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ON THE PROOF-THEORY OF A FIRST-ORDER EXTENSION OF GL

Abstract. We introduce a first order extension of GL, called ML^3, and develop its proof theory via a proxy cut-free sequent calculus GLTS. We prove the highly nontrivial result that cut is a derived rule in GLTS, a result that is unavailable in other known first-order extensions of GL. This leads to proofs of *weak reflection* and the related *conservation result* for ML^3, as well as proofs for Craig's interpolation theorem for GLTS.

Turning to semantics we prove that ML^3 is sound with respect to arithmetical interpretations and that it is also sound *and* complete with respect to converse well-founded and transitive finite Kripke models. This leads us to expect that a Solovay-like proof of arithmetical completeness of ML^3 is possible.

Keywords: Gentzen-style logic; Gentzen's Hauptsatz; Hilbert-style logic; GL; K4; provability logic; provability predicate; modal first-order logic; sequent calculus; cut-elimination; reflection theorem; Craig interpolation; arithmetical interpretation; soundness; arithmetical completeness

1. Introduction

This paper develops the proof theory of a first-order extension of GL, the ML^3 introduced in Section 3. This logic also extends the first-order modal logic M^3 of [20, 21] that is in turn an extension of K4.

To motivate the introduction of ML^3 we need briefly to explain the genesis of M^3, which was introduced to settle a question posed in [5] that, *essentially*, was: "can we add $\Box$ to classical predicate logic so that — for classical A and B — the provability of $\Box A \to \Box B$ is tantamount to the

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classical derivability of B from A?" Using model-theoretic tools (Kripke models), [20, 21] proved "the conservation result" for M^3 that answers the question affirmatively: For classical Γ, A, B, we have that B is provable from $\Gamma \cup \{A\}$ classically iff M^3 establishes that the premises Γ, $\Box\Gamma$ prove $\Box A \to \Box B$. In other words, $\Box$ in this context is a general-purpose (as opposed to one specific to Peano arithmetic) provability operator for classical predicate logic.

The design criterion for M^3 — to solve a specific problem — dictated that just as the classical $\vdash$, in something like $\vdash A$, is oblivious to the free variables of A,[1] so must $\Box$ be in something like $\Box A$. *Thus the latter expression is always a sentence.* Nevertheless, M^3 *does* have free and bound object variables, quantifiers and predicates and is a first-order system that is however not interested in prying inside the scope of the "box" $\Box$.

ML^3 is an extension of M^3 over the same modal language, obtained by adding Löb's axiom schema $\Box(\Box A \to A) \to \Box A$, and thus this logic extends GL as well. Our motivation to study it was on one hand to introduce a *well behaved* first-order extension of GL, and on the other hand obtain a logic that still simulates classical provability $\vdash$ via the modal box, just as M^3 does.

In what sense is ML^3 "*well behaved*"? The design criterion that it must support simulation of general provability — i.e., "$\Box$ is classical $\vdash$" — entails that all object variables in a box's scope must be bound, and this in turn entails that ML^3 (just as was the case with M^3) has a cut-free Gentzenisation in which the cut rule is derivable. This in turn leads to a user-friendly proof theory. Moreover we are hoping to prove that ML^3, or some closely related variant of it, are arithmetically complete. As a first instalment toward this goal we establish here that ML^3 is semantically complete with respect to transitive and converse well-founded finite Kripke models (Subsection 7.2).

How well behaved are the various known predicative extensions of GL? First let us compare ML^3 with the QML, that is, the first-order modal logic — where $\Box A$ is not closed if A is not — whose theorems, *by definition*, are precisely those formulae of the language whose every arithmetical interpretation is Peano-provable. QML is *by construction* arithmetically complete, however, it is not recursively axiomatisable as

[1] This is true in a foundation of first-order classical logic that supports strong generalisation — that is, A unconditionally derives $\forall x A$ — as in [9, 16, 19].

shown by Vardanyan [23], and thus is *unusable* as a tool to *do* logic with. On the other hand, ML^3 is axiomatised by its very construction, and is eminently usable.[2]

One does not need to go to extreme first-order examples such as QML. There have been other predicate modal logics in the literature, beyond QML and ML^3, for example, QGL (quantified GL), that is, the "straightforward" first-order extension of GL where $\Box A$ has as free variables precisely those of A. It is known ([1]) that (a) QGL admits no cut elimination[3] and (b) it is not arithmetically complete ([10]).

Thus it appears that insisting that the language must have access to the free variables of A behind a box — in $\Box A$ — introduces undesirable side-effects and severely limits the resulting first-order modal calculus in that, apart from an artificial *appearance* of "generality" in the behaviour of its quantifiers in relation to the box $\Box$, the calculus *fails to support important metatheoretical tools, such as cut elimination and Craig interpolation*, a fact that also limits the *usability* of said "predicate" modal logics. We will return to this point shortly.

We next outline how Gentzenisation of M^3 was done in [15] and how it is accomplished here for ML^3. Thus, loc. cit. set out to develop a proof theory for M^3 and, in particular, to prove the conservation theorem without the use of semantical tools. The main proof-theoretic tool was to obtain a cut-elimination result for a Gentzen-style logic that we built as a proxy for M^3 and we called GTKS. Such an approach, building a sequent calculus proxy for a Hilbert-style logic *in order to develop the latter's proof theory*, has been common in the modal logic literature, albeit in the Boolean domain, for example, [8, 11, 22].

The modal rule "TR" below — where $\forall\Gamma$ denotes the set of all universal closures of formulae in Γ,

$$\frac{\forall\Gamma, \Box\Gamma \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$$

was central in GTKS and is a first-order version of the rule TR that appears in [11]. Of course, the added complexity of a Gentzen-style system helps as long as it admits cut-elimination. We proved in [15] that this is the case, using an adaptation of Schütte's [14] "cut-elimination" proofs in the (first-order) classical and intuitionistic settings, that is,

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[2] [20, 21] contain examples of uses of the related M^3 and so does [6].

[3] That is, its Gentzenisation does not.

rather than adopting cut as a primitive and then trying to get rid of it, we started instead without a cut rule and proved that GTKS can simulate cut (it is a derived rule). In that respect our "cut-elimination" proof was drastically different from the ones in [11, 22]. The rule TR originated in [8, 11, 22] in the form $\dfrac{\Gamma, \Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$ where it was used toward obtaining a propositional modal sequent calculus system equivalent to K4. Our version was a modification aimed to fit a predicate context (dictating the presence of $\forall\Gamma$) and we included the Ψ and Φ to avoid the inclusion of weakening/strengthening *primary* rules, thus simplifying the *cut simulation* proof.

Given that one can trivially show that M^3 is sound under arithmetical interpretations, the next natural question was: can we extend this logic so that it also becomes *complete* with respect to such interpretations and thus serve as a predicate *provability logic* — and do so without sacrificing cut eliminability?

The present paper is a contribution toward precisely this direction, introducing a proper extension of M^3, the ML^3, via the addition of the Löb axiom schema $\Box(\Box A \to A) \to \Box A$. While we do not have yet a polished proof of arithmetical completeness to include in this paper, we nevertheless include here a key result that is expected to lead us there, namely a proof that ML^3 is sound and complete with respect to *finite and converse well-founded* transitive Kripke structures. Moreover, we develop the proof theory of ML^3 by introducing a sequent calculus proxy — called GLTS this time — which while it does not include cut as a primary rule, it nevertheless can simulate cut (as a derived rule). We use the freedom from cut of GLTS to show that the modal operator in ML^3 can also act — just as it did in M^3 — as a "general provability operator" for classical logic in that the conservation result *is preserved* as we pass from M^3 to ML^3.

Related results obtained are a proof of weak reflection (essentially, if ML^3 can prove $\Box A$ from certain assumptions, then it can also prove A from closely related assumptions), that Craig Interpolation holds for both GTKS and GLTS, and we also prove two negative results: Neither schema $\Box A \to A$ (strong reflection) nor $A \to \Box A$ (strong necessitation) are provable in ML^3.

Our GLTS includes a modified "GLR" rule that we adapted from [8, 22]. This rule proves the Löb schema in this sequent calculus. The cut simulation proof that we include here has similarities to the one in

[22] but is considerably more complex due to the presence of quantifiers. This time we had to borrow techniques from loc. cit. since the Schütte-like induction that we used in [15] did not work here due to the presence of the "diagonal formula" $\Box A$ in the modified GLR:

$$\frac{\forall\Gamma, \Box\Gamma, \Box A \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$$

Nevertheless, following Schütte [14], we have chosen some seemingly "nonstandard" sequent calculus rules. Suffice it to say that our GTKS (and GLTS of the current paper) are as they should be in terms of proof power, since we have proved them to be equivalent to two semantically complete Hilbert-style systems, M^3 (and ML^3 respectively).

We can now add some final remarks to our preceding discussion regarding the choice that $\Box A$ be always closed. Rather than accept an a priori "right way" to go about this, which would be oblivious to specific design goals, we note that our choice, which was made all the way back in [20, 21], was goal-driven: What do we *want* our logic *to be able do*?

Well, our logic should force $\Box A$ to behave like $\vdash A$ whenever A is classical. It should also make $\Box A$ behave like $Pr(\ulcorner A^{*}\urcorner)$ — where Pr is Gödel's provability predicate — to obtain an arithmetical interpretation. *Neither of these two goals require* one to reach inside the scope of $\Box$, notwithstanding the assertion in [4], that

> "At Grade 3, $\Box$ is allowed to attach to open formulae, as in $\Box(x > 7)$. This is the *level needed* [our italics] to combine modality with quantifiers, for *we need* [our italics] to say such things as 'something is such that it is necessarily greater than 7' "

This assertion must be seen in the context where it is offered, for if it is applied universally, then the word "need" that occurs twice in it is too strong. The assertion applies to the use of $\Box$ as a *necessity operator* — and Quine disputes the assertion even in this context. The italicised verbs above purport to apply to all contexts. Yet, in our context — where $\Box$ can be at once a general and a Peano-specific provability operator — we note that in both statements "for some x, $\vdash A(\ldots, x, \ldots)$" (classical A) and "for some x, $Pr(\ulcorner A(\ldots, x, \ldots)\urcorner)$ (modal A) is Peano provable" the result *is independent of* x.

Our choice of syntax for $\Box A$ has subsequently led to two predicate sequent calculi, GTKS and GLTS, that are cut-free and yet can each simulate the cut rule. This much cannot be said for "the more general" systems, such as QGL, that allow $\Box A$ to have free variables.

Our approach has led to *one solution* of *how* to add the box to classical predicate calculus and still have a *usable* system, with powerful proof-theoretic tools such as "cut-elimination" and Craig interpolation — for both GTKS and GLTS, using a proof that, just like that of Maehara-Takeuti, hinges on the absence of cut in both systems — and the ability to simulate classical proofs via the conservation result. It is notable that Craig Interpolation fails in other predicate modal logics (cf. [3]). Moreover ML^3, with its finite converse well-founded Kripke models is most likely, as we hope to prove, complete under arithmetical interpretations.

Finally, we briefly mention another choice that we made in [20, 21], which we preserved here. This too makes eminent sense from the end-result, or design, point of view: Our Kripke semantics in loc. cit. and in the present paper, use the varying domain semantics ([4]), with one important *variation*: While in such semantics, "normally", $\forall x A$ is evaluated in the current world, yet $A(y)$ is evaluated over all worlds. As a result, $\forall x A \to A(y)$ does not evaluate as true, and that is unacceptable for our first-order logics and our *intended semantics* that include the goal of simulating the classical $\vdash$. Thus, in our case, $A(y)$ *is evaluated in the current world* just as $\forall x A$ is, and thus the "substitution axiom" (axiom (2) in 3.1) is true, as intended.

