such that $\operatorname{Ext}^k_R(M, A) = 0$ for all $i = 1, \dots, r+1$, then $\operatorname{Ext}^1_R(M, B) = 0$.

We can now give a proof of Theorem 1.4(ii):

The first proof of Theorem 1.4(ii). It is enough to assume *R* is local and $pd_R(M \otimes_R N) \le n$, and prove *M* is free. Note that we have $pd_R(\Omega_R(M \otimes_R N)) \le n-1$, and hence $n - pd_R(\Omega_R(M \otimes_R N)) \ge 1$. Therefore Lemma 3.2 implies that $\text{Ext}_R^1(M, \Omega_R(M \otimes_R N)) = 0$. Consequently *M* is free due to Lemma 3.1.

Now we aim to give distinct proofs for the second part of Theorem 1.4 that are of independent interest. Before proceeding to the proof we establish a theorem:

Theorem 3.3. Let *R* be a local ring and let *X* and *Y* be *R*-modules such that $X \neq 0$, $pd_R(X) < \infty$, and $Tor_i^R(X,Y) = 0$ for all $i \ge 1$. Assume there exists an *R*-module *M* such that there is a surjective *R*-module homomorphism $M \xrightarrow{\alpha} X \otimes_R Y$ and $Ext_R^i(M,Y) = 0$ for all i = 1, ..., r, where $r = pd_R(X)$. Then there exists a surjective *R*-module homomorphism $M \rightarrow Y$.

Proof. Let $0 \to R^{\oplus n_r} \to \cdots \to R^{\oplus n_0} \to X \to 0$ be a minimal free resolution of *X*. Note, since $X \neq 0$, we have that $n_0 \neq 0$. As $\operatorname{Tor}_i^R(X, Y) = 0$ for all $i \ge 1$, we obtain the exact sequence

$$0 \to Y^{\oplus n_r} \to \cdots \xrightarrow{\pi} Y^{\oplus n_0} \to X \otimes_R Y \to 0,$$

where $\operatorname{im}(Y^{\oplus n_{i+1}} \to Y^{\oplus n_i}) \subseteq \mathfrak{m}Y^{\oplus n_i}$ for all $i = 0, \dots, r-1$. Then we consider the exact sequences:

$$0 \to Y^{\oplus n_r} \to \dots \to Y^{\oplus n_1} \to \operatorname{im}(\pi) \to 0 \text{ and } 0 \to \operatorname{im}(\pi) \xrightarrow{f} Y^{\oplus n_0} \xrightarrow{p} X \otimes_R Y \to 0$$

We look at the following pullback (commutative) diagram:

$$\begin{array}{ccc} 0 & \longrightarrow \operatorname{im}(\pi) & \longrightarrow \exists P & \longrightarrow M & \longrightarrow 0 \\ & & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow \operatorname{im}(\pi) & \xrightarrow{f} & Y^{\oplus n_0} & \xrightarrow{p} & X \otimes_R Y & \longrightarrow 0 \end{array}$$

As $\operatorname{Ext}_{R}^{i}(M,Y) = 0$ for all i = 1, ..., r, we use Lemma 3.2 with A = Y and $B = \operatorname{im}(\pi)$ and conclude that $\operatorname{Ext}_{R}^{1}(M, \operatorname{im}(\pi)) = 0$. Hence the top short exact sequence in the diagram splits, that is, there is a splitting map $M \to P$. Hence taking the composition of this splitting map with the map $P \to Y^{\oplus n_0}$ in the diagram, we obtain an *R*-module homomorphism $\beta : M \to Y^{\oplus n_0}$ such that $\alpha = p\beta$.

Note that, since $\operatorname{im}(f) \subseteq \mathfrak{m} Y^{\oplus n_0}$, it follows that the map $f \otimes 1_k : \operatorname{im}(\pi) \otimes_R k \to Y^{\oplus n_0} \otimes_R k$ is the zero map. Therefore $0 = \operatorname{im}(f \otimes 1_k) = \operatorname{ker}(p \otimes 1_k)$, that is, $p \otimes 1_k$ is an isomorphism. This implies that $\beta \otimes 1_k$ is surjective since $\alpha \otimes 1_k = (p \otimes 1_k) \circ (\beta \otimes 1_k)$. Consequently, by Nakayama's lemma, β is also surjective. Thus we obtain a surjection $M \to Y^{\oplus n_0}$ as claimed.

