INDEX SLIP.

PHILOSOPHICAL TRANSACTIONS.

I. *On the Propagation of Tremors over the Surface of an Elastic Solid.*

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Received June 11,—Read June 11,—Revised October 28, 1903.

Introduction.

1. This paper treats of the propagation of vibrations over the surface of a "semi-infinite" isotropic elastic solid, *i.e.*, a solid bounded only by a plane. For purposes of description this plane may be conceived as horizontal, and the solid as lying below it, although gravity is not specially taken into account.*

The vibrations are supposed due to an arbitrary application of force at a point. In the problem most fully discussed this force consists of an impulse applied vertically to the surface; but some other cases, including that of an internal source of disturbance, are also (more briefly) considered. Owing to the complexity of the problem, it has been thought best to concentrate attention on the vibrations as they manifest themselves at the free surface. The modifications which the latter introduces into the character of the waves propagated into the interior of the solid are accordingly not examined minutely.

The investigation may perhaps claim some interest on theoretical grounds, and also in relation to the phenomena of earthquakes. Writers on seismology have naturally endeavoured from time to time to interpret the phenomena, at all events in their broader features, by the light of elastic theory. Most of these attempts have been based on the laws of wave-propagation in an unlimited medium, as developed by Green and Stokes; but Lord Rayleigh's discovery† of a special type of surface-waves has made it evident that the influence of the free surface in modifying the character of the vibrations is more definite and more serious than had been suspected. The present memoir seeks to take a further step in the adaptation of theory to actual conditions, by investigating cases of *forced* waves, and by abandoning (ultimately) the restriction to simple-harmonic vibrations. Although the circumstances of actual earthquakes must differ greatly from the highly idealized state of

* Professor Bromwich has shown ('Proc. Lond. Math. Soc.,' vol. 30, p. 98 (1898)) that in such problems as are here considered the effect of gravity is, from a practical point of view, unimportant.

† 'Proc. Lond. Math. Soc.,' vol. 17, p. 4 (1885); 'Scientific Papers,' vol. 2, p. 441.

things which we are obliged to assume as a basis of calculation, it is hoped that the solution of the problems here considered may not be altogether irrelevant.

It is found that the surface disturbance produced by a single impulse of short duration may be analysed roughly into two parts, which we may distinguish as the "minor tremor" and the "main shock," respectively. The minor tremor sets in at any place, with some abruptness, after an interval equal to the time which a wave of longitudinal displacement would take to traverse the distance from the source. Except for certain marked features at the inception, and again (to a lesser extent) at an epoch corresponding to that of direct arrival of transversal waves, it may be described, in general terms, as consisting of a long undulation leading up to the main shock, and dying out gradually after this has passed. Its time-scale is more and more protracted, and its amplitude is more and more diminished, the greater the distance from the source. The main shock, on the other hand, is propagated as a solitary wave (with one maximum and one minimum, in both the horizontal and vertical displacements); its time-scale is constant; and its amplitude diminishes only in accordance with the usual law of annular divergence, so that its total energy is maintained undiminished. Its velocity is that of free Rayleigh waves, and is accordingly somewhat less than that of waves of transversal displacement in an unlimited medium.*

The paper includes a number of subsidiary results. It is convenient to begin by attacking the problems in their two-dimensional form, calculating (for instance) the effect of a pressure applied uniformly along a *line* of the surface. The interpretation of the results is then comparatively simple, and it is found that a good deal of the analysis can be utilized afterwards for the three-dimensional cases. Again, the investigation of the effects of a simple-harmonic source of disturbance is a natural preliminary to that of a source varying according to an arbitrary law.

Incidentally, new solutions are given of the well-known problems where a periodic force acts transversely to a line, or at a point, in an unlimited solid. These serve, to some extent, as tests of the analytical method, which presents some features of intricacy.

2. A few preliminary formulæ and conventions as to notation may be put in evidence at the outset, for convenience of reference.

The usual notation of BESSEL's Functions "of the first kind" is naturally adhered to; thus we write:

$$J_0(\zeta) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(\zeta \cos \omega)\, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

* Compare the concluding passage of Lord RAYLEIGH'S paper:

"It is not improbable that the surface-waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."

The calculations indicate that the preponderance is much greater than would be inferred from a mere comparison of the ordinary laws of two-dimensional and three-dimensional divergence.