2. Language and terminology

This paper is a continuation of [15], and thus it is recommended that the reader reviews the first few pages of section 1 in loc. cit. that cover the specifics of the first-order language and (some of) the terminology that will be used here. However, for convenience's sake, we will restate the following points:

- We do not include equality in our logic and we employ neither constants nor functions in its alphabet, but we do allow predicates — in particular, 0-ary predicates, that is, propositional variables. Also, as in Schütte [14], we distinguish between bound $x, y, x''_{11}, \ldots$ and free $a, b, a'_2, \ldots$ variables.
- Formulae are built from *atomic* first-order formulae by the connectives $\forall, \to$ and $\perp$ and the modal $\Box$.

- Our interest in a variable a (resp. x) that may or may not occur in an expression F is denoted by $F[a]$ (resp. $F[x]$).
- For any formula, A, $\forall A$ will denote the (canonical) *universal closure* of A — "canonical" requiring the $\forall x \forall y \ldots$ prefix to be sorted in ascending lexicographic order of the bound variables.
- We say that a first-order formula is *classical* if it does not contain the $\Box$ symbol.
- We say that a first-order proof is *classical* if all the formulae involved in the proof are classical.
- For any formula, A, $\Box A$ is considered to be *closed*. The specific formation rules that guarantee closure by $\Box$ are detailed in [15]. This convention, already utilised in [20, 21, 6], is consistent with the intended semantics of the $\Box$ as general (classical) first-order provability: A is provable iff $\forall A$ is. Thus, $\forall \Box A$ is simply a verbose manner of writing $\Box A$.
- For any set of formulae, Γ, the notations $\Box\Gamma$ and $\forall\Gamma$ mean $\{\Box A : A \in \Gamma\}$ and $\{\forall A : A \in \Gamma\}$ respectively.

3. The system $\mathbf{ML}^3$

In [15] we showed that the proof theory of the logical axioms of the Hilbert-style formal systems BM ([6]) and M^3 ([20, 21]) can be investigated from within the Gentzen system GTKS. Indeed, GTKS, BM and M^3 are closely related with respect to deductive power.

We will now introduce a new Hilbert-style formal first-order extension of GL (and of M^3 of [15]), that we call ML^3, and will start developing its proof theory in the next section via a proxy Gentzen system introduced there, GLTS.

Definition 3.1 (Axioms and rules of inference for ML^3). The set of logical axioms of ML^3 is $\Lambda \cup \Box\Lambda \cup \Box\Box\Lambda$[4], where Λ consists of all instances of the following basic schemata:

(1) All tautologies
(2) $\forall x A[x] \to A[a]$
(3) $A[a] \to \forall x A[x]$, provided a does not occur in A.
(4) $\forall x(A \to B) \to (\forall x A \to \forall x B)$
(5) $\Box(A \to B) \to \Box A \to \Box B$

[4] The presence of $\Box\Box\Lambda$ helps in proving the derivability of $\Box A \to \Box\Box A$.

(6) $\Box(\Box A \to A) \to \Box A$
(7) $\Box A \to \Box \forall x A$

There are two *primary rules of inference.* Modus ponens (MP) "if A and $A \to B$, then infer B", and generalisation (Gen) "if A, then infer $(\forall x)A$". Derivability in ML^3 is denoted by $\Gamma \vdash_{ML^3} A$, that is, A is derived from hypotheses Γ. $\dashv$

Remark 3.2. Axiom schema (7) is rendered unremarkable by the necessity of its presence: If ML^3 is to achieve conservation, that for classical A and B, if $A \vdash B$, then $\Box A \to \Box B$ is ML^3-provable, then it *must* be that $\Box A \to \Box \forall x A$ is ML^3-provable since $A \vdash \forall x A$ classically (and, by the way, modally as well). The reader will note that this schema is a special case of the Barcan formula (schema) $\forall x \Box A \to \Box \forall x A$ in view of schema (3). Neither 7 nor the general Barcan formula are adopted in QGL. Rather, QGL has as axiom schemata those of GL, augmented by the classical schemata 1–4 of 3.1.

The M^3 of [15, 20, 21] uses $\Box A \to \Box\Box A$ instead of (6) and its logical axiom set is $\Lambda' \cup \Box\Lambda'$, the prime reflecting the disagreement regarding axiom (6). As is well known, axiom group (1) allows *proof by tautological implication*, that is, if $A, B, \ldots \models_{taut} X$, then X is derivable (syntactically) from the hypotheses $A, B, \ldots$[5] We use the following notation:

(i) $\Gamma \vdash_{\mathrm{ML}^3} A$ is short for "the formula A is derivable from Γ in ML^3".
(ii) $\Gamma \vdash_{\mathrm{M}^3} A$ is short for "the formula A is derivable from Γ in M^3" of [20, 21, 15]. $\dashv$

In the following series of lemmata we shall explore some of the properties of ML^3 and its relation to M^3.

We will use $A \wedge B$ as an abbreviation for $(A \to (B \to \bot)) \to \bot$. $\dashv$

Lemma 3.3. $\vdash_{\mathrm{ML}^3} \Box(\Box(A \wedge B) \to (\Box A \wedge \Box B))$

Proof. I. $\Box\big(\Box(A \wedge B \to A) \to (\Box(A \wedge B) \to \Box A)\big) \in \Box\Lambda$ (Boxed (5))

II. $\Box\Box(A \wedge B \to A) \in \Box\Box\Lambda$
III. $\Box\Box(A \wedge B \to A) \to \Box\big(\Box(A \wedge B) \to \Box A)\big)$ (By I. using (5))
IV. $\Box\big(\Box(A \wedge B) \to \Box A\big)$ (MP of II. and III.)
V. $\Box\big(\Box(A \wedge B) \to \Box B\big)$ (Similarly)
VI. $\Box\big[(\Box(A \wedge B) \to \Box A) \to \big((\Box(A \wedge B) \to \Box B) \to (\Box(A \wedge B) \to \Box A \wedge \Box B)\big)\big] \in \Box\Lambda$

[5] "$A, B, \ldots \models_{taut} X$" is short for "$A \to B \to \ldots \to X$ is a tautology".

VII. $\Box(\Box(A \wedge B) \to (\Box A \wedge \Box B))$ (Using (5) and then MP with IV. and V.) $\dashv$

Corollary 3.4. $\vdash_{\mathrm{ML}^3} \Box(\Box A \wedge \Box B \to C) \to \Box(\Box(A \wedge B) \to C)$.

Proof. $\Box\Big([\Box(A \wedge B) \to \Box A \wedge \Box B] \to [(\Box A \wedge \Box B \to C) \to (\Box(A \wedge B) \to C)]\Big) \in \Box\Lambda$, now using (5) and MP with the conclusion of the previous lemma we get $\Box(\Box A \wedge \Box B \to C) \to \Box(\Box(A \wedge B) \to C)$. $\dashv$

Lemma 3.5. $\vdash_{\mathcal{X}} \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$ *when* $\mathcal{X}$ *is* M^3 *or* ML^3.

Proof. This is well-known and uses only MP, (1), and (5). $\dashv$

Lemma 3.6. $\vdash_{\mathrm{ML}^3} \Box A \to \Box\Box A$

Proof. Relying on 3.4 and 3.5, the proof is basically the one appearing in [2]. $\dashv$

Theorem 3.7 (Weak Necessitation). *If* $\Gamma \vdash_{\mathrm{ML}^3} A$, *where* $\Gamma = \Gamma' \cup \Box\Gamma'$ *or* $\Gamma = \Box\Gamma'$, *then* $\Gamma \vdash_{\mathrm{ML}^3} \Box A$

Proof. By induction on Γ-theorems.

1. If $A \in \Lambda \cup \Box\Lambda \cup \Gamma'$ then $\Box A \in \Box\Lambda \cup \Box\Box\Lambda \cup \Box\Gamma'$ thus $\Box\Gamma' \vdash_{\mathrm{ML}^3} \Box A$ and so $\Gamma \vdash_{\mathrm{ML}^3} \Box A$.
2. If $A \in \Box\Box\Lambda \cup \Box\Gamma'$ then $A = \Box B$ for some $B \in \Box\Lambda \cup \Gamma'$. Then since $\Gamma \vdash_{\mathrm{ML}^3} \Box B$ and since $\vdash_{\mathrm{ML}^3} \Box B \to \Box\Box B$ (3.6), then using MP we get $\Gamma \vdash_{\mathrm{ML}^3} \Box\Box B$ i.e. $\Gamma \vdash_{\mathrm{ML}^3} \Box A$.
3. If $\Gamma \vdash_{\mathrm{ML}^3} B \to A$ and $\Gamma \vdash_{\mathrm{ML}^3} B$, then by the I.H. we get $\Gamma \vdash_{\mathrm{ML}^3} \Box(B \to A)$. Now using (5) and MP we get $\Gamma \vdash_{\mathrm{ML}^3} \Box B \to \Box A$. Another application of the I.H. gives us $\Gamma \vdash_{\mathrm{ML}^3} \Box B$ and now using MP we get $\Gamma \vdash_{\mathrm{ML}^3} \Box A$.
4. If $A = \forall x B$ and $\Gamma \vdash_{\mathrm{ML}^3} B$, then by the I.H. $\Gamma \vdash_{\mathrm{ML}^3} \Box B$, but since $\Gamma \vdash_{\mathrm{ML}^3} \Box B \to \Box\forall x B$ (7) then by using MP we get $\Gamma \vdash_{\mathrm{ML}^3} \Box A$. $\dashv$

Note. Let us introduce the abbreviation $L(A)$ for $\Box(\Box A \to A) \to \Box A$, i.e., axiom (6), and $L_n(A_1, \ldots, A_m)$ for the sequence $\Box^n L(A_1), \ldots, \Box^n L(A_m)$. Note that $L_n(A)$ is simply the formula $\Box^n L(A)$, and, in particular, $L_0(A) = L(A)$.

Proposition 3.8. $\vdash_{\mathrm{M}^3} L_n(A) \to L(A)$, *for* $n \geqslant 0$.

Proof. This is Lemma 2.2. in [13] $\dashv$

Corollary 3.9. *$\Gamma \vdash_{ML^3} A$ if and only if there are formulae $A_1, \ldots, A_n$ such that $L_1(A_1, \ldots, A_n), \Gamma \vdash_{M^3} A$.*

Proof. Let the axioms of ML^3 be $\Lambda \cup \Box\Lambda \cup \Box\Box\Lambda$ (Λ as in 3.1) and the axioms of M^3 be $\Lambda' \cup \Box\Lambda'$ ([20, 21]). Let us show the *only if*. We have cases:

1. A is in $\Lambda \cup \Box\Lambda$. Since both Λ and Λ' share axioms (1)-(5) and (7), we only need to consider the subcases where A is $\Box L(B)$ or $L(B)$ (boxed or unboxed axiom (6)). Now, the first subcase is immediate since trivially $\Box L(B) \vdash_{M^3} A$. On the other hand, by 3.8 and MP we have $\Box L(B) \vdash_{M^3} L(B)$, which settles the second case.
2. A is in $\Box\Box\Lambda$. Then $A = \Box\Box B$ where $\Box B \in \Box\Lambda$ and so by the previous part there are $A_1, \ldots, A_n$ such that $L_1(A_1, \ldots, A_n) \vdash_{M^3} \Box B$. Now using the fact that $\vdash_{M^3} \Box B \to \Box\Box B$, and MP, we get $L_1(A_1, \ldots, A_n) \vdash_{M^3} A$.
3. A is in Γ. Obviously $\Gamma \vdash_{M^3} A$.
4. $\Gamma \vdash_{ML^3} A$ as a result of MP on $B \to A$ and B. Then by the I.H. there are $A_1, \ldots, A_m$ and $A_{m+1}, \ldots, A_n$ such that $L_1(A_1, \ldots, A_m), \Gamma \vdash_{M^3} B \to A$ and $L_1(A_{m+1}, \ldots, A_n), \Gamma \vdash_{M^3} B$. Thus, $L_1(A_1, \ldots, A_n), \Gamma \vdash_{M^3} B \to A$ and $L_1(A_1, \ldots, A_n), \Gamma \vdash_{M^3} B$. Now apply MP to reach the sought for conclusion.
5. $A = \forall x B$ and $\Gamma \vdash_{ML^3} B$. Follows directly from the I.H.