We proceed by assembling some basic results which play an important role in the sequel:

3.4. Let *R* be a ring and let $0 \to A \xrightarrow{J} B \xrightarrow{g} C \to 0$ be a short exact sequence of *R*-modules. Consider the syzygy exact sequence $0 \to \Omega_R C \to G \xrightarrow{\pi} C \to 0$, where *G* is a free *R*-module. Then, by taking the pullback of the maps *g* and π , we obtain an *R*-module homomorphism $h: G \to B$ and a short exact sequence of *R*-modules of the form

$$0 \to \Omega_R C \to A \oplus G \xrightarrow{[f,h]} B \to 0,$$

where [f, h](x, y) = f(x) + h(y) for all $x \in A$ and $y \in G$.

3.5. Let *R* be a local ring and let $f : X \to Y$ be an *R*-module homomorphism. Assume *X* has no nonzero free summand and *Y* is free. Then it follows that $im(f) \subseteq \mathfrak{m}Y$.

To establish this, we set $Y = R^{\oplus n}$ for some $n \ge 0$. Then $f = i_1 p_1 f + \ldots + i_n p_n f$, where $p_j : Y \to R$ and $i_j : R \to Y$ are the natural projections and injections, respectively. Suppose $\operatorname{im}(p_j f) \nsubseteq m$ for some j with $1 \le j \le n$. Then $\operatorname{im}(p_j f) = R$ and this gives a surjective R-module homomorphism $p_j f : X \to R$, which shows that R is a direct summand of X. Therefore $\operatorname{im}(p_j f) \subseteq \mathfrak{m}$ for each j, and this implies that $\operatorname{im}(f) \subseteq i_1(\mathfrak{m}) + \cdots + i_n(\mathfrak{m}) \subseteq \mathfrak{m} Y$.

We are now ready to provide two distinct proofs for Theorem 1.4(ii). Note that, for both of these proofs, it is enough to assume *R* is local, $pd_R(M \otimes_R N) \leq n$, and prove that *M* is free. We can write $M = H \oplus P$ for some free *R*-module *P* and an *R*-module *H* which has no nonzero free summand. Hence it suffices to show H = 0. Note also that, since $H \otimes_R N$ is a direct summand of $M \otimes N$, it follows that $pd_R(H \otimes_R N) \leq n$.

The second proof of Theorem 1.4(*ii*). Suppose $H \neq 0$ and seek a contradiction. Set $r = \mu(N)$. Then there is a surjection $H^{\oplus r} \to H \otimes_R N$. Moreover we have that $\operatorname{Ext}^i_R(H,R) = 0$ for all i = 1, ..., n. Hence we use Theorem 3.3 by setting $X = H \otimes_R N$ and Y = R, and obtain a surjection $H^{\oplus r} \twoheadrightarrow R$. This implies that *R* is a direct summand of $H^{\oplus \mu(N)}$. Then *R* is also a direct summand of *H*; see, for example [31, 1.2]. Thus H = 0 as required.

Next is another proof of Theorem 1.4(ii) which does not appeal to Theorem 3.3; we keep the same setup discussed in the paragraph preceding the second proof of Theorem 1.4(ii).

The third proof of Theorem 1.4(ii). We consider the syzygy exact sequence $0 \to \Omega_R N \xrightarrow{\alpha} F \xrightarrow{\beta} N \to 0$, where *F* is a free *R*-module. We tensor this short exact sequence with *H* and obtain the following short exact sequence:

$$(3.4.1) 0 \to \ker(1_{M'} \otimes \beta) \xrightarrow{\gamma} H \otimes F \xrightarrow{1_H \otimes \beta} H \otimes N \to 0.$$

In view of the fact recorded in 3.4, the exact sequence (3.4.1) yields an exact sequence of the form

$$(3.4.2) 0 \to \Omega_R(H \otimes_R N) \to \ker(1_H \otimes \beta) \oplus G \xrightarrow{[\gamma,\delta]} H \otimes_R F \to 0.$$

where *G* is a free *R*-module and $\delta : G \to H \otimes_R F$ is an *R*-module homomorphism.