By a known theorem we have also

$$J_0(\zeta) = \frac{2}{\pi}\int_0^\infty \sin(\zeta \cosh u)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

provided ζ be real and positive. For our present purpose it is convenient to follow H. WEBER* in adopting as the standard function "of the second kind"

$$K_0(\zeta) = \frac{2}{\pi}\int_0^\infty \cos(\zeta \cosh u)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

It is further necessary to have a special symbol for that combination of the two functions (2) and (3) which is appropriate to the representation of a diverging wave-system; we write, after Lord RAYLEIGH,†

$$D_0(\zeta) = \frac{2}{\pi}\int_0^\infty e^{-i\zeta \cosh u}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

so that

$$D_0(\zeta) = K_0(\zeta) - iJ_0(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We shall also write, in accordance with the usual conventions,

$$J_1(\zeta) = -J'_0(\zeta), \quad K_1(\zeta) = -K'_0(\zeta), \quad D_1(\zeta) = -D'_0(\zeta) \quad . \quad . \quad (6).$$

For large values of ζ we have the asymptotic expansion

$$D_0(\zeta) = \sqrt{\frac{2}{\pi\zeta}} \cdot e^{-i(\zeta + \frac{1}{4}\pi)} \left\{1 - \frac{1^2}{1!\,(8i\zeta)} + \frac{1^2 \cdot 3^2}{2!\,(8i\zeta)^2} - \ldots\right\} \quad . \quad . \quad . \quad (7).$$

In the two-dimensional problems of this paper we shall have to deal with a number of solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + h^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

constructed from the type

$$\phi = Ae^{-\alpha y}\, e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

where ξ is real, and

$$\alpha = \sqrt{(\xi^2 - h^2)}, \quad \text{or} = i\sqrt{(h^2 - \xi^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

* 'Part. Diff.-Gleichungen d. math. Physik,' Brunswick, 1899–1901, vol. 1, p. 175. HEINE ('Kugelfunctionen,' Berlin, 1878–1881, vol. 1, p. 185) omits the factor $2/\pi$. In terms of the more usual notation,

$$K_0 = \frac{2}{\pi}\{-Y_0 + (\log 2 - \gamma) J_0\},$$

where γ is EULER'S constant. The function $\frac{1}{2}\pi K_0$ has been tabulated (see J. H. MICHELL, 'Phil. Mag.,' Jan., 1898).

† 'Phil. Mag.,' vol. 43, p. 259 (1897); 'Scientific Papers,' vol. 4, p. 283. I have introduced the factor $2/\pi$, and reversed the sign.

according as $\xi^2 \gtrless h^2$, the radicals being taken positively. In particular, we shall meet with the solution

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha y}\, e^{i\xi x}\, d\xi}{\alpha} = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-\alpha y} \cos \xi x \, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

and it is important to recognize that this is identical with $D_0(hr)$, where $r = \sqrt{(x^2 + y^2)}$. To see this, we remark that ϕ, as given by (11), is an even function of x, and that for $x = 0$ it assumes the form

$$\phi = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-\alpha y}\, d\xi}{\alpha} = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-i\sqrt{(h^2+\eta^2)}y}\, d\eta}{\sqrt{(h^2 + \eta^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

by the method of contour-integration.* This is obviously equal to $D_0(hy)$. Again, the mean value of any function ϕ which satisfies (8), taken round the circumference of a circle of radius r which does not enclose any singularities, is known to be equal to $J_0(kr).\phi_0$, where ϕ_0 is the value at the centre.† We can therefore adapt an argument of THOMSON and TAIT‡ to show that a solution of (8) which has no singularities in the region $y > 0$, and is symmetrical with respect to the axis of y, is determined by its values at points of this axis. We have, accordingly,

$$D_0(hr) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha y}\, e^{i\xi x}\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Again, in some three-dimensional problems where there is symmetry about the axis of z, we have to do with solutions of

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + h^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

based on the type

$$\phi = A e^{-\alpha z} J_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $\varpi = \sqrt{(x^2 + y^2)}$, and α is defined as in (10). In particular, we have the solution

$$\phi = \int_0^{\infty} \frac{e^{-\alpha z} J_0(\xi\varpi)\, \xi\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

which (again) reduces to a known function. At points on the axis of symmetry ($\varpi = 0$) it takes the form

$$\phi = \int_0^{\infty} \frac{e^{-\alpha z}\, \xi\, d\xi}{\alpha} = \int_{ih}^{\infty} e^{-\alpha z}\, d\alpha = \frac{e^{-ihz}}{z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

* If we equate severally the real and imaginary parts in the second and third members of (12), we reproduce known results.

† H. WEBER, 'Math. Ann.,' vol. 1 (1868).

‡ 'Natural Philosophy,' § 498.