We now turn to the *if*.

1. A is in $\Lambda' \cup \Box\Lambda'$. Again we only need to show the claim for the case $A = \Box B \to \Box\Box B$ or $A = \Box(\Box B \to \Box\Box B)$. The first case is by 3.6, and the second by 3.7 applied to the first case.
2. A is in Γ or $L_1(A_1, \ldots, A_n)$ for some $i = 1, \ldots, n$. But $\Box L(A_i) \in \Lambda \cup \Box\Lambda$ for all such i and so $\Gamma \vdash_{ML^3} A$.
3. A was obtained via MP or generalisation. Similar to the *only if*. $\dashv$

4. The Gentzen system GLTS

We now introduce the earlier announced Gentzen-style system that we will prove to be equivalent to ML^3 in terms of deductive power.

The formal rules (*schemata*) of GLTS are:

(1) Initial rules: $\Gamma, A \vdash \Delta, A$ — where A is atomic[6] — and $\Gamma, \bot \vdash \Delta$.

(2) $\rightarrow$-*left* rule: $$\frac{\Gamma, A \rightarrow \bot \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$$

(3) $\rightarrow$-*right* rule: $$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

(4) $\bot$-*right* rule: $$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A \rightarrow \bot}$$

(5) $\bot$-*left* rule: $$\frac{\Gamma \vdash \Delta, A}{\Gamma, A \rightarrow \bot \vdash \Delta}$$

(6) $\forall$-*right* rule: $\dfrac{\Gamma \vdash \Delta, A[a]}{\Gamma \vdash \Delta, \forall x A[x]}$ — as long as a, the *eigenvariable of the rule*, does not occur in the conclusion ("denominator") of the rule.[7]

(7) $\forall$-*left* rule: $$\frac{\Gamma, A[a] \vdash \Delta}{\Gamma, \forall x A[x] \vdash \Delta}$$

(8) The *modified* "GLR"[8] modal rule: $\dfrac{\forall\Gamma, \Box\Gamma, \Box A \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$ $\dashv$

The Γ and Δ in the rules are called the "side formulae" (s.f.); the resulting single formula in the "denominator" in rules (2)–(8) is the "principal formula" (p.f.) of the rule; rule (1) has A as principal formula. The single formulae displayed in the "numerators" of (2)–(8) are the "minor formulae" (m.f.). A numerator sequent is a *premise* while the denominator sequent is the *conclusion* of the rule. (3) is, intuitively, the "deduction theorem". The (arbitrary) sets Φ and Ψ in rule (8) are *weakening* and *strengthening parts* respectively (they help us easily achieve 4.4 and 4.5).

For convenience's sake, let us restate the following definitions, theorems and corollaries from [15]. Whenever a proof is supplied for said results here, it will only address the modal case, since the other cases are identical to the corresponding parts in loc. cit.

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[6] The intuition behind the rule is evident. The requirement for atomic A facilitates the basis step of (meta)proofs by induction on the "order" of (formal) theorems, as it will soon become evident.

[7] "x" is thought of as a *metavariable* that denotes *any* bound variable.

[8] Cf. [8, 11, 12] — the modifications are two: a) we require the universal closure of Γ in the numerator, rather than Γ itself, which helps to simulate axiom (7) of ML^3 b) we allow the rule to introduce "weakening" and "strengthening" parts to its conclusion.

Definition 4.1 (Theorems). A *theorem*, or *derived sequent*, is defined inductively to be one of:

(1) A sequent of the type of rule (1). We say it is derived with *order* 0 or that it is an *axiom*.
(2) A sequent of the same type as in the denominator of rule (2) provided the two corresponding sequents in the numerator are also theorems. If the latter two are derived with orders $\leqslant m$ and $\leqslant n$, then the former is derived with order $\leqslant 1 + \max(m, n)$.
(3) A sequent of the same type as in the denominator of rules (3–8) provided the corresponding sequent in the numerator is also a theorem. If the latter is derived with order $\leqslant m$, then the former is derived with order $\leqslant 1 + m$.

We say that "$\Gamma \vdash \Delta$ is (a theorem) provable (derivable) with order $\leqslant m$". $\dashv$

Remark 4.2. The concept of order in the definition above is indebted to [14]. A sequent derivable with order $\leqslant m$ is, of course, also derivable with order $\leqslant m + 1$, $\leqslant 2^{m+1}$, etc. $\dashv$

In what follows we extend the notation $F[a]$ to $\Gamma[a]$ and $(\Gamma \vdash \Delta)[a]$ when attention is to be drawn to the free variable a (that may or may not actually occur).

Theorem 4.3. *If $(\Gamma \vdash \Delta)[a]$ is provable with order $\leqslant m$ and b is some other free variable, then $(\Gamma \vdash \Delta)[b]$ is also provable with order $\leqslant m$.*

Proof. Induction on theorems. Let $(\Gamma \vdash \Delta)[a]$ be the result of rule (8). This sequent has the form $\Phi[a], \Box\Xi[a] \vdash \Box A[a], \Psi[a]$, where $\forall\Xi[a], \Box\Xi[a], \Box A[a] \vdash A[a]$ is derivable with order $< m$. By the I.H. so is $\forall\Xi[b], \Box\Xi[b], \Box A[b] \vdash A[b]$. Thus $\Phi[b], \Box\Xi[b] \vdash \Box A[b], \Psi[b]$ is derivable with order $\leqslant m$ by an application of rule (8). It is noted that, since boxed formulae have no free variables, $\Box A[a] = \Box A[b]$ and $\Box\Xi[a] = \Box\Xi[b]$; moreover $\forall\Xi[a] = \forall\Xi[b]$ since $\forall\Xi$ has no free variables either. $\dashv$

Theorem 4.4 (Weakening). *If $\Gamma \vdash \Delta$ is derived with order $\leqslant m$ then so is $\Theta, \Gamma \vdash \Delta$.*

Proof. Induction on the derivation of $\Gamma \vdash \Delta$. Our sequent $\Gamma \vdash \Delta$ was derived with order $\leqslant m$, is the result of rule (8) and has the form $\Sigma, \Box\Phi \vdash \Box A, \Pi$ ($\Box A$ p.f.) where $\forall\Phi, \Box\Phi, \Box A \vdash A$ is derived with order

$< m$. By an application of rule (8) we can obtain $\Phi, \Sigma, \Box\Phi \vdash \Box A, \Pi$ with order $\leqslant m$ (the I.H. was not needed here).

Corollary 4.5 (Strengthening). *If* $\Gamma \vdash \Delta$ *is derived with order* $\leqslant m$ *then so is* $\Gamma \vdash \Delta, \Theta$.

Corollary 4.6. $\Gamma, A \vdash \Delta, A$ *is derivable for any* A.

Note. In what follows, $\Longrightarrow$ and $\Longleftrightarrow$ are often employed inside a proof as abbreviations of the informal "if ... then" and "iff" respectively.

Proof. By 4.4 and 4.5 it suffices to show the provability of $A \vdash A$. We do induction on the complexity of A.[9] We only show the case where A is $\Box C$. Then

$$\text{I.H.} \Longrightarrow C \vdash C \overset{(4.4)}{\Longrightarrow} C, \Box C \vdash C \overset{\text{repeated } \forall\text{-left}}{\Longrightarrow} \forall C, \Box C \vdash C \overset{\text{GLR-rule}}{\Longrightarrow} \Box C \vdash \Box C \quad \dashv$$

The following are important *inversion* results. They say that our rules are, in some weak sense, reversible.

Theorem 4.7. (a) *If* $\Gamma, A \to B \vdash \Delta$ *is derivable with order* $\leqslant m$, *then each of* $\Gamma, A \to \bot \vdash \Delta$ *and* $\Gamma, B \vdash \Delta$ *are derivable with order* $\leqslant m$.

(b) *If* $\Gamma \vdash \Delta, A \to B$ *is derivable with order* $\leqslant m$, *then* $\Gamma, A \vdash \Delta, B$ *is derivable with order* $\leqslant m$.

(c) *If* $\Gamma \vdash \Delta, A \to \bot$ *is derivable with order* $\leqslant m$, *then* $\Gamma, A \vdash \Delta$ *is derivable with order* $\leqslant m$.

(d) *If* $\Gamma, A \to \bot \vdash \Delta$ *is derivable with order* $\leqslant m$, *then* $\Gamma \vdash \Delta, A$ *is derivable with order* $\leqslant m$.

(e) *If* $\Gamma \vdash \Delta, \forall x A[x]$ *is derivable with order* $\leqslant m$, *then* $\Gamma \vdash \Delta, A[a]$ *is derivable with order* $\leqslant m$ *(for any choice of* a*).*

Gentzen's Hauptsatz

Definition 4.8. A proof gives rise to a *directed graph* in a natural way. Its sequents are the graph nodes. *Edges* are introduced as follows: For every application of inference (2) — a "V-type" inference — we introduce two edges, one connecting the left premise, the other connecting the right premise, to the conclusion. For all other inferences applied in the proof — each an "I-type" inference — we introduce one edge that connects the premise to the conclusion.

[9] Number of $\to, \forall, \Box$ in the formula.

a. The *end-sequent* of a proof is the final (i.e. the bottommost) sequent of the proof.
b. A sequence of sequents $(S_i)_{0\leqslant i\leqslant n}$ in a proof P is called a *path* iff it is so in the graph-theoretic sense: For $i = 0, 1, 2, \ldots, n-1$, there is an edge connecting S_i *to* S_{i+1}. If $i < j$, we say that S_i is *above* S_j in the proof and, correspondingly, S_j is *below* S_i. We say that the path $(S_i)_{0\leqslant i\leqslant n}$ has length n—that is, the path length equals the number of participating edges. We write $l(S_0, S_n) = n$.
c. A *thread* (of P) is a path that connects an axiom to the end-sequent.
d. An inference I that is applied in a proof P *belongs to a path* $(S_i)_{0\leqslant i\leqslant n}$ iff it contributes an edge to the path.
e. Let I be an inference rule[10] and S be a sequent. We say that I is *above* S if S is either the conclusion of I or is *below* said conclusion. If I is a GLR inference and S is a sequent, then we say that I is *directly above* S if I is above S and no GLR inferences belong to the path that connects the conclusion of I to S.[11]
f. Let I_1 and I_2 be two *distinct* inferences that belong to the same path $(S_i)_{0\leqslant i\leqslant n}$ in a proof P. We say that I_1 is *above* I_2 (I_2 is *below* I_1) in P iff I_1 contributes nodes S_i and S_{i+1} and I_2 contributes nodes S_j and S_{j+1} and $i + 1 \leqslant j$. ⊣

As is usual, a Gentzen-style proof leads to a tree rather than to an undirected acyclic graph since rather than reusing nodes we use new copies (e.g., several copies of the same axiom).

Given the nomenclature introduced by the above definition we may now freely use colloquialisms such as "I is the *second* GLR rule above sequent S on some thread", meaning that there is a GLR rule I' on the thread that is directly above S, while I is directly above the premise of I'.