Note that, since $H \otimes_R N$ is a direct summand of $M \otimes_R N$, it follows that $\operatorname{Ext}^i_R(H \otimes_R F, R)$ is a direct summand of $\operatorname{Ext}^i_R(M \otimes_R F, R)$ for all i = 1, ..., n. Hence we have

$$(3.4.3) \qquad \qquad \operatorname{Ext}_{R}^{i}(H \otimes_{R} F, R) = 0 \text{ for all } i = 1, \dots, n.$$

We have $\operatorname{pd}_R(\Omega_R(M'\otimes_R N)) < n$ as $\operatorname{pd}_R(M\otimes_R N) \leq n$. Thus, by letting A = R, $B = \Omega_R(H\otimes_R N)$, and $r = \operatorname{pd}_R(B)$, we conclude from (3.4.3) and Lemma 3.2 that $\operatorname{Ext}_R^1(H\otimes_R F, B) = 0$. Therefore the exact sequence (3.4.2) splits and yields a splitting map $[\varepsilon, \phi]^T \colon H \otimes_R F \to \ker(1_H \otimes \beta) \oplus G$, where $[\gamma, \delta] \circ [\varepsilon, \phi]^T = 1_{H \otimes_R F}$, that is, $\gamma \varepsilon + \delta \phi = 1_{H \otimes_R F}$. This implies that

$$(3.4.4) H \otimes_R F = \operatorname{im}(1_{M' \otimes_R F}) = \operatorname{im}(\gamma \varepsilon + \delta \phi) \subseteq \operatorname{im}(\gamma \varepsilon) + \operatorname{im}(\delta \phi) \subseteq \operatorname{im}(\gamma \varepsilon) + \mathfrak{m}(M' \otimes_R F).$$

Here, in (3.4.4), the last inclusion holds due to the fact stated in 3.5: as $\phi : H \otimes_R F \to G$ is an *R*-module homomorphism, $H \otimes_R F$ has no nonzero free summand, and *G* is free, it follows from 3.5 that $\operatorname{im}(\phi) \subseteq \mathfrak{m}G$. Now we use Nakayama's lemma and deduce from (3.4.4) that $\operatorname{im}(\gamma \varepsilon) = H \otimes_R F$. So, $\gamma \varepsilon : M' \otimes_R F \to M' \otimes_R F$, being a surjective map, is injective, that is, $\gamma \varepsilon$ is bijective. So γ is surjective and $0 = \operatorname{coker}(\gamma) \cong H \otimes_R N$; see (3.4.2). This forces H = 0 since $N \neq 0$.

A proof of Theorem 1.6. This subsection is devoted to a proof of Theorem 1.6; see Corollary 3.10. Prior to giving a proof of the theorem, we prepare some preliminary results.

3.6. Let *R* be a local ring and let *M* and *N* be *R*-modules such that *M* is maximal Cohen-Macaulay. If $pd_R(N) < \infty$, then $Tor_i^R(M,N) = 0$ for all $i \ge 1$; see [38, 2.2]. Moreover, if $id_R(N) < \infty$, then $Ext_R^i(M,N) = 0$ for all $i \ge 1$; see [28, 2.6(b)].

To establish the next corollary, we use Theorem 3.3, 3.6, and a beautiful result of Sharp [36].

Corollary 3.7. Let R be a Cohen-Macaulay local ring with a canonical module ω , and let M and N be nonzero R-modules. Assume M is maximal Cohen-Macaulay and $id_R(M \otimes_R N) < \infty$. Then there is a surjective R-module homomorphism $M^{\oplus \mu(N)} \to \omega$.