Since the mean value of a function ϕ which satisfies (14), taken over the surface of a sphere of radius r not enclosing any singularities, is equal to

$$\frac{\sin hr}{hr} \cdot \phi_0,$$

where ϕ_0 is the value at the centre,* the argument already borrowed from THOMSON and TAIT enables us to assert that

$$\frac{e^{-ihr}}{r} = \int_0^\infty \frac{e^{-\alpha z} \mathrm{J}_0(\xi\varpi)\, \xi\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),\dagger$$

where

$$r = \sqrt{(\varpi^2 + z^2)} = \sqrt{(x^2 + y^2 + z^2)}.$$

Finally, we shall require FOURIER's Theorem in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(\lambda)\, e^{i\xi(x-\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),\ddagger$$

and the analogous formula

$$f(\varpi) = \int_0^\infty \mathrm{J}_0(\xi\varpi)\, \xi\, d\xi \int_0^\infty f(\lambda)\, \mathrm{J}_0(\xi\lambda)\, \lambda\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (20).\S$$

As particular cases, if in (19) we have $f(x) = 1$ for $x^2 < a^2$, and $= 0$ for $x^2 > a^2$, then

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \xi a}{\xi} e^{i\xi x}\, d\xi = \frac{2}{\pi} \int_0^\infty \frac{\sin \xi a}{\xi} \cos \xi x\, d\xi \quad . \quad . \quad . \quad . \quad (21);$$

and, if in (20) $f(\varpi) = 1$ for $\varpi < a$, and $= 0$ for $\varpi > a$, then

$$f(\varpi) = a \int_0^\infty \mathrm{J}_0(\xi\varpi)\, \mathrm{J}_1(\xi a)\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

These are of course well-known results.||

* H. WEBER, 'Crelle,' vol. 69 (1868).

† If in (18) we put $z = 0$, and then equate separately the real and imaginary parts, we deduce

$$\int_0^\infty \mathrm{J}_0(\zeta \cosh u) \cosh u\, du = \frac{\cos \zeta}{\zeta},$$

$$\int_0^\infty \mathrm{J}_0(\zeta \sin u) \sin u\, du = \frac{\sin \zeta}{\zeta}.$$

These are known results. *Cf.* RAYLEIGH, 'Scientific Papers,' vol. 3, pp. 46, 98 (1888); HOBSON, 'Proc. Lond. Math. Soc.,' vol. 25, p. 71 (1893); and SONINE, 'Math. Ann.,' vol. 16).

‡ H. WEBER, 'Part. Diff.-Gl. etc.,' vol. 2, p. 190. Since λ occurs here and in (20) only as an intermediate variable, no confusion is likely to be caused by its subsequent use to denote an elastic constant.

§ H. WEBER, 'Part. Diff.-Gl. etc.,' vol. 1, p. 193.

|| It may be noticed that if in (20) we put $f(\varpi) = e^{-ih\varpi}/\varpi$, we reproduce formulæ given in the foot-note † above.

PART I.

Two-Dimensional Problems.

3. The equations of motion of an isotropic elastic solid in two dimensions (x, y) are

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu\nabla^2 u, \quad \rho\frac{\partial^2 v}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial y} + \mu\nabla^2 v \quad . \quad . \quad (23),$$

where u, v are the component displacements, ρ is the density, λ, μ are the elastic constants of Lamé, and

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

These equations are satisfied by

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),^*$$

provided

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\lambda + 2\mu}{\rho}\nabla^2\phi, \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{\mu}{\rho}\nabla^2\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

In the case of simple-harmonic motion, the time-factor being e^{ipt}, the latter equations take the forms

$$(\nabla^2 + h^2)\,\phi = 0, \quad (\nabla^2 + k^2)\,\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

where

$$h^2 = \frac{p^2\rho}{\lambda + 2\mu} = p^2a^2, \quad k^2 = \frac{p^2\rho}{\mu} = p^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

the symbols a, b denoting (as generally in this paper) the wave-*slownesses*,† *i.e.*, the reciprocals of the wave-velocities, corresponding to the irrotational and equivoluminal types of disturbance respectively.

The formulæ (25) now give, for the component stresses,

$$\left.\begin{aligned}
\frac{p_{xx}}{\mu} &= \frac{\lambda}{\mu}\Delta + 2\frac{\partial u}{\partial x} = -k^2\phi - 2\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial^2\psi}{\partial x\,\partial y} \\
\frac{p_{xy}}{\mu} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial^2\phi}{\partial x\,\partial y} - k^2\psi - 2\frac{\partial^2\psi}{\partial x^2} \\
\frac{p_{yy}}{\mu} &= \frac{\lambda}{\mu}\Delta + 2\frac{\partial v}{\partial y} = -k^2\phi - 2\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\psi}{\partial x\,\partial y}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29).$$

* Green, 'Camb. Trans.,' vol. 6 (1838); 'Math. Papers,' p. 261.

† The introduction of special symbols for wave-slownesses rather than for wave-velocities is prompted by analytical considerations. The term "wave-slowness" is accredited in Optics by Sir W. R. Hamilton.