Definition 4.9 (Width [22]). Let $S = \Gamma \vdash A, \Delta$ be a sequent in a proof P. We denote by $\mathcal{W}(S, A)$ the set of all rules I above S (along some thread through S) that satisfy all the following

(1) I is the second GLR rule above S on the relevant thread
(2) the GLR rule I' that is between I and S on that thread has A as its diagonal formula

10 A "rule" means an *application* of a rule in this context.

11 In particular, if S is the conclusion of a GLR inference, I, then I is directly above S.

(3) No rule on the path $I \rightsquigarrow I' \rightsquigarrow S$ introduces A by weakening.[12]
(4) A is not the p.f. of I.

For a given S, the cardinality of $\mathcal{W}(S, A)$ is called the *width* of A—denoted by $w(S, A)$ with lower case w. ⊣

Clearly, if A is not boxed then (2) in 4.9 fails, and thus $w(S, A) = 0$. On the other hand, if $w(S, A) > 0$, then first off $A = \Box C$ for some C and, secondly, each I in $\mathcal{W}(S, A)$ has $\Box C$ occur in the antecedent of its conclusion. Indeed,

- $\Box C$ occurs in the antecedent of the I' premise by (2).
- By (3) in 4.9, $\Box C$ cannot be introduced by weakening, thus I must have the form $\dfrac{\forall\Phi, \Box\Phi, \forall C, \Box C, \Box B \vdash B}{\Psi, \Box\Phi, \Box C \vdash \Box B, \Xi}$ where, by (4), B is not the same as C.

Definition 4.10. If S is a sequent in a proof P, then we denote by $\mathrm{Sub}_S(P)$ the sub-proof of P that derives S. We then define:

(1) If P is a proof and S is a sequent in P, then a *cover* of S in P is a set of sequents in $\mathrm{Sub}_S(P)$ such that each thread in $\mathrm{Sub}_S(P)$ contains exactly one member of the set.
(2) A cover of S with the property that there are no GLR inferences in the paths between any of its members and S, is called a *classical cover*.
(3) If $\mathcal{U}$ is a cover of S then we define $m(\mathcal{U}) = \sum_{S' \in \mathcal{U}} l(S', S)$.
(4) A cover $\mathcal{U}$ of S is called a *maximal classical cover* if $\mathcal{U}$ is classical and $m(\mathcal{U}) = max\{m(\mathcal{U}') | \mathcal{U}'$ is a classical cover of $S\}$.[13]

Note that for any S, $\{S\}$ is a (classical) cover of S, and therefore a maximal classical cover of S always exists.

Lemma 4.11. *Let P be a proof and $S = \Gamma, \Box A \vdash \Delta$ a sequent in P, and let $\mathcal{U} = \{S_1, \ldots, S_n\}$ be a maximal classical cover of S in P, then:*

(a) *Each S' on the path from S_i to S has the form $\Gamma', \Box A \vdash \Delta'$ for $i = 1 \ldots n$. (In particular $S_i = \Gamma_i, \Box A \vdash \Delta_i$ for $i = 1 \ldots n$).*
(b) *If, for each $i = 1 \ldots n$, there is a proof of $\bar{S}_i = \Gamma_i, \forall\Sigma, \Box\Sigma \vdash \Delta_i$, then there is a proof of $\bar{S} = \Gamma, \forall\Sigma, \Box\Sigma \vdash \Delta$.*

---

[12] The only rule that can introduce weakening formulae is the GLR.

[13] Note that if we remove the word *classical* from this definition, then a maximal cover of S is always the set of initial sequents of $\mathrm{Sub}_S(P)$.

Proof. (a) Fix an i. We proceed by induction on $l = l(S_i, S)$. If $l = 0$ we are done. If $l > 0$, then, since boxed formulae cannot be introduced by a non-GLR inference — and S is the conclusion of such an inference, I — $\Box A$ must be present in the antecedent(s) of the premise(s) of I. Now, S_i must either be a premise of I and we are done, or S_i is above (one of) the premise(s) S' of I. Since $l(S_i, S') < l$ we are done by the I.H.

(b) By induction on $m = m(\mathcal{U})$. If $m = 0$, then it must be that $\mathcal{U} = \{S\}$, and the conclusion is identical to the hypothesis.

Let next $m > 0$. We examine the case where S is the conclusion of rule (2). Thus, let the premises be $S' = \Gamma', \Box A, C \vdash \Delta$ and $S'' = \Gamma', \Box A, B \to \bot \vdash \Delta$ (so $\Gamma = \Gamma', C \to B$). Let us focus on S': We have either $S' \in \mathcal{U}$, or there is an $S_k \in \mathcal{U}$, which is above S' along a path $S_k \rightsquigarrow S' \rightsquigarrow S$. Let us take the latter case first. Then — since we omit members of $\mathcal{U}$ on threads that pass through S'' — we have a *proper* subset of $\mathcal{U}$, say $\mathcal{U}'$, that is a maximal classical cover of S'. Thus $m(\mathcal{U}') < m(\mathcal{U})$ and so, by the I.H., we have a proof of $\Gamma', \forall\Sigma, \Box\Sigma, C \vdash \Delta$ if $C \neq \Box A$, or of $\Gamma', \forall\Sigma, \Box\Sigma \vdash \Delta$ if $C = \Box A$; but we can always use 4.4 to turn the second sequent to the first. If $S' \in \mathcal{U}$ then we can use the hypothesis (and possibly 4.4) to reach the same conclusion. Similarly, we have a proof of $\Gamma', \forall\Sigma, \Box\Sigma, B \to \bot \vdash \Delta$, and now applying rule (2) we get $\bar{S}$.

The other cases are done in a similar manner: if $\frac{S'}{S}$ was the rule responsible for S, and it is that $S' \notin \mathcal{U}$, then $\mathcal{U}$ is a maximal classical cover of S' as well, but its m-value is smaller and hence the I.H. applies. If $S' \in \mathcal{U}$, then the I.H. is not needed. Moreover note that, since $\forall\Sigma$ and $\Box\Sigma$ are closed, they cannot interfere with the application of rule (6). $\dashv$

Theorem 4.12 (Gentzen's *Hauptsatz*). *If $\Gamma \vdash \Delta, A$ and $\Theta, A \vdash \Phi$ are derivable, then so is $\Gamma, \Theta \vdash \Delta, \Phi$.*

Note. The formula A above is called the *cut formula*.

Proof. This proof is a slightly more involved version of the proof of the Hauptsatz given in [15], the complication arising from the presence of the diagonal formula in GLR.

Now, the proof is by induction on the ordinal

$$\alpha = \omega^3 \cdot c + \omega^2 \cdot w + \omega \cdot m + n\,, \tag{$\star$}$$

where c is the *modified complexity* of the "cut formula" A. By "modified complexity" we mean the ordinal $\omega \cdot k + r$ where k counts $\Box$ occurrences and r counts the total of all $\to, \forall$ occurrences in A. Thus $(k, r) < (k+1, r')$ and $(k, r) < (k, r+1)$ for all k, r, r'.[14] Thus the induction on α can be visualised as a quadruple induction: We do a *primary induction* on c as in the analogous proof in [15]. Induction on $w = w(\Gamma \vdash \Delta, A)$ is a *secondary* induction — S.I. — on the *width* of the cut formula A that occurs to the right of $\vdash$. A *tertiary* and even a *quaternary induction* — T.I. and Q.I. — on derivation orders for $\Gamma \vdash A, \Delta$ (order $\leqslant m$) and $\Theta, A \vdash \Phi$ (order $\leqslant n$) will be called upon at appropriate points. Because of the "weights" attached to c, w, m and n, a reduction of c by 1 reduces α even if the other parameters increase. A reduction of w also reduces α as long as c does not increase.

In all inductions — P.I., S.I., T.I. and Q.I. — Γ, Δ, Θ and Φ are parameters (i.e., *the P.I.H., S.I.H., T.I.H. and Q.I.H. hold for all* $\Gamma, \Delta, \Theta, \Phi$).

First, we deal with the cases where A is not a boxed formula. The only thing we need to show is that cuts in the proof in [15] that rely on induction on the order of the derivation, i.e. tertiary and quaternary induction, do not involve a cut formula with a higher width, and thus the corresponding argument in [15] is also valid here. This task is relatively easy since a quick examination of the proof in [15] reveals that in all the cuts whose feasibility relied on induction on the order of derivation the cut formula was the original A; and since A is not boxed then the width remained at 0.

Assume now that $A = \Box C$. In this case too we start with a tertiary induction, this time on the derivation order $\leqslant m$ of $\Gamma \vdash \Box C, \Delta$.

1. Say $\Gamma \vdash \Box C, \Delta$ is an axiom. Since $\Box C$ is not atomic, so is $\Gamma, \Theta \vdash \Phi, \Delta$.

2. We assume first that $\Box C$ is not the p.f. in the last rule applied to derive $\mathbf{S} = \Gamma \vdash \Box C, \Delta$.

Suppose that it was obtained via rule (2) from $S_3 = \Gamma' \vdash \Box C, \Delta$ and $S_4 = \Gamma'' \vdash \Box C, \Delta$ (orders $< m$). It is immediate that $w(S_3, \Box C)$, $w(S_4, \Box C) \leqslant w(S, \Box C)$ thus, using the T.I.H., both of $\Theta, \Gamma' \vdash \Delta, \Phi$ and $\Theta, \Gamma'' \vdash \Delta, \Phi$ are derivable. Applying to these two the rule (2) we derive $\Theta, \Gamma \vdash \Delta, \Phi$.

For applicable rules among (3)–(7) the argument is precisely like that for (2), except that it involves one premise.

[14] This definition of "complexity" is needed at step IV(d). The whole fuss in the definition is to have "more complexity" in $\Box A$ than in $\forall A$.

3. Case where GLR was applied — still $\Box C$ is not the p.f. thus it is a strengthening formula. The premise, obtained with order $< m$, is of the form $\forall\Gamma', \Box\Gamma', \Box D \vdash D$, where $\Gamma = \Gamma_1, \Box\Gamma_2$ and $\Gamma' \subseteq \Gamma_2$ while $\Phi = \Box D, \Phi'$. Reapplying GLR to the last sequent, with a judicious choice of weakening/strengthening, yields

$$\Theta, \overbrace{\Gamma_1, \Box\Gamma_2 - \Box\Gamma', \Box\Gamma'}^{\Gamma} \vdash \overbrace{\Box D, \Phi'}^{\Phi}, \Delta$$

4. Let now $\Box C$ be the p.f. in the last step of the derivation of $\Gamma \vdash \Box C, \Delta$: Then $\Gamma = \Xi, \Box\Psi_1$ while the premise has the following form, with $\Psi \subseteq \Psi_1$:

$$\forall\Psi, \Box\Psi, \Box C \vdash C, \text{derived with order} < m \tag{$*$}$$

Our key objective is to show that

$$\forall\Psi, \Box\Psi \vdash C \tag{$*_1$}$$

is derivable. Once we have ($*_1$) we can conclude the proof exactly as in [15].

Now, an application of GLR to ($*$), without weakening/strengthening parts, yields

$$\Box\Psi \vdash \Box C, \text{derived with order} \leqslant m. \tag{$**$}$$

This has not increased the induction variable ($\star$).

We now turn to the analysis of the derivation of $\Theta, \Box C \vdash \Phi$ by a quaternary induction (Q.I.) on its derivation order $\leqslant n$.[15] For the basis, if this sequent is an axiom, then so is $\Gamma, \Theta \vdash \Delta, \Phi$ since $\Box C$ is not atomic.

For the Q.I. step we only have the case that $\Box C$ is not the p.f. in the rule that derived $\Theta, \Box C \vdash \Phi$.