Proof. Set $T = M \otimes_R N$. Then we obtain a surjective *R*-module map $M^{\oplus r} \to T$ by tensoring the canonical surjection $R^{\oplus r} \to N$ with *M*, where $r = \mu(N)$. Note that, since $id_R(T) < \infty$, it follows that $T \cong X \otimes_R \omega$, where $X = Hom_R(\omega, T)$ and $pd_R(X) < \infty$; see [36, 2.9]. Moreover, by 3.6, we have that $Tor_i^R(X, \omega) = 0$ and $Ext_R^i(M, \omega) = 0$ for all $i \ge 1$. Therefore, by setting $Y = \omega$, we conclude from Theorem 3.3 that there is a surjective *R*-module homomorphism $M^{\oplus r} \to \omega$.

Theorem 3.8. Let R be a Cohen-Macaulay local ring and let M and N be nonzero R-modules. Assume M is maximal Cohen-Macaulay. Assume further there is an R-regular sequence $\{\underline{x}\} \subseteq \mathfrak{m}$ such that $M/\underline{x}M$ is not faithful over $R/\underline{x}R$. Then $\mathrm{id}_R(M \otimes_R N) = \infty$.

Proof. Let *N* be a nonzero *R*-module and suppose $id_R(M \otimes_R N) < \infty$. Then, by Corollary 3.7, there is a surjective *R*-module homomorphism $M^{\oplus r} \to \omega$, where $r = \dim_k(N \otimes_R k)$. Thus we obtain a surjective $R/\underline{x}R$ -module homomorphism $(M/\underline{x}M)^{\oplus r} \to \omega/\underline{x}\omega$, where $\omega/\underline{x}\omega$ is a canonical module of $R/\underline{x}R$. This surjection implies, since $\omega/\underline{x}\omega$ is faithful over $R/\underline{x}R$, that $M/\underline{x}M$ is faithful over $R/\underline{x}R$.

The following fact, which is used to prove Corollary 3.10, can be deduced from [10, 9.6.5]. Here we provide a short proof for the convenience of the reader.

3.9. Let *R* be a Cohen-Macaulay local ring with a canonical module ω and let *M* be an *R*-module such that $pd_R(M) < \infty$. Let $0 \to F_n \to \cdots \to F_0 \to M \to 0$ be a free resolution of *M*. Note, by 3.6, we have

that $\operatorname{Tor}_i(M, \omega) = 0$ for all $i \ge 1$. Hence $0 \to F_n \otimes_R \omega \to \cdots \to F_0 \otimes_R \omega \to M \otimes_R \omega \to 0$ is an exact sequence so that $\operatorname{id}_R(M \otimes_R \omega) < \infty$.

Theorem 1.6 is subsumed by the following consequence of Theorem 3.8:

Corollary 3.10. Let R be a Cohen-Macaulay local ring and let M and N be R-modules such that $M = \Omega_R L$ for some nonfree maximal Cohen-Macaulay R-module L. Then the following are equivalent:

- (i) N = 0
- (ii) $\operatorname{pd}_R(M \otimes_R N) < \infty$.
- (iii) $\operatorname{id}_R(M \otimes_R N) < \infty$.

Proof. We first assume (i) and (iii) are equivalent and prove that (ii) implies (i). For that we may assume *R* is complete and hence *R* admits a canonical module ω . Suppose $pd_R(M \otimes_R N) < \infty$. Then it follows that $id_R(M \otimes_R (N \otimes_R \omega)) < \infty$; see 3.9. This yields $N \otimes_R \omega = 0$, that is, N = 0 due to our assumption.