In the applications which we have in view, the vibrations of the solid are supposed due to prescribed forces acting at or near the plane $y = 0$. We therefore assume as a typical solution of (27), applicable to the region $y > 0$,

$$\phi = Ae^{-\alpha y}e^{i\xi x}, \quad \psi = Be^{-\beta y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

where ξ is real, and α, β are the positive real, or positive imaginary,* quantities determined by

$$\alpha^2 = \xi^2 - h^2, \quad \beta^2 = \xi^2 - k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

For the region $y < 0$, the corresponding assumption would be

$$\phi = A'e^{\alpha y}e^{i\xi x}, \quad \psi = B'e^{\beta y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

The time-factor is here (and often in the sequel) temporarily omitted.

The expressions (30), when substituted in (25) and (29), give for the displacements and stresses at the plane $y = 0$

$$u_0 = (i\xi A - \beta B)\, e^{i\xi x}, \quad v_0 = (-\alpha A - i\xi B)\, e^{i\xi x} \quad . \quad . \quad . \quad . \quad (33),$$

and

$$\left.\begin{aligned} [p_{xy}]_0 &= \mu\,\{-2i\xi\alpha A + (2\xi^2 - k^2)\,B\}\, e^{i\xi x} \\ [p_{yy}]_0 &= \mu\,\{(2\xi^2 - k^2)\,A + 2i\xi\beta B\}\, e^{i\xi x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34).$$

The forms corresponding to (32) would be obtained by affixing accents to A and B, and reversing the signs of α, β.

4. In order to illustrate, and at the same time test, our method, it is convenient to begin with the solution of a known problem, viz., where a periodic force acts transversally on a line of matter, in an unlimited elastic solid.†

Let us imagine, in the first instance, that an extraneous force of amount $Ye^{i\xi x}$ per unit area acts parallel to y on a thin stratum coincident with the plane $y = 0$. The normal stress will then be discontinuous at this plane, viz.,

$$[p_{yy}]_{y=+0} - [p_{yy}]_{y=-0} = -Ye^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

whilst the tangential stress is continuous. These conditions give, by (34),

$$\left.\begin{aligned} (2\xi^2 - k^2)(A - A') + 2i\xi\beta\,(B + B') &= -\frac{Y}{\mu} \\ -2i\xi\alpha\,(A + A') + (2\xi^2 - k^2)(B - B') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36).$$

Again, the continuity of u and v requires

$$\left.\begin{aligned} i\xi\,(A - A') - \beta\,(B + B') &= 0 \\ \alpha\,(A + A') + i\xi\,(B - B') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

* This convention should be carefully attended to; it runs throughout the paper.

† RAYLEIGH, 'Theory of Sound,' 2nd ed., § 376.

Hence

$$A = -A' = \frac{Y}{2k^2\mu}, \quad B = B' = \frac{i\xi}{\beta} \cdot \frac{Y}{2k^2\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

We have, then, for $y > 0$,

$$\phi = \frac{Y}{2k^2\mu} e^{-\alpha y} e^{i\xi x}, \quad \psi = \frac{Y}{2k^2\mu} \cdot \frac{i\xi}{\beta} e^{-\beta y} e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

To pass to the case of an extraneous force Q concentrated on the line $x = 0$, $y = 0$, we make use of (19). Assuming that the $f(\lambda)$ of this formula vanishes for all but infinitesimal values of λ, for which it becomes infinite in such a way that

$$\int_{-\infty}^{\infty} f(\lambda)\, d\lambda = Q,$$

we write, in (39), $Y = Q\, d\xi/2\pi$, and integrate with respect to ξ from $-\infty$ to $+\infty$.* We thus obtain, for $y > 0$,

$$\phi = \frac{Q}{4\pi k^2\mu} \int_{-\infty}^{\infty} e^{-\alpha y} e^{i\xi x}\, d\xi, \quad \psi = \frac{iQ}{4\pi k^2\mu} \int_{-\infty}^{\infty} \frac{\xi e^{-\beta y} e^{i\xi x}\, d\xi}{\beta} \quad . \quad . \quad . \quad (40),$$

or, on reference to (13),

$$\phi = -\frac{Q}{4k^2\mu} \frac{\partial}{\partial y} D_0(hr), \quad \psi = \frac{Q}{4k^2\mu} \frac{\partial}{\partial x} D_0(kr) \quad . \quad . \quad . \quad . \quad (41),$$

where $r = \sqrt{(x^2 + y^2)}$.

If we put $x = r\cos\theta$, $y = r\sin\theta$, we find from (25), on inserting the time-factor, that for large values of r the radial and transverse displacements are

$$\left.\begin{aligned} \frac{\partial\phi}{\partial r} + \frac{\partial\psi}{r\,\partial\theta} &= \frac{Q}{4(\lambda + 2\mu)} \sqrt{\frac{2}{\pi hr}} \cdot e^{i(pt - hr - \frac{1}{4}\pi)} \sin\theta \\ \frac{\partial\phi}{r\,\partial\theta} - \frac{\partial\psi}{\partial r} &= \frac{Q}{4\mu} \sqrt{\frac{2}{\pi kr}} \cdot e^{i(pt - kr - \frac{1}{4}\pi)} \cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (42),$$

respectively.† Use has here been made of (7).