(i) $\Theta, \Box C \vdash \Phi$ was obtained via an applicable rule, let us call it "**R**", among (2)–(7); say, it was (2). The premises have the forms $S_3 = \Theta', \Box C \vdash \Phi$ and $S_4 = \Theta'', \Box C \vdash \Phi$, each derived with order $< n$. By the Q.I.H. $\Theta', \Gamma \vdash \Phi, \Delta$ and $\Theta'', \Gamma \vdash \Phi, \Delta$ are derivable. We can now apply rule **R** to them to derive $\Theta, \Gamma \vdash \Phi, \Delta$.

The cases for the other rules are similar, but with one premise. We note that for rule (6) we can employ 4.3, if necessary, to guarantee that the eigenvariable used does not appear free in Γ.

[15] Note that since the induction is on the width of the formula in the *left* upper sequent, this induction parameter does not come into play here.

(ii) Rule (8) was used to derive $\Theta, \Box C \vdash \Phi$. The only possible scenarios are that the premise was either $\forall\Omega, \Box\Omega, \Box D \vdash D$ (subcase where $\Box C$ is a weakening formula), or $\forall\Omega, \Box\Omega, \forall C, \Box C, \Box D \vdash D$, for some D; either sequent being derived with order $< n$. Since weakening does not add to derivation order, without loss of generality we accept that *the* premise was

$$\forall\Omega, \Box\Omega, \forall C, \Box C, \Box D \vdash D, \text{derived with order} < n. \qquad (***)$$

Thus $\Theta = \Theta_1, \Box\Theta_2$ and $\Omega \subseteq \Theta_2$, while $\Phi = \Box D, \Phi'$.

We now embark on establishing $(*_1)$. Notice that by remarks following Definition 4.9, the set $\mathcal{W}(\Box\Psi \vdash \Box C, \Box C)$ — if not empty — will contain rules such as

$$J = \frac{\forall\Lambda, \Box\Lambda, \forall C, \Box C, \Box B \vdash B}{\Pi, \Box\Lambda, \Box C \vdash \Box B, \Xi}$$

We thus have two cases:

Case $w(\Box\Psi \vdash \Box C, \Box C) = 0$. Let us set $S = \forall\Psi, \Box\Psi, \Box C \vdash C$, and let $\mathcal{U}$ be a maximal classical cover of S in the proof. Let $S' \in \mathcal{U}$. By Lemma 4.11(a), S' has the form $\Gamma', \Box C \vdash \Delta'$. We have three subcases:

1. $S' = \Gamma', \Box C \vdash \Delta'$ is an axiom. Then so is $S'' = \forall\Psi, \Box\Psi, \Gamma' \vdash \Delta'$.
2. S' is the conclusion of a GLR inference of the form

$$\frac{\forall\Gamma', \Box\Gamma', \Box D \vdash D}{\Box C, \Xi', \Box\Gamma' \vdash \Box D, \Theta'},$$

   where $\Box C$ is a weakening formula; cf. 4.9.
   Reusing the GLR differently, we obtain a proof for $S'' = \forall\Psi, \Box\Psi, \Xi', \Box\Gamma' \vdash \Box D, \Theta'$.
3. S' is the conclusion of a GLR inference that has $\Box C$ as its p.f. (diagonal):

$$\frac{\forall\Lambda, \Box\Lambda, \forall C, \Box C \vdash C}{\Pi, \Box\Lambda, \Box C \vdash \Box C, \Xi}.$$

   By $(**)$ via 4.4, we also have a proof of $S'' = \forall\Psi, \Box\Psi, \Pi, \Box\Lambda \vdash \Box C, \Xi$.

In all cases we succeeded to go from "the provable S'" to "the also provable S''", replacing $\Box C$ in the antecedent of S' by "$\forall\Psi, \Box\Psi$". By Lemma 4.11(b) we have a proof of $(*_1)$.

Case $w(\Box\Psi \vdash \Box C, \Box C) > 0$. We will derive $(*_1)$ once more. By the condition governing this case, we have at least one GLR inference above

(∗) that has the form J above. Let us concentrate on one such actual member of $\mathcal{W}(\Box\Psi \vdash \Box C, \Box C)$ and *eliminate* it while showing that we can still derive (∗). Below we have a partial view of the path where J and $\Box\Psi \vdash \Box C$ belong in the proof. By Definition 4.9(4), C is not the same as B — the former cannot be the p.f.

$$
\begin{array}{c}
\vdots\ \pi_0 \\
J\ \dfrac{\forall C, \Box C, \forall\Lambda, \Box\Lambda, \Box B \vdash B}{\Pi, \Box C, \Box\Lambda \vdash \Box B, \Xi} \\
\vdots\ \pi \\
\dfrac{\forall\Psi, \Box\Psi, \Box C \vdash C}{\Box\Psi \vdash \Box C}
\end{array}
$$

Examine the following diagram where although we employ cuts we will explain that they can be simulated by the rules (1)–(8) alone. To reduce notational clutter we use the abbreviation $\hat{\Gamma}$ for "$\forall\Gamma, \Box\Gamma$". The π-labels are used to keep track of paths that are "copied-and-pasted".

$$
\begin{array}{c}
\vdots\ \text{cf. } 4.6 \\
\mathrm{GLR}_1\ \dfrac{\forall B, \Box B \vdash B}{\forall B, \Box B, \Pi, \underbrace{\Box C}_{\text{weaken.}}, \Box\Lambda \vdash \Box B, \Xi} \\
\vdots\ \pi;\ \hat{B} \text{ is s.f.} \\
\mathrm{GLR}''\ \dfrac{\hat{B}, \hat{\Psi}, \Box C \vdash C}{\Box\Psi, \Box B \vdash \cancel{\Box C}}
\end{array}
\qquad
\begin{array}{c}
\vdots\ \pi \\
\hat{\Psi}, \Box C \vdash C \\
\vdots\ \forall\text{-right} \\
\hat{\Psi}, \Box C \vdash \cancel{\forall C}
\end{array}
\quad
\begin{array}{c}
\vdots\ \pi_0 \\
\cancel{\forall C}, \Box C, \hat{\Lambda}, \Box B \vdash B
\end{array}
$$

$$
\dfrac{\hat{\Psi}, \Box C \vdash \cancel{\forall C} \qquad \cancel{\forall C}, \Box C, \hat{\Lambda}, \Box B \vdash B}{\cancel{\Box C}, \hat{\Psi}, \hat{\Lambda}, \Box B \vdash B}\ \mathrm{Cut}_1
$$

$$
\dfrac{\Box\Psi, \Box B \vdash \cancel{\Box C} \qquad \cancel{\Box C}, \hat{\Psi}, \hat{\Lambda}, \Box B \vdash B}{\hat{\Psi}, \hat{\Lambda}, \Box B \vdash B}\ \mathrm{Cut}_2
$$

$$
\begin{array}{c}
\mathrm{GLR}'''\ \dfrac{\hat{\Psi}, \hat{\Lambda}, \Box B \vdash B}{\underbrace{\Box C, \Pi,}_{\text{Weaken.}}\ \Box\Psi, \Box\Lambda \vdash \Box B, \Xi} \\
\vdots\ \pi;\ \Box\Psi \text{ is s.f.}
\end{array}
$$

$$
\mathrm{Cut}_3\ \dfrac{\dfrac{\hat{\Psi}, \Box C \vdash C}{\Box\Psi \vdash \cancel{\Box C}} \qquad \begin{array}{c}\vdots\ \pi \\ \hat{\Psi}, \cancel{\Box C} \vdash C\end{array}}{\hat{\Psi} \vdash C}
$$

- The conclusion of Cut_1 is derivable by the P.I.H.
- The conclusion of Cut_2 is derivable because GLR_1 does not contribute to the width (cf. 4.9(3)) since $\Box C$ is weakening, and thus $w(\Box\Psi, \Box B \vdash \Box C, \Box C) <$ *original* $w(\Box\Psi \vdash \Box C, \Box C)$, allowing the use of the S.I.H.

- The conclusion of Cut_3 is derivable, again utilising the S.I.H. Notice that we removed the GLR J — depicted above at the end of the π_0-path — from $\mathcal{W}(\Box\Psi \vdash \Box C, \Box C)$ and GLR_1 was *not* added. Thus we have reduced the *original* $w(\Box\Psi \vdash \Box C, \Box C)$ by 1.

By reference to the proof in [15], this concludes the present proof. $\dashv$

5. Comparing ML^3 and GLTS

In this section and later we use the notational convention

(i) $\Gamma \vdash_{\mathrm{GLTS}} \Delta$ is short for "the sequent $\Gamma \vdash \Delta$ is derivable in GLTS".
(ii) $\Gamma \vdash_{\mathrm{GTKS}} \Delta$ is short for "the sequent $\Gamma \vdash \Delta$ is derivable in GTKS of [15]".

The main theorem of this section is:

THEOREM 5.1. *If* $\Gamma \vdash_{\mathrm{GLTS}} \Delta$ *then* $\Gamma \vdash_{\mathrm{ML}^3} \bigvee \Delta$, *and if* $\Gamma \vdash_{\mathrm{ML}^3} A$, *then* $\forall\Gamma \vdash_{\mathrm{GLTS}} A$.

As in [15], $\bigvee \Delta$ means $\bigvee_{A\in\Delta} A$. Furthermore, $\Gamma \vdash_X \Delta$ means that the sequent $\Gamma \vdash \Delta$ is derivable in X, if X is a Gentzen-style system. If Y is a Hilbert-style system, then $\Gamma \vdash_Y A$ means that A is derivable in Y from hypotheses Γ.

LEMMA 5.2. 1. $\Gamma \vdash_{\mathrm{GTKS}} \Delta \implies \Gamma \vdash_{\mathrm{GLTS}} \Delta$.
2. $\Gamma \vdash_{\mathrm{GLTS}} \Delta \iff \exists A_1 \ldots \exists A_n$ *such that* $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\mathrm{GTKS}} \Delta$.

PROOF. 1. Since both share the same non-modal inference rules, we only need to show that GLTS can simulate the rule TR of GTKS.

Indeed, if $\forall\Gamma, \Box\Gamma \vdash_{\mathrm{GLTS}} A$ then by 4.4, $\forall\Gamma, \Box\Gamma, \Box A \vdash_{\mathrm{GLTS}} A$ and thus (8) derives $\Sigma, \Box\Gamma \vdash_{\mathrm{GLTS}} \Box A, \Xi$.

2. First assume the left hand side. The proof is by induction on the order of derivation.

a. If $\Gamma \vdash_{\mathrm{GLTS}} \Delta$ is an axiom then so is $L_1(A), \Gamma \vdash_{\mathrm{GTKS}} \Delta$ for any A.

b. If $\Gamma \vdash_{\mathrm{GLTS}} \Delta', \forall x A(x)$ is the conclusion of rule (6) applied to the premise $\Gamma \vdash_{\mathrm{GLTS}} \Delta', A(a)$, then by the I.H. there are $A_1, \ldots, A_n$ such that $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\mathrm{GTKS}} \Delta', A(a)$. Now apply rule (6),[16] bearing in mind that $L_1(A_1, \ldots, A_n)$ is a set of closed formulae, to obtain $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\mathrm{GTKS}} \Delta', \forall x A(x)$.

[16] Which both systems share.

(The other non-modal rules are done similarly.)

c. If $\Sigma, \Box\Gamma \vdash_{\text{GLTS}} \Box A, \Xi$ is the conclusion of rule (8) applied to the premise $\forall\Gamma, \Box\Gamma, \Box A \vdash_{\text{GLTS}} A$, then, by the I.H., $L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma, \Box A \vdash_{\text{GTKS}} A$ for some $A_1, \ldots, A_n$.