Next we proceed to prove (iii) implies (i). Let $\underline{x} \subseteq \mathfrak{m}$ be a maximal *M*-regular sequence. Then \underline{x} is also *L*-regular since *L* is maximal Cohen-Macaulay. It follows that $M/\underline{x}M = \Omega_{R/\underline{x}R}(L/\underline{x}L) \subseteq \mathfrak{m}(F/\underline{x}F)$ for some finitely generated free *R*-module *F*. Therefore $0 \neq \operatorname{Soc}(R/\underline{x}R) \subseteq \operatorname{Ann}_{R/\underline{x}R}(M/\underline{x}M)$ since $R/\underline{x}R$ is Artinian. So, if $\operatorname{id}_R(M \otimes_R N) < \infty$, then Theorem 3.8 implies that N = 0. This completes the proof. \Box

Some corollaries. We finish this section by giving two corollaries that demonstrate how our results can be used. The first corollary we establish yields an affirmative answer to Question 1.3 when one of the modules considered is a finite direct sum of high syzygy modules of the residue field. First we recall:

3.11. Let *R* be a Cohen-Macaulay local ring with a canonical module ω . Then *R* is regular if a finite direct sum of copies of syzygy modules of the residue field of *R* maps surjectively onto ω ; see [23, 3.6].

Corollary 3.12. Let *R* be a *d*-dimensional Cohen-Macaulay local ring and let $M = \bigoplus_{i=d}^{d+n} (\Omega_R^i k)^{\oplus a_i}$, where $n \ge 0$ and some $a_i \ge 1$. If $pd_R(M \otimes_R N) < \infty$ or $id_R(M \otimes_R N) < \infty$ for some nonzero *R*-module *N*, then *R* is regular so that $pd_R(M) < \infty$ and $pd_R(N) < \infty$.

Proof. We may assume, without loss of generality, that *R* is complete and hence admits a canonical module ω . If $\text{pd}_R(M \otimes_R N) < \infty$, then we know by 3.9 that $\text{id}_R(M \otimes_R T) < \infty$, where $T = N \otimes_R \omega$. Note that *M* is maximal Cohen-Macaulay. Hence, if either $\text{pd}_R(M \otimes_R N) < \infty$ or $\text{id}_R(M \otimes_R N) < \infty$, then Proposition 3.7 implies that a finite direct sum of copies of *M* maps surjectively onto ω . Therefore 3.11 shows that *R* is regular.

A maximal Cohen-Macaulay module over a Cohen-Macaulay local ring is said to be *Ulrich* with respect to an m-primary ideal *I* of *R* provided that M/IM is free over R/I, and the multiplicity e(I,M) of *M* with respect to *I* equals the length of M/IM. Ulrich modules [9] are of current research interest, they have been studied extensively, and examples of such modules are abundant in the literature. For example, if *R* is a one-dimensional local domain, then m^i is Ulrich with respect to m for all $i \gg 0$; see [24, 25] for definitions, details, and further examples of Ulrich modules.

Corollary 3.13. Let *R* be a Cohen-Macaulay local ring, *I* an m-primary ideal of *R*, and let *M* and *N* be nonzero *R*-modules such that $pd_R(M \otimes_R N) < \infty$. If *M* is Ulrich with respect to *I*, then *M* is free, $pd_R(N) < \infty$, and $pd_R(I) < \infty$. Therefore, if *M* is Ulrich with respect to m, then *R* is regular.

Proof. We can consider the faithfully flat extension $R \to R[x]_{\mathfrak{m}[x]}$ and hence assume R is complete with canonical ω and has infinite residue field; see, for example [20, page 48]. We pick a minimal reduction $\underline{x} = (x_1, \ldots, x_d)$ of I, where $d = \dim(R)$, so that $IM = \underline{x}M$; see [24, 3.1]. Thus the ideal $I/\underline{x}R$ of $R/\underline{x}R$ annihilates $M/\underline{x}M$. Note, by 3.9, we have that $\mathrm{id}_R(M \otimes_R T) < \infty$, where $T = N \otimes_R \omega$. As \underline{x} is a regular sequence on R, Theorem 3.3 implies that $M/\underline{x}M$ must be faithful over $R/\underline{x}R$. Thus $I/\underline{x}R = 0$, that is, $I = \underline{x}R$. Hence $\mathrm{pd}_R(I) < \infty$. Now, since $M/\underline{x}M$ is free over $R/\underline{x}R$, we conclude that M is free over R. \Box

APPENDIX A. PROOF OF THEOREM 1.2 AND SOME MISCELLANEOUS OBSERVATIONS

This section contains a proof of Theorem 1.2 as well as some observations yielding affirmative answers to Questions 1.1 and 1.3 in some special cases. We should note that one can find various conditions in the literature under which special cases of these questions are true. Here, in this appendix, we point out only a few such results which motivate us and which are interesting for us.