A simple expression can be obtained for the rate (W, say) at which the extraneous

* The indeterminateness of the formula (19) in this case may be evaded by supposing, in the first instance, that the force Q, instead of being concentrated on the line $x = 0$, is uniformly distributed over the portion of the plane $y = 0$ lying between $x = \pm a$. It appears from (21) that we should then have

$$Y = \frac{Q}{2\pi} \cdot \frac{\sin \xi a}{\xi a}\, d\xi.$$

If we finally make $a = 0$ we obtain the results (40).

† The second of these results is equivalent to that given by RAYLEIGH, *loc. cit.*, for the case of incompressibility ($\lambda = \infty$).

force does work in generating the cylindrical waves which travel outwards from the source of disturbance. The formulæ (40) give, for the value of $\partial v/\partial t$ at the origin,

$$\frac{\partial v_0}{\partial t} = \frac{ipQe^{ipt}}{4\pi k^2\mu}\int_{-\infty}^{\infty}\left(\frac{\xi^2}{\beta} - \alpha\right)d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

This expression is really infinite, but we are only concerned with the part of it in the same phase with the force,* which is finite. Taking this alone, we have

$$\frac{\partial v_0}{\partial t} = \frac{pQe^{ipt}}{4\pi k^2\mu}\left\{\int_{-k}^{k}\frac{\xi^2 d\xi}{\sqrt{(k^2 - \xi^2)}} + \int_{-h}^{h}\sqrt{(h^2 - \xi^2)}\,d\xi\right\} = (k^2 + h^2)\frac{pQe^{ipt}}{8k^2\mu} \quad . \quad (44).$$

Discarding imaginary parts, we find that the mean rate, per unit length of the axis of z, at which a force Q cos pt does work is

$$W = \left(1 + \frac{h^2}{k^2}\right)\frac{pQ^2}{16\mu} = \frac{\lambda + 3\mu}{16\mu(\lambda + 2\mu)}\,pQ^2 \,. \quad . \quad . \quad . \quad . \quad . \quad (45).$$

5. We may proceed to the case of a "semi-infinite" elastic solid, bounded (say) by the plane $y = 0$, and lying on the positive side of this plane. We examine, in the first place, the effect of given periodic forces applied to the boundary.

As a typical distribution of *normal* force, we take

$$[p_{xy}]_0 = 0, \quad [p_{yy}]_0 = \mathrm{Y}e^{i\xi x} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

the factor e^{ipt} being as usual understood. The constants A, B in (30) are determined by means of (34), viz.:

$$\left.\begin{aligned} -2i\xi\alpha\mathrm{A} + (2\xi^2 - k^2)\,\mathrm{B} &= 0, \\ (2\xi^2 - k^2)\,\mathrm{A} + 2i\xi\beta\mathrm{B} &= \frac{\mathrm{Y}}{\mu} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).$$

Hence

$$\mathrm{A} = \frac{2\xi^2 - k^2}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Y}}{\mu}, \quad \mathrm{B} = \frac{2i\xi\alpha}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Y}}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where, for shortness,

$$\mathrm{F}(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2\alpha\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

We shall find it convenient, presently, to write also

$$f(\xi) = (2\xi^2 - k^2)^2 + 4\xi^2\alpha\beta \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

* The awkwardness is evaded if (as in a previous instance) we distribute the force uniformly over a length $2a$ of the axis of x. This will introduce a factor $\left(\frac{\sin \xi a}{\xi a}\right)^2$ under the integral signs in the second member of (44).

The surface-values of the displacements are now given by (33), viz. :

$$\left.\begin{aligned} u_0 &= \frac{i\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}}{F(\xi)} \cdot \frac{Y}{\mu}, \\ v_0 &= \frac{k^2\alpha e^{i\xi x}}{F(\xi)} \cdot \frac{Y}{\mu} \end{aligned}\right\} \quad \ldots \ldots \quad (51).$$

The effect of a concentrated force Q acting parallel to y at points of the line $x = 0$, $y = 0$ is deduced, as before, by writing $Y = -Qd\xi/2\pi$, and integrating from $-\infty$ to ∞; thus

$$\left.\begin{aligned} u_0 &= -\frac{iQ}{2\pi\mu}\int_{-\infty}^{\infty} \frac{\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}\,d\xi}{F(\xi)}, \\ v_0 &= -\frac{Q}{2\pi\mu}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{F(\xi)} \end{aligned}\right\} \quad \ldots \ldots \quad (52).$$