Using rule (3) we get $L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma \vdash_{\text{GTKS}} \Box A \to A$ and, via 4.4, $\forall\Psi, \Box\Gamma \vdash_{\text{GTKS}} \Box A \to A$ with the appropriate Ψ to make TR applicable. The latter then yields $L_1(A_1, \ldots, A_n), \Sigma, \Box\Gamma \vdash_{\text{GTKS}} \Box(\Box A \to A), \Xi$. Also, it is readily seen that $\Box L(A), \Box(\Box A \to A) \vdash_{\text{GTKS}} \Box A$. Now, by cutting the formula $\Box(\Box A \to A)$ we derive $L_1(A, A_1, \ldots, A_n), \Sigma, \Box\Gamma \vdash_{\text{GTKS}} \Box A, \Xi$.

Next, assume the right hand side. Thus, we have $A_1, \ldots, A_n$ such that $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{GTKS}} \Delta$. By part 1, $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{GLTS}} \Delta$.

Now, 3.8 and the results in [15] yield $\Box L(A_i) \vdash_{\text{GTKS}} L(A_i)$ and, by part 1, $\Box L(A_i) \vdash_{\text{GLTS}} L(A_i)$. Thus, applying GLR, we obtain $\vdash_{\text{GLTS}} \Box L(A_i)$, for all $i = 1 \ldots n$. It follows that we can remove $L_1(A_1, \ldots, A_n)$ from $L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{GLTS}} \Delta$ using repeated cuts. $\dashv$

Proof of 5.1. $\Gamma \vdash_{\text{GLTS}} \Delta \overset{5.2}{\Longleftrightarrow} \exists A_1 \ldots \exists A_n : L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{GTKS}} \Delta \overset{[15]}{\Longrightarrow} \exists A_1 \ldots \exists A_n : L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{M}^3} \bigvee \Delta \overset{3.9}{\Longleftrightarrow} \Gamma \vdash_{\text{ML}^3} \bigvee \Delta$.

Conversely,

$\Gamma \vdash_{\text{ML}^3} A \overset{3.9}{\Longleftrightarrow} \exists A_1 \ldots \exists A_n : L_1(A_1, \ldots, A_n), \Gamma \vdash_{\text{M}^3} A \overset{[15]}{\Longrightarrow} \exists A_1 \ldots \exists A_n : L_1(A_1, \ldots, A_n), \forall\Gamma \vdash_{\text{GTKS}} A \overset{5.2}{\Longrightarrow} \forall\Gamma \vdash_{\text{GLTS}} A$. $\dashv$

6. Weak reflection

Toward a proof of 6.4 below, we state without proof the following two results (proved in [15]):

PROPOSITION 6.1 (GTKS vs M^3). *If $\Gamma \vdash_{GTKS} \Delta$, then $\Gamma \vdash_{\text{M}^3} \bigvee \Delta$. In the other direction, if $\Gamma \vdash_{\text{M}^3} A$, then $\forall\Gamma \vdash_{GTKS} A$.*

PROPOSITION 6.2. *If Σ, Δ are classical, the GTKS-derivability of $\Sigma, \Box\Gamma \vdash \Delta$ implies that of $\Sigma \vdash \Delta$ with a proof that does not use the TR rule and employs only classical formulae.*

Note. In loc. cit. the additional assumption that Γ is classical was never used in the proof of 6.2, thus it is not made here.

PROPOSITION 6.3. *If Γ, Δ are classical, then the derivability of $\Gamma, \Box\Theta \vdash \Box A, \Delta$ in GTKS implies that of $\Gamma, \forall\Theta, \Box\Theta \vdash A, \Delta$.*

Note. Again, there was a restriction in the statement of 6.3 as given in loc. cit. that Θ is classical. Given that the said restriction was not used anywhere in the proof, we have also lifted it in the proposition statement here. We emphasise, as it was also emphasised in loc. cit., that we do not require A to be classical either.

PROPOSITION 6.4 (Weak Reflection for ML^3). *If* $\Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} \Box A$, *where* Γ *is classical, then* $\Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} A$ *as well.*

Note. A is not assumed to be classical.

PROOF. $\Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} \Box A \stackrel{3.9}{\Longrightarrow} \exists A_1, \ldots, A_n : L_1(A_1, \ldots, A_n), \Gamma, \Box\Gamma \vdash_{\mathrm{M}^3} \Box A \stackrel{6.1}{\Longrightarrow} \exists A_1, \ldots, A_n : L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma \vdash_{\mathrm{GTKS}} \Box A \stackrel{6.3}{\Longrightarrow} \exists A_1, \ldots, A_n : \forall L_0(A_1, \ldots, A_n), L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma \vdash_{\mathrm{GTKS}} A \stackrel{3.8\ \&\ 4.12}{\Longrightarrow} \exists A_1, \ldots, A_n : L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma \vdash_{\mathrm{GTKS}} A \stackrel{5.2}{\Longrightarrow} \forall\Gamma, \Box\Gamma \vdash_{\mathrm{GLTS}} A \stackrel{5.1}{\Longrightarrow} \Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} A$. A short explanation for step $\stackrel{6.3}{\Longrightarrow}$: The premise $L_1(A_j)$ — i.e., $\Box L_0(A_j)$ — is a "$\Box\Theta$", thus, by 6.3, contributes a $\forall L_0(A_j)$ to the premise. But $L_0(A_j)$ is closed. ⊣

COROLLARY 6.5 (Conservation Theorem for ML^3). *If* Γ *and* A *are classical, then* $\Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} \Box A$ *implies that we have a classical proof of* A *from* Γ.

PROOF. As in the proof of the previous proposition, we can deduce that $\exists A_1, \ldots, A_n : L_1(A_1, \ldots, A_n), \forall\Gamma, \Box\Gamma \vdash_{\mathrm{GTKS}} A$. By 6.2 we obtain a classical proof of $\forall\Gamma \vdash_{\mathrm{GTKS}} A$ and hence (using 6.1) of $\Gamma \vdash_{\mathrm{M}^3} A$ and, by 3.9, of $\Gamma \vdash_{\mathrm{ML}^3} A$.[17] ⊣

Note. The converse is trivially true, as the assumption leads to $\Gamma, \Box\Gamma \vdash_{\mathrm{ML}^3} A$ and then we apply weak necessitation.

What about strong reflection and strong necessitation?

Neither $\Box p \to p$ (strong reflection) nor $p \to \Box p$ (strong necessitation) are provable within ML^3, where p is a propositional variable: Note that if $\vdash_{\mathrm{ML}^3} \Box p \to p$ then $\vdash_{\mathrm{GLTS}} \Box p \to p$, hence $\Box p \vdash_{\mathrm{GLTS}} p$. By rule (8) we have $\vdash_{\mathrm{GLTS}} \Box p$ and cutting it with the first sequent we obtain $\vdash_{\mathrm{GLTS}} p$ which is untenable: $\vdash p$ is neither a GLTS axiom, nor the consequence (denominator) of any rule.

[17] It is quite simple to establish that any classical proof in M^3 is a classical proof in ML^3.

Similarly, if $\vdash_{\mathbf{ML}^3} p \to \Box p$, then $\vdash_{\mathbf{GLTS}} p \to \Box p$ and hence $p \vdash_{\mathbf{GLTS}} \Box p$. Now, $p \vdash \Box p$ is not a (GLTS) axiom and p cannot be the p.f. of any rule. Can $\Box p$ be such a p.f.? Well, if so, the premise must be of the form $\forall\Gamma, \Box\Gamma, \Box p \vdash p$. Now $\Gamma \neq \emptyset$ since $\Box p \vdash p$ is not derivable as we have just seen. But then the result — $p \vdash \Box p$, must contain at least one boxed formula to the left of $\vdash$, which is not the case.

7. Semantic completeness of $\mathbf{ML}^3_+$ and $\mathbf{ML}^3$

7.1. Preliminaries

The standard semantic approach to modal logic is due to Kripke, who in 1959 introduced his "possible worlds" semantics [7]. This approach involves a *frame*, which is a pair, $\langle W, R\rangle$, consisting of a non-empty set of *possible worlds*, W, and a binary *accessibility* relation, R, on W. Thus, for example, if w_1 and w_2 are possible worlds in W then we say that w_2 is *accessible from* w_1 iff $w_1 R w_2$.

We will use here a particular type of a Kripke frame, called a *pointed Kripke frame* (cf. [17]), which is a triplet $\langle W, R, w_0\rangle$ such that $\langle W, R\rangle$ is a frame and $w_0 \in W$ is an R-minimum, that is, for all $w \in W$, $w_0 = w$ or $w_0 R w$.

A *Kripke structure*, $\mathcal{M}$, is a pair $(\langle W, R, w_0\rangle, \{(M_w, I_w) : w \in W\})$ where $\langle W, R, w_0\rangle$ is a pointed frame, and I_w is an *interpretation* (on each $w \in W$ — cf. [21]) where M_w is the *domain* of individuals associated with w. Basically, an interpretation (or sometimes, a *valuation*), I_w, effects a truth-assignment, for each world $w \in W$, from closed formulae to the set $\{f, t\}$, so that for any closed formula A in w, $I_w(A) = t$ indicates that A is true in w. For all modal formulae B, not necessarily closed, B is true in w iff $I_w(\forall B) = t$.[18] We write, $w \models B$ ([17] writes $w \Vdash B$ and says "w forces B", terminology that we often use here).

For the record (cf. [21]), each interpretation I_w acts on a countable (finite or enumerable) domain M_w associated with w, and is defined by induction on the complexity of *closed formulae*. The interesting points to note are two: one, for each w, $I_w(\forall x B[x])$ is defined "locally" — that

[18] The W_w are just a sets. The functions I_w effect a first-order modal interpretation of formulae. Apart from the slight departure mentioned in the last paragraph of the Introduction, the definition of the various I_w is unremarkable, and details are given in [20, 21].

is, it is true iff all of $B[x := i]$ ($i \in M_w$) are forced by w; two, $I_\omega(\Box B)$ is defined "globally" — that is, it is true iff all w' satisfying wRw' force $\forall B$.

We say that B is *true in a structure* $\mathcal{M}$ iff $w_0 \models B$ — i.e., is forced by the "start-world" of the frame. We also say that the structure is a *model* of B and write $\models_{\mathcal{M}} B$. If, for some $\mathcal{S}$, the foregoing holds for all $B \in \mathcal{S}$, we call $\mathcal{M}$ a model of $\mathcal{S}$ and write $\models_{\mathcal{M}} \mathcal{S}$. We write $\models \mathcal{S}$ — no subscript — iff every structure is a model of $\mathcal{S}$. Semantic implication, $\mathcal{S} \models B$, means that every model of $\mathcal{S}$ is a model of B as well.

The parallel notion (to truth) of *validity* of C in $\mathcal{M}$ — i.e., *truth in all worlds* of $\mathcal{M}$ — will not concern us since it is tantamount to the truth of $\Box C \wedge C$ at w_0.

In what follows, and in analogy with the term *well-founded*, we will call a (binary) relation R *converse well-founded* iff its converse, R^{-1}, is well-founded. The next result is easily verifiable and well-known:

PROPOSITION 7.1. *Let $\langle W, R\rangle$ be a frame such that W is finite and R is transitive, then R is converse well-founded if and only if R is irreflexive.*

Now, returning to our development, it is known that GL is sound with respect to Kripke models equipped with converse well-founded accessibility relations.

It is also known that the converse to this property is true for K4 and GL: that is, if A is true in all (pointed) Kripke models whose accessibility relation is transitive (resp. transitive and converse well-founded), then A is a theorem of K4 (resp. of GL).