A proof of Theorem 1.2. In this subsection we give a proof of Theorem 1.2 which is due to Roger Wiegand [37]. The proof relies upon the facts A.1, A.2, and A.3 stated next.

A.1. Let *R* be a commutative ring, and let *M* and *N* be *R*-modules. If $M \otimes_R N$ is nonzero and free, then *M* and *N* are both projective; see, for example [8, 3.4.7].

A.2. Let *R* be a local ring and let *I* be a proper ideal of *R* such that *I* contains a non zero-divisor on *R*. Then *I* can be (minimally) generated by non zero-divisors on *R*. Here we justify this fact by giving a brief argument (taken from [37]).

We proceed by induction on the minimal number v of generators required for I. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be the set of all associated primes of R. Choose $x_1 \in I - ((\mathfrak{m}I) \cup (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t))$. Then x_1 is a non zero-divisor on R. If v = 1, then $I = Rx_1$, and hence we are done. So we assume $v \ge 2$ and choose a minimal generating set for I, say x_1, \ldots, x_v . Let J be the ideal of R generated by x_1, \ldots, x_{v-1} . Then, since J contains the non zero-divisor x_1 , it follows by the induction hypothesis that J is minimally generated by some elements y_1, \ldots, y_{v-1} , where each y_i is a non zero-divisor on R. So I is minimally generated by y_1, \ldots, y_v , where y_v is an element in $I - ((J + \mathfrak{m}I) \cup (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t))$.

The fact recorded in A.2 implies:

A.3. Let *R* be a local ring of positive depth. Then there is a sequence $\underline{x} = \{x_1, \dots, x_n\} \subseteq \mathfrak{m}$ such that each x_i is a non zero-divisor on *R* and $k \cong \bigotimes_{i=1}^n (R/x_iR)$ for some positive integer *n*. Therefore *R* is regular if the following condition holds: $\operatorname{pd}_R(R/\underline{x}R) < \infty$ whenever \underline{x} is a sequence of elements in \mathfrak{m} , each of which is a non zero-divisor on *R*.

We are now ready to give a proof of Theorem 1.2:

A proof of Theorem 1.2. ([37]) Note that, in view of A.1, it is enough to prove (i) \implies (ii) \implies (iii). The implication (i) \implies (ii) is due to the fact that, if M and N are R-modules, then $M \otimes_R N$ is a direct summand of $(M \oplus N) \otimes_R (M \oplus N)$. As A.3 establishes the implication (ii) \implies (iiii), the conclusions of the theorem hold.

Remark A.4. The argument used for the proof of Theorem 1.2 also characterizes Gorenstein rings (respectively, complete intersection rings) by using the Gorenstein dimension [2] (respectively, the complete intersection dimension [7]) instead of the projective dimension. For example a local ring *R* of positive depth must be Gorenstein if $G-\dim_R(M \otimes_R M) < \infty$ for each *R*-module *M* with $G-\dim_R(M) < \infty$.

An affirmative answer for Question 1.1. In this subsection we establish an observation advertised in the introduction, and obtain an affirmative answer for Question 1.1 in a special case; see Proposition A.7.

An *R*-module *N* over a ring *R* satisfies (S_n) for some $n \ge 0$ if depth_{*R*_p} $(N_p) \ge \min\{n, \text{depth}(R_p)\}$ for all $p \in \text{Supp}_R(N)$ (recall that depth $(0) = \infty$). Note that, over Cohen-Macaulay rings, the condition (\widetilde{S}_n) is nothing but a condition (S_n) of Serre; see, for example [21, page 3].

A.5. Let *R* be a local ring and let *M* and *N* be nonzero *R*-modules such that $pd_R(M) < \infty$. Assume at least one of the following conditions holds:

(i) N satisfies (\tilde{S}_h) , where $h = \text{pd}_R(M)$.