In a similar manner, corresponding to the *tangential* surface forces :

$$[p_{xy}]_0 = Xe^{i\xi x}, \quad [p_{yy}]_0 = 0 \quad \ldots \ldots \quad (53),$$

we should find

$$A = -\frac{2i\xi\beta}{F(\xi)} \cdot \frac{X}{\mu}, \quad B = \frac{2\xi^2 - k^2}{F(\xi)} \cdot \frac{X}{\mu} \quad \ldots \quad (54).$$

And, for the effect of a concentrated force P acting parallel to x at the origin,

$$\left.\begin{aligned} u_0 &= -\frac{P}{2\pi\mu}\int_{-\infty}^{\infty} \frac{k^2\beta e^{i\xi x}\,d\xi}{F(\xi)}, \\ v_0 &= \frac{iP}{2\pi\mu}\int_{-\infty}^{\infty} \frac{\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}\,d\xi}{F(\xi)} \end{aligned}\right\} \quad \ldots \ldots \quad (55).$$

The comparison of u_0 in (52) with v_0 in (55) gives an example of the general principle of reciprocity.*

We may also consider the case of an *internal* source of disturbance, resident (say) in the line $x = 0$, $y = f$, the boundary $y = 0$ being now entirely free. The simplest type of source is one which would produce symmetrical radial motion (in two dimensions) in an unlimited solid, say

$$\phi = D_0(hr), \quad \psi = 0 \quad \ldots \ldots \quad (56),$$

where r, $= \sqrt{\{x^2 + (y-f)^2\}}$, denotes distance from the source. If we superpose on this an equal source in the line $x = 0$, $y = -f$, we obtain

$$\phi = D_0(hr) + D_0(hr'), \quad \psi = 0 \quad \ldots \ldots \quad (57),$$

* RAYLEIGH, 'Theory of Sound,' vol. 1, § 108.

where $r' = \sqrt{\{x^2 + (y+f)^2\}}$. It is evident, without calculation, that the condition of zero tangential stress at the plane $y = 0$ is already satisfied; the normal stress, however, does not vanish. It appears from (13) that in the neighbourhood of the plane $y = 0$ the preceding value of ϕ is equivalent to

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha(y-f)}\, e^{i\xi x}\, d\xi}{\alpha} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha(y+f)}\, e^{i\xi x}\, d\xi}{\alpha}$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\cosh \alpha y}{\alpha}\, e^{-\alpha f}\, e^{i\xi x}\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Substituting in (29) we find that this makes

$$[p_{xy}]_0 = 0, \quad [p_{yy}]_0 = \frac{2\mu}{\pi}\int_{-\infty}^{\infty} \frac{2\xi^2 - k^2}{\alpha}\, e^{-\alpha f}\, e^{i\xi x}\, d\xi \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Comparing with (46), we see that the desired condition of zero stress on the boundary will be fulfilled, provided we superpose on (57) the solution obtained from (30) and (48) by putting

$$\mathrm{Y} = -\frac{2\mu}{\pi} \cdot \frac{2\xi^2 - k^2}{\alpha}\, e^{-\alpha f}\, d\xi,$$

and afterwards integrating with respect to ξ from $-\infty$ to ∞. The surface-displacements corresponding to this auxiliary solution are obtained from (51), and if we incorporate the part of u_0 due to (58), we find, after a slight reduction,

$$\left.\begin{aligned} u_0 &= -\frac{4ik^2}{\pi}\int_{-\infty}^{\infty} \frac{\beta\xi e^{-\alpha f}\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} \\ v_0 &= -\frac{2k^2}{\pi}\int_{-\infty}^{\infty} \frac{(2\xi^2 - k^2)\, e^{-\alpha f}\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

These calculations might be greatly extended. For example, it would be easy, with the help of Art. 4, to work out the case where a vertical or a horizontal periodic force acts on an internal line parallel to z. And, by means of the reciprocal theorem already adverted to, we could infer the horizontal or vertical displacement at an internal point due to a given localized surface force.

6. It remains to interpret, as far as possible, the definite integrals which occur in the expressions we have obtained.

It is to be remarked, in the first place, that the integrals, as they stand, are to a certain extent indeterminate, owing to the vanishing of the function $\mathrm{F}(\xi)$ for certain real values of ξ. It is otherwise evident *à priori* that on a particular solution of any of our problems we can superpose a system of free surface waves having the wave-length proper to the imposed period $2\pi/p$. The theory of such waves has been given by Lord RAYLEIGH,* and is moreover necessarily contained implicitly in our analysis.