Reference [21] introduced the system (which we here call) M^3_+ whose language and axioms are those of M^3 but is also equipped with the equality predicate, constant and function symbols, and also the axioms:

1. $a = a$
2. $\Box(a = a)$
3. $s = t \to A[a := s] \leftrightarrow A[a := t]$, for all terms s, t.
4. $\Box(s = t \to A[a := s] \leftrightarrow A[a := t])$, for all terms s, t.

The cited paper showed the completeness of M^3_+ with respect to all pointed Kripke frames whose accessibility relation is transitive.

In the next subsection, we build on techniques from loc. cit. and show that the system ML^3_+ (which is obtained from ML^3 in the identical manner that M^3_+ was from M^3) is complete with respect to the class of pointed Kripke frames whose accessibility relation is transitive and converse well-founded. In what follows we view ML^3_+ as M^3_+ with the

Löb axiom added, thus the logical axioms are the schemata of Definition 3.1 plus $\Box A \to \Box\Box A$ (explicitly listed), along with their (singly, *not* singly *and* doubly) boxed versions: in what follows, we denote all these axioms by $\widetilde{\Lambda} \cup \Box\widetilde{\Lambda}$. The reason we opt to add $\Box A \to \Box\Box A$ explicitly is convenience: Makes the inclusion of the $\Box\Box$ versions of the schemata in 3.1 redundant, as the only technical purpose for including $\Box\Box\Lambda$ was toward deriving the schema $\Box A \to \Box\Box A$ from the remaining schemata.

Since all the preliminary definitions and major parts of the proof in [21] apply without modification here, we will only elaborate on the parts where the two proofs diverge. Therefore, it is recommended that the readers familiarise themselves with the said paper.

7.2. Semantic completeness of $\mathbf{ML}^3_+$

The following is borrowed from [21].

Lemma 7.2 (Main Semantic Lemma for M^3_+). *Let $\mathcal{T}$ be a consistent set of wfmf over the language L of M^3_+, and let N be an arbitrary enumerable set. Then there is a finite or enumerable subset M of N and a consistent Henkin completion Γ of $\mathcal{T}$ over $L(M)$ — the language L extended by adding all the members of M as new constants.*

That is, Γ is a set of wfmf over $L(M)$ such that

i. *Γ is consistent and the set-difference $\Gamma - \mathcal{T}$ is a set of sentences.*
ii. (Maximality) *For any sentence A over $L(M)$, either $A \in \Gamma$ or $\neg A \in \Gamma$, where we henceforth write "$\neg A$" for "$A \to \bot$".*
iii. (Henkin property) *If Γ proves $\exists x A$ over $L(M)$ then it also proves $A[c]$ for some $c \in M$; c is called a Henkin (witnessing) constant.*
iv. (Distinguishing Constants) *If $a \neq b$ in M (metamathematically), then $\Gamma \vdash \neg a = b$.*

Note that the maximality of Γ implies that Γ is deductively closed: for all A, if $\Gamma \vdash A$, then $A \in \Gamma$. The converse is, of course, trivial.

As in [21], *strong completeness* — "for M^3_+: if $\mathcal{T} \models A$, then $\mathcal{T} \vdash_{\mathrm{M}^3_+} A$" — is proved by arguing the contrapositive: "If, for a closed A, $\mathcal{T} \nvdash A$, then there is a model $\mathcal{M}$ of $\mathcal{T}$ that falsifies A; $I_{w_0}(A) = f$".

This is taken as is from [21] for M^3_+ and using the "standard trick" (cf. [17]) of cutting down a K4 model into a *finite* GL model — adapted here to the first-order case — we will cut down a M^3_+ model into a finite ML^3_+ model.

First let us define the *reduced set of subformulae* of a given closed formula A.

DEFINITION 7.3. For any closed formula A over the language L, the *reduced* set of closed subformulae of A, $S_r(A)$, is defined as follows:

$$S_r(A) = \{A\} \cup \begin{cases} \emptyset & A \text{ is atomic} \\ S_r(B) \cup S_r(C) & A \text{ is } B \to C \\ S_r(B) & A \text{ is } \neg B \\ S_r(\forall B) & A \text{ is } \Box B \\ \{B[x := t] : t \text{ is a closed term over } L\} & A \text{ is } \forall x B[x] \end{cases}$$

Note. The aim in the last clause is not to miss any subformulae of A of the form $\Box E$ by virtue of them "hiding" in the scope of a $\forall$. It should be obvious that as all such subformulae are closed, we will get multiple identical copies of boxed subformulae, one for each t. Thus, any one, "canonically chosen", t will do as well, rather than taking all the (infinitely many) t.

It is immediate from 7.3 that:

PROPOSITION 7.4. *For any closed formula A, the set $S_{\Box,\forall}(A)$, defined as $\{\Box B, \forall B \mid \Box B \in S_r(A)\}$, is finite.*

Let now A be closed such that $\mathcal{S} \nvdash_{\mathrm{ML}^3_+} A$. Then it is also the case that $\mathcal{S} \nvdash_{\mathrm{M}^3_+} A$. Thus, by [21], let $\mathcal{M} = (\langle W, R, w_0\rangle, \{(M_w, I_w) : w \in W\})$ be a model of $\mathcal{S}$ such that $I_{w_0}(A) = f$. Here each w is a consistent Henkin completion of $\widetilde{\Lambda} \cup \Box\widetilde{\Lambda}$ — taking $\mathcal{T} = \emptyset$ in 7.2 — while w_0 is a consistent Henkin completion of $\mathcal{S} \cup \{\neg A\}$. The I_w are defined as in loc. cit. starting with

$$\text{for each Boolean variable } q,\ \mathcal{I}_w(q) = \boldsymbol{t} \text{ iff } q \in w \tag{$\diamondsuit$}$$

and for each n-ary predicate ϕ, and $i_1, \ldots i_n$ in M_w,

$$\mathcal{I}_w(\phi(\vec{i}_n)) = \boldsymbol{t} \text{ iff } \phi(\vec{i}_n) \in w, \tag{$\heartsuit$}$$

where M_w is the "domain M" associated with the Henkin completion w as in 7.2.

We now *identify* any two worlds of $\mathcal{M}$ that contain precisely the *same subset* of $S_{\Box,\forall}(A)$ — that is, if $w', w'' \in W$ then $w' \sim w''$ iff $w' \cap S_{\Box,\forall}(A) = w'' \cap S_{\Box,\forall}(A)$. Next, we first pick w_0 as the *representative* of the (equivalence) class of which it is a member, and then we arbitrarily pick a *representative* from each one of the remaining classes. We will call these world-representatives α, β, γ, etc. rather than w, w', w_2, etc. In particular, we will let $\alpha_0 = w_0$. Thus, $A \notin \alpha_0$. By 7.4, we obtain a finite set of worlds $W_r = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$. The corresponding domains are naturally called M_{α_j}.

Now that we have cut down the set of worlds into a finite subset, our task is to redefine the accessibility relation R, to get an irreflexive and transitive relation, $\widehat{R}$ on the set W_r.

Definition 7.5 (The Accessibility Relation modulo a fixed A). Let $\beta, \gamma \in W_r$. Then $\beta \widehat{R} \gamma$ iff

i. For every $\Box B \in \beta$ that *is a reduced subformula of* A we have that $\forall B$ and $\Box B$ are in γ.
ii. There is some reduced subformula $\Box E$ of A such that $\Box E \in \gamma$ but $\Box E \notin \beta$.

Note. With 7.5 recorded, we now further reduce W_r, *if applicable*, by dropping all the α_j that are not $\widehat{R}$-accessible from α_0. We continue calling it "W_r".

Proposition 7.6. *$\widehat{R}$ is irreflexive and transitive.*

Proof. If $\beta \widehat{R} \beta$ for some $\beta \in W_r$ then, by 7.5, there must be some $\Box E \in \beta$ such that $\Box E \notin \beta$ which is impossible, therefore $\widehat{R}$ is irreflexive.

Now, assume that $\beta \widehat{R} \gamma \widehat{R} \delta$ for β, γ, δ in W_r. Let $\Box B$ be a subformula of A such that $\Box B \in \beta$. Then, by 7.5, $\Box B \in \gamma$ and thus $\{\forall B, \Box B\} \subseteq \delta$. Now, since $\beta \widehat{R} \gamma$ there is some $\Box E \in \gamma$ such that $\Box E \notin \beta$, but, because $\gamma \widehat{R} \delta$, we know that $\Box E \in \delta$. Thus, by definition, $\beta \widehat{R} \delta$. ⊣

We now state:

Theorem 7.7. *For the aforementioned pair $\mathcal{S}$ and A such that $\mathcal{S} \nvdash_{\mathrm{ML}^3_+} A$, there is a Kripke model $\widehat{\mathcal{M}} = (\langle W_r, \widehat{R}, \alpha_0 \rangle, \{(M_\beta, I_\beta) : \beta \in W_r\})$ of $\mathcal{S}$ such that W_r is finite, $\widehat{R}$ is transitive and irreflexive (hence also converse well-founded) and $I_{\alpha_0}(A) = f$.*

The proof hinges on the following lemma that is the counterpart of Lemma 6.3 in [21]:

Lemma 7.8. *We have fixed a sentence A as in the foregoing, as well as an $\widehat{\mathcal{M}}$ obtained by cutting down a model $\mathcal{M}$ of $\mathcal{S}$ for M^3_+ such that w_0 does* not *force A.*

For each $\alpha \in W_r$ and any given reduced subformula B of A, we have $\mathcal{I}_\alpha(B) = t$ iff $B \in \alpha$.

Proof. The proof will be done using induction on the *modified complexity* of a formula (cf. the proof of 4.12), a natural choice since we took $\forall B$ to be a "subformula" of $\Box B$.

The basis is the definition of I_α for $\alpha \in W_r$ as in ($\diamondsuit$) and ($\heartsuit$) on p. 355. For the induction step we will only develop the two interesting cases, where $B = \forall x C$ or $B = \Box C$. The other cases are handled similarly to those in Lemma 6.3 of [21].

1. $B = \forall x C$. This is argued precisely as in [21]:

Let first $I_\alpha(\forall x C) = t$. Then (Kripke semantics) it must be that $I_\alpha(C[x := i]) = t$ for all $i \in W_\alpha$, and therefore, by the I.H., $C[x := i] \in \alpha$ for all such i. Now, if $\forall x C \notin \alpha$, then, by maximality, $\neg \forall x C$ is in α; that is, $\exists x \neg C$ is. It follows that for some Henkin constant $c \in M_\alpha$, $\neg C[x := c]$ is in α contradicting the latter's consistency.

Conversely, assume that $\forall x C \in \alpha$. By Axiom (2) in 3.1, modus ponens, and deductive closure, $C[x := i]$ is in α, for all $i \in M_\alpha$. By the I.H., $I_\alpha(C[x := i]) = t$ for all such i, and by the definition of I_α we get that $I_\alpha(\forall x C) = t$.

2. $B = \Box C$. Let first $\Box C \in \alpha$. To conclude that B is forced by α, examine the arbitrary β such that $\alpha \widehat{R} \beta$ — of course, if none such exists, then there is nothing to prove. As $\Box C$ is a subformula of our "reference", A, we have (cf. 7.5) that $\Box C$ and $\forall C$ are in β. The I.H. applies to $\forall C$ (lower modified complexity than B), and thus it is forced by (is true in) β. By Kripke semantics, $I_\alpha(\Box C) = t$.