(ii) $\operatorname{G-dim}_R(N) < \infty$ and M satisfies (\widetilde{S}_h) , where $h = \operatorname{G-dim}_R(N)$.

Then $\operatorname{Tor}_{i}^{R}(M,N) = 0$ for all $i \geq 1$. Therefore $\operatorname{pd}_{R}(M \otimes_{R} N) < \infty$ if and only if $\operatorname{pd}_{R}(N) < \infty$. \Box

Proof. We have, since $pd_R(M) < \infty$, the following equality of Jorgensen [29, 2.2]:

 $\sup\{n \ge 0 \mid \operatorname{Tor}_n^R(M,N) \neq 0\} = \sup\{\operatorname{depth}(R_{\mathfrak{p}}) - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M \otimes_R N)\}.$

Let $\mathfrak{p} \in \operatorname{Supp}_R(M \otimes_R N)$ and proceed to show that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \ge \operatorname{depth}(R_\mathfrak{p})$; note that establishing this inequality is sufficient to conclude the vanishing of all $\operatorname{Tor}^R(M,N)$ modules due to the equality of Jorgensen.

First assume the condition in part (i) holds. Then $\operatorname{G-dim}_{R_p}(N_p) \leq h$ and hence the claim follows if $h \geq \operatorname{depth}(R_p)$ since N satisfies (\widetilde{S}_h) . If, on the other hand, $\operatorname{depth}(R_p) \geq h$, then the claim follows due to the following (in)equalities:

$$\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{pd}_{R}(M) \geq \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}}).$$

Next assume the conditions in part (ii) hold. Then it follows that $\operatorname{G-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \leq h$ and hence $\operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}}) - \operatorname{G-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \operatorname{depth}(R_{\mathfrak{p}}) - h$. So, in view of this inequality, since M satisfies (\widetilde{S}_h) , we deduce that $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \operatorname{depth}(R_{\mathfrak{p}})$. This proves the claim. \Box

Remark A.6. If the condition in part (i) of A.5 holds, then an alternative way of proving the vanishing of $\text{Tor}_{i}^{R}(M,N)$ for all $i \ge 1$ is to make use of the following interesting results:

(a) Let *R* be a local ring and let *M* be an *R*-module such that $pd_R(M) = h < \infty$. Then there exists an *R*-regular sequence $\underline{x} = \{x_1, \dots, x_h\} \subseteq \mathfrak{m}$ with the following property: $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \ge 1$ whenever *N* is an *R*-module such that \underline{x} is a regular sequence on *N*; see [19, 2.5].

(b) If *R* is a local ring and *N* is an *R*-module satisfying (\tilde{S}_h) for some $h \ge 0$, then each *R*-regular sequence of length at most *h* is also an *N*-regular sequence; see [33, 2.1].

Our observation in A.5 can be compared with Example 2.1(ii): there are *R*-modules *M* and *N* such that depth(*R*) – depth_{*R*}(*M*) = depth(*R*) – depth_{*R*}(*N*) = 1 = pd_{*R*}(*M*) = pd_{*R*}(*N*) < ∞ = pd_{*R*}(*M* $\otimes_R N$), and G-dim_{*R*}(*N*) < ∞ , but neither *M* nor *N* satisfies (\widetilde{S}_1).

Next we use A.5 and obtain an affirmative answer to Question 1.1 for a special case.

Proposition A.7. Let *R* be a local ring and let *M* be an *R*-module which is locally free on the punctured spectrum of *R*. If $pd_R(M) \le depth(R)/2$, then $pd_R(M \otimes_R M) < \infty$.

Proof. Set $d = \operatorname{depth}(R)$ and $h = \operatorname{pd}_R(M)$. Then $\operatorname{depth}_R(M) \ge d/2 \ge h = \min\{d,h\}$. Hence, since $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M) - \{\mathfrak{m}\}$, it follows that M satisfies (\widetilde{S}_h) . Therefore the claim follows from A.5.