Thus, if we put $Y = 0$ in (47), we find that the conditions of zero surface-stress are satisfied, provided

$$A : B = 2\kappa^2 - k^2 : 2i\kappa\alpha_1 = -2i\kappa\beta_1 : 2\kappa^2 - k^2 \quad . \quad . \quad . \quad . \quad (61),$$

where κ is a root of $F(\xi) = 0$, and α_1, β_1, denote the corresponding values of α, β, Now, in the notation of (49) and (50),

$$F(\xi) f(\xi) = (2\xi^2 - k^2)^4 - 16(\xi^2 - h^2)(\xi^2 - k^2)\xi^4$$

$$= k^8 \left\{1 - 8\frac{\xi^2}{k^2} + \left(24 - 16\frac{h^2}{k^2}\right)\frac{\xi^4}{k^4} - 16\left(1 - \frac{h^2}{k^2}\right)\frac{\xi^6}{k^6}\right\}. \quad . \quad (62).$$

Equating this to zero, we have a cubic in ξ^2/k^2, and since $k^2 > h^2$, it is plain that there is a real root between 1 and ∞. It may also be shown without much difficulty that the remaining roots, when real, lie between 0 and h^2/k^2. The former root makes α, β real and positive, and therefore cannot make $f(\xi) = 0$. The latter roots make α, β positive imaginaries, and therefore cannot make $F(\xi) = 0$. This latter equation has accordingly only two real roots $\xi = \pm\kappa$, where $\kappa > k$.

Thus, in the case of incompressibility ($\lambda = \infty$, $h = 0$) it is found that

$$\kappa/k = 1{\cdot}04678 \ldots,$$

and that the remaining roots of (62) are complex.† On POISSON's hypothesis as to the relation between the elastic constants ($\lambda = \mu$, $h^2 = \frac{1}{3}k^2$), the roots of (62) are all real, viz., they are

$$\xi^2/k^2 = \tfrac{1}{4}, \quad \tfrac{1}{4}(3 - \sqrt{3}), \quad \tfrac{1}{4}(3 + \sqrt{3}),$$

so that

$$\kappa/k = \tfrac{1}{2}\sqrt{(3 + \sqrt{3})} = 1{\cdot}087664 \ldots;$$

this will usually be taken as the standard case for purposes of numerical illustration.

In analogy with (28), it will be convenient to write

$$\kappa = pc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where c denotes the wave-slowness of the Rayleigh waves. The corresponding wave-velocity is

$$c^{-1} = \frac{k}{\kappa} \cdot b^{-1} = \frac{k}{\kappa} \cdot \sqrt{\frac{\mu}{\rho}}.$$

According as we suppose $\lambda = \infty$, or $\lambda = \mu$, this is ·9553 times, or ·9194 times, the velocity of propagation of plane transverse waves in an unlimited solid.

The further properties of free Rayleigh waves are contained in the formulæ (61)

* 'Proc. Lond. Math. Soc.,' vol. 17 (1885); 'Scientific Papers,' vol. 2, p. 441.

† *Cf.* RAYLEIGH (*loc. cit.*), where it is also shown (virtually) that they are roots of $f(\xi)$, not of $F(\xi)$, if α, β be chosen so as to have their real parts positive.

and (30). We merely note, for purposes of reference, that if in (33) we put $\xi = \pm \kappa$, and accordingly, from (61),

$$A = (2\kappa^2 - k^2)\,C, \quad B = \pm\, 2i\kappa\alpha_1 C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

we obtain by superposition a system of standing waves in which

$$u_0 = -\,2\kappa\,(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\,C \sin \kappa x \,.\, e^{ipt}, \quad v_0 = 2k^2\alpha_1 C \cos \kappa x \,.\, e^{ipt} \quad . \quad (65).$$

The theory here recapitulated indicates the method to be pursued in treating the definite integrals of Art. 5. We fix our attention, in the first instance, on their "principal values," in CAUCHY's sense, and afterwards superpose such a system of free Rayleigh waves as will make the final result consist solely of waves travelling outwards from the origin of disturbance.

It may be remarked that an alternative procedure is possible, in which even temporary indeterminateness is avoided. This consists in inserting in the equations of motion (23) frictional terms proportional to the velocities, and finally making the coefficients of these terms vanish. This method has some advantages, especially as regards the positions of the "singular points" to be referred to. The chief problem of this paper was, in fact, first worked through in this manner; but as the method seemed rather troublesome to expound as regards some points of detail, it was abandoned in favour of that explained above.

7. The most important case, and the one here chiefly considered, is that of a concentrated *vertical* force applied to the surface, to which the formulæ (52) relate. The case of a *horizontal* force, expressed by the formulæ (55), could be treated in an exactly similar manner.

Since u_0 is evidently an odd, and v_0 an even, function of x, it will be sufficient to take the case of x positive.