Conversely, let (arguing contrapositively; cf. [21]) $\Box C \notin \alpha$. Consider the set $X = \{\Box C, \neg \forall C\} \cup \{\forall D, \Box D \mid \Box D$ is a reduced subformula of A and $\Box D \in \alpha\}$. We write X as

$$\{\forall D_1, \Box D_1, \ldots, \forall D_r, \Box D_r, \Box C, \neg \forall C\}$$

and claim that it is consistent. If not, using propositional logic, we get:

$$\vdash \forall D_1 \wedge \Box D_1 \wedge \ldots \wedge \forall D_r \wedge \Box D_r \to (\Box C \to \forall C)$$

By $\vdash \forall C \to C$ (Axiom 2), the above yields:

$$\vdash \forall D_1 \wedge \Box D_1 \wedge \ldots \wedge \forall D_r \wedge \Box D_r \to (\Box C \to C)$$

Using weak necessitation (3.7), Axiom (5) and 3.5 we get:

$$\vdash \Box \forall D_1 \wedge \Box\Box D_1 \wedge \ldots \wedge \Box \forall D_r \wedge \Box\Box D_r \to \Box(\Box C \to C)$$

Noting that $\vdash \Box D_i \to \Box\forall D_i \wedge \Box\Box D_i$[19] and that $\vdash \Box(\Box C \to C) \to \Box C$, we obtain:

$$\vdash (\Box D_1 \wedge \ldots \wedge \Box D_r) \to \Box C$$

However, by construction, the (deductively closed) world α contains all of $\Box D_1, \ldots, \Box D_r$ but does not contain $\Box C$ — a contradiction. Therefore, the set X is *consistent*.

This fact and Lemma 7.2 imply that there must be a world, say w, in W that contains X. Now, since every world in W that is equivalent to w must contain the same subset of $S_{\Box,\forall}(A)$ as w does (and, by maximality, they also must contain the same subset of negations of members of $S_{\Box,\forall}(A)$), we immediately conclude that all the members of the equivalence class of w contain X. Let us call β the representative picked from this equivalence class for membership in W_r. In particular, notice that $\forall C \notin \beta$.

By construction of X and β, if a subformula $\Box D$ of A is in α, then it is in β. Moreover, all $\forall D$ are in β. Finally, $\Box C \in \beta$ but $\Box C \notin \alpha$, the latter by this case's main assumption. Thus, $\alpha \widehat{R} \beta$. But, $\forall C$ is not in β because $\neg\forall C$ is. Thus β forces the falsehood of $\forall C$, and we conclude that $I_\alpha(\Box C) = f$. $\dashv$

Proof of 7.7. Since $A \notin \alpha_0 = w_0$, then by the previous lemma (recall that we assume that A is closed), $\mathcal{M}$ is a Kripke model of $\mathcal{S}$ for M^3_+ such that W_r is finite, $\widehat{R}$ is transitive and converse well-founded, and $I_{\alpha_0}(A) = f$. But this model of $\mathcal{S}$ is also a model for ML^3_+ since the properties of $\widehat{R}$ make the Löb axiom (schema) true at α_0, as is well-known (e.g., [17]). $\dashv$

Remark 7.9. 1. That ML^3_+ is sound with respect to finite transitive-irreflexive Kripke structures is true with an unremarkably standard proof (adapted from that for GL) that we omit.

2. In the case of ML^3, it is easier to obtain a maximal Henkin set as in 7.2 that only satisfies the first four requirements.[20] Having that in mind, the proof of the semantic completeness of ML^3 is almost identical to the one above. $\dashv$

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[19] Boolean logic, Axiom 7 and $\vdash \Box D \to \Box\Box D$

[20] Recall that neither the equality symbol, $=$, nor constants or functions are part of the language of ML^3.

8. Interpolation

This section considers the reverse of cut for GTKS and GLTS: If we can prove $\vdash A \to B$, then we can find a formula C — the *interpolant* of A and B — that bears some relationship to A and B, so that both $\vdash A \to C$ and $\vdash C \to B$ are provable. To make "bears some relationship" precise, let us indicate by the symbol $\langle A \rangle$ (cf. [14]), the *support of* A, that is, the set of all free variables, Boolean variables and predicate letters appearing in A. For example, $\langle p \rangle = \{p\}$, $\langle \phi a \rangle = \{\phi, a\}$, $\langle \Box \phi a \rangle = \{\phi\}$, and $\langle \bot \rangle = \emptyset$. We extend the symbol so that $\langle \Sigma \rangle = \bigcup_{A \in \Sigma} \langle A \rangle$.

We now state a general version of Craig's interpolation theorem. A partition of a sequent $\Gamma \vdash \Delta$ is an expression $(\Gamma_1, \Delta_1; \Gamma_2, \Delta_2)$, where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = \Delta_1 \cup \Delta_2$. A Γ_i and a Δ_j can be empty, consistently however with the previous two equalities. Moreover, $\Gamma_1 \cap \Gamma_2 = \emptyset = \Delta_1 \cap \Delta_2$.

Theorem 8.1 (Craig's interpolation theorem for GTKS and GLTS). *If $\Gamma \vdash_{\mathcal{X}} \Delta$ where $\mathcal{X} \in \{\mathrm{GTKS}, \mathrm{GLTS}\}$ then, for any chosen partition $(\Gamma_1, \Delta_1; \Gamma_2, \Delta_2)$ of $\Gamma \vdash_{\mathcal{X}} \Delta$, an A (called an interpolant of $\{\Gamma_1, \Delta_1; \Gamma_2, \Delta_2\}$) can be found satisfying:*

(1) $\Gamma_1 \vdash_{\mathcal{X}} A, \Delta_1$ *and* $\Gamma_2, A \vdash_{\mathcal{X}} \Delta_2$.
(2) $\langle A \rangle \subseteq \langle \Gamma_1 \cup \Delta_1 \rangle \cap \langle \Gamma_2 \cup \Delta_2 \rangle$.

Proof. The proof is done by induction on the order of derivation of $\Gamma \vdash \Delta$; however, since it follows very similar lines to the well-known Maehara-Takeuti proof [18], it will be omitted. $\dashv$

9. Arithmetical soundness of ML^3

First, let us suppose that the (free) variables $v_0, v_1, \ldots$ and the (bound) variables $x_0, x_1, \ldots$ are common to the languages of ML^3 and of arithmetic. Second, let $Pr(\ulcorner \cdot \urcorner)$ denote the provability predicate in Peano Arithmetic. Following [2] we define:

Definition 9.1 (Realisation). 1. A *realistion* is a function $*$ from a set of predicate letters to formulas (not necessarily atomic) in the language of arithmetic such that for all n, if π is an n-place predicate letter in the domain of $*$, then π^* is a formula in which exactly the first n variables (i.e. $v_0, \ldots, v_{n-1}$) occur free.

2. We say that F is a *formulaic expression* if $F = F'(q_1, \ldots, q_n)$ where F' is a formula and $q_1, \ldots, q_n$ are variables (free or bound). For example, if π is a predicate letter then $\pi(x_2, v_4)$ is a formulaic expression.
3. A *realisation of a formulaic expression* F of ML^3 is a realisation whose domain contains all predicate letters occurring in F. $\dashv$

DEFINITION 9.2 (Arithmetical Interpretation — ML^3). For every *formulaic expression* F of ML^3 and realisation $*$ of F, we define the arithmetical interpretation (or *translation*) F^* of F under $*$ as follows:

1. $\perp^* = \perp$.
2. If F is the *formulaic expression* $\pi(q_1, \ldots, q_n)$ where π is a sentence letter, then F^* is the result $\pi^*(q_1 \ldots, q_n)$ of respectively substituting $q_1, \ldots, q_n$ for $v_0, \ldots, v_{n-1}$ in π^* (while systematically changing, if necessary, the bound variables of π^* in order to avoid any bound variable among $q_1, \ldots, q_n$ from being captured by a quantifier).[21]
3. If $F = A \to B$ then $F^* = A^* \to B^*$.
4. If $F = \forall x_i A[v_k := x_i]$ then $F^* = \forall x_j (A[v_k := x_j])^*$. Where, in the case that x_i occurs in $(A[v_k])^*$, then x_j is the first bound variable that does not. Otherwise $i = j$.[22]
5. If $F = \Box A$ then $F^* = Pr(\ulcorner \forall A^* \urcorner)$. $\dashv$

We only state the following lemma since its proof is quite standard.

LEMMA 9.3. *Let A and B be formulae. Then:*

1. *If* $\vdash_{\mathcal{PA}} A$ *then* $\vdash_{\mathcal{PA}} Pr(\ulcorner \forall A \urcorner)$.
2. $\vdash_{\mathcal{PA}} Pr(\ulcorner \forall (A \to B) \urcorner) \to (Pr(\ulcorner \forall A \urcorner) \to Pr(\ulcorner \forall B \urcorner))$.
3. $\vdash_{\mathcal{PA}} Pr(\ulcorner \forall (Pr(\ulcorner \forall A \urcorner) \to A) \urcorner) \to Pr(\ulcorner \forall A \urcorner)$.
4. $\vdash_{\mathcal{PA}} Pr(\ulcorner \forall A \urcorner) \to Pr(\ulcorner \forall Pr(\ulcorner \forall A \urcorner) \urcorner)$.

One can easily derive that $\vdash_{\mathcal{PA}} (\forall A)^* \leftrightarrow \forall A^*$ and that if x_i does not occur in $(A[v_j])^*$ then $(A[v_j := x_i])^* = (A[v_j])^*[v_j := x_i]$. These two facts are helpful in the proof of the main result of our section (which we omit):

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[21] Thus, if π is a 1-place sentence letter, and $\pi^* = \forall x_1(v_0 + v_0 = x_1)$ then $(\pi(x_1))^* = \forall x_2(x_1 + x_1 = x_2)$.

[22] Thus, if π is a 2-place sentence letter and $\pi^* = \forall x_1(v_0 + x_1 = v_1)$, then in order to find $(\forall x_1 \pi(v_5 := x_1, v_9))^*$ we first note that $\pi(v_5, v_9)$ contains x_1 and thus we need to examine $\forall x_2(\pi(x_2, v_9))^*$ which is equal to $\forall x_2 \forall x_1(x_2 + x_1 = v_9)$.

Lemma 9.4 (Arithmetical Soundness of ML^3). *If* $\vdash_{\mathrm{ML}^3} A$ *then for every realisation* $*$, $\vdash_{\mathcal{PA}} A^*$.

Proof. The proof can be done by induction on proofs in ML^3 (3.1): (Sketch) The interpretation, $*$, preserves tautologies since it commutes with Boolean connectives. Furthermore, the definition of $*$ (and the fact the $\mathcal{PA}$ is a first order system itself) will ensure that the interpretation of any instance of (2)–(4) in 3.1 is provable in $\mathcal{PA}$. Similarly, 9.3 implies that the interpretation of any instance of (5)–(7) is provable in $\mathcal{PA}$, and 9.3(1) implies that interpretations of boxed instances of (1)–(7) are also provable in $\mathcal{PA}$. Finally, $*$ preserves implications by MP and Gen since it commutes with Boolean connectives and (practically) commutes with $\forall$. $\dashv$

10. Conclusions

We have introduced a first-order extension ML^3 of GL and an equivalent to it Gentzen-style system GLTS.

Unlike QGL, the "natural" first-order extension of GL, ML^3 supports cut-elimination (its Gentzen-style proxy is cut free). The precise technical demonstration that cut elimination fails in QGL is found in [1] and it has its root in the fact that the underlying first-order language, unlike ours, allows $\Box A$ to have free variables.

Moreover, we proved that ML^3 supports Craig Interpolation, its modal box simulates the classical provability $\vdash$ on classical formulae, is sound with respect to arithmetical interpretations, and is semantically complete with respect to converse well-founded transitive finite Kripke structures. Remains to be seen whether the latter property may prove to be a stepping stone toward proving ML^3 to be arithmetically complete — as it was in Solovay's proof for the case of GL — thus providing the first example of a usable first-order provability logic.

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