Example A.8. ([12, 3.5]) Let $R = k[[x, y, z]]/(xy - z^2)$ and let *I* be the ideal of *R* generated by *x* and *y*. Then *R* is a two-dimensional Cohen-Macaulay ring and *I* is locally free on the punctured spectrum of *R* such that $pd_R(I) = 1$. Hence one can use, for example, A.7 and conclude that $pd_R(I \otimes_R I) < \infty$.

Some affirmative answers for Question 1.3. In this subsection we record some observations giving affirmative answers to Question 1.3; see, for example, Proposition A.12.

The first and the second parts of the next result are essentially due to Celikbas-Takahashi [16] and Gheibi [22], respectively.

A.9. Let *R* be a local ring and let *M* and *N* be nonzero *R*-modules. Assume $pd_R(M \otimes_R N) < \infty$.

- (i) If $M = \mathfrak{m}X$ for some nonzero *R*-module *X*, then *R* is regular, and $pd_R(M) < \infty$ and $pd_R(N) < \infty$.
- (ii) If $id_R(M) < \infty$, then *R* is Gorenstein and $pd_R(M) < \infty$.

Proof. The conclusion in part (i) follows from two facts: $\mathfrak{m}X \otimes_R N \cong \mathfrak{m}C$ for some nonzero *R*-module *C* [16, 2.8]. Moreover, if $pd_R(\mathfrak{m}C) < \infty$; then *R* is regular; see [32, 1.1].

For part (ii), note that we have a surjection $M^{\oplus r} \to M \otimes_R N$ for some $r \ge 1$. Hence if $id_R(M) < \infty$, then [22, 4.1] shows that *R* is Gorenstein; in that case $pd_R(M) < \infty$ since $id_R(M) < \infty$; see [32, 2.2]. \Box

A special case of A.9(ii) is:

A.10. Let *R* be a Cohen-Macaulay local ring with a canonical module ω and let *N* be a nonzero *R*-module such that $pd_R(\omega \otimes_R N) < \infty$. Then *R* is Gorenstein and hence $pd_R(N) < \infty$.

The next observation is straightforward due to A.5 and A.9:

A.11. Let *R* be a local ring and let *M* and *N* be nonzero *R*-modules such that $pd_R(M \otimes_R N) < \infty$ and $id_R(M) < \infty$. Then $pd_R(M) < \infty$ and $pd_R(N) < \infty$ if at least one of the following holds:

(i) N satisfies (\widetilde{S}_h) , where $h = \operatorname{depth}(R) - \operatorname{depth}_R(M)$.

(ii) *M* satisfies (\widetilde{S}_h) , where $h = \operatorname{depth}(R) - \operatorname{depth}_R(N)$.

The next result is essentially contained in [14, 4.7 and 4.8]; here it is reformulated in terms of the projective dimension. We add a brief argument for completeness.

Proposition A.12. Let $R = \mathbb{C}[[x_0, ..., x_d]]/(f)$ be a simple singularity, where $0 \neq f \in (x_0, ..., x_d)^2$ and *d* is a positive even integer. Let *M* and *N* be *R*-modules such that $pd_R(M \otimes_R N) < depth(R)$. If *M* and *N* are both locally free on the punctured spectrum of *R*, then $pd_R(M) < depth(R)$ and $pd_R(N) < depth(R)$. Therefore, if *M* and *N* are maximal Cohen-Macaulay, then *M* and *N* are free.

Proof. Assume *M* and *N* are both locally free on the punctured spectrum of *R*. Then $M \otimes_R N$ is torsion-free since depth_{*R*} $(M \otimes_R N) \ge 1$. Hence $M \otimes_R N \cong \overline{M} \otimes_R N \cong M \otimes_R \overline{N}$, where $\overline{(-)}$ denotes the torsion-free part of the module in question; see [26, 1.1]. So, in view of [18, 2.8 and 3.16] and [27, 1.9], we can use an argument similar to [13, 2.11] and conclude that both *M* and *N* are torsion-free, and Tor^{*R*}_{*i*}(M, N) = 0 for all $i \ge 1$. The case where *M* and *N* are maximal Cohen-Macaulay follow similarly because *R* is an isolated singularity, that is, R_p is regular for each non-maximal prime ideal p of *R*.

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