As regards the horizontal* displacement u_0, we consider the integral

$$\int \Phi(\zeta)\, d\zeta = \int \frac{\zeta\,\{(2\zeta^2 - k^2) - 2\sqrt{(\zeta^2 - h^2)}\,\sqrt{(\zeta^2 - k^2)}\}\, e^{i\zeta x}\, d\zeta}{(2\zeta^2 - k^2)^2 - 4\sqrt{(\zeta^2 - h^2)}\,\sqrt{(\zeta^2 - k^2)}\,\zeta^2} \quad . \quad . \quad (66),$$

taken round a suitable contour in the plane of the complex variable $\zeta, = \xi + i\eta$. If this contour does not include either "poles" $(\pm\,\kappa,\ 0)$, or "branch-points" $(\pm\,h,\ 0)$, $(\pm\,k,\ 0)$ of the function to be integrated, the result will be zero.

A convenient contour for our purpose is a rectangle, one side of which consists of the axis of ξ except for small semicircular indentations surrounding the singular points specified, whilst the remaining sides are at an infinite distance on the side $\eta > 0$. It is easily seen that the parts of the integral due to these infinitely distant sides will vanish of themselves. If we adopt for the radicals $\sqrt{(\zeta^2 - h^2)}$ and $\sqrt{(\zeta^2 - k^2)}$,

* The sense in which the terms "horizontal" and "vertical" are used is indicated in the second sentence of the Introduction.

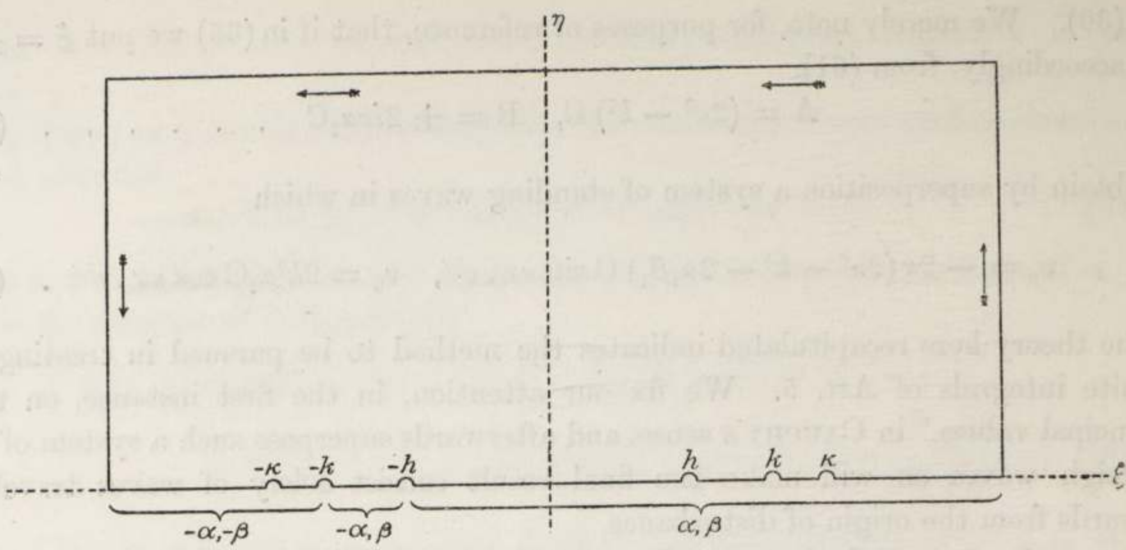

Fig. 1.

at points of the axis of ξ, the consistent system of values indicated in fig. 1,* we find, for the various parts of the first-mentioned side,†

$$\int_{-\infty}^{-k} \Phi(\zeta)\, d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi\, \frac{-\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\, e^{-i\kappa x}}{\mathrm{F}'(-\kappa)},$$

$$\int_{-k}^{-h} \Phi(\zeta)\, d\zeta = \int_{-k}^{-h} \frac{\xi(2\xi^2 - k^2 + 2\alpha\beta)\, e^{i\xi x}\, d\xi}{f(\xi)},$$

$$\int_{-h}^{\infty} \Phi(\zeta)\, d\zeta = \mathfrak{P}\int_{-h}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi\, \frac{\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\, e^{i\kappa x}}{\mathrm{F}'(\kappa)},$$

where the terms with $\mathrm{F}'(-\kappa)$ and $\mathrm{F}'(\kappa)$ in the denominator are due to the small semicircles about the points $(\pm\kappa, 0)$. Equating the sum of these expressions to zero, we find, since $\mathrm{F}'(-\kappa) = -\mathrm{F}'(\kappa)$,

$$\begin{aligned} \mathfrak{P}\int_{-\infty}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} &= -2i\pi \mathrm{H} \cos \kappa x \\ &\quad + \int_{-k}^{-h} \left\{ \frac{2\xi^2 - k^2 - 2\alpha\beta}{\mathrm{F}(\xi)} - \frac{2\xi^2 - k^2 + 2\alpha\beta}{f(\xi)} \right\} \xi e^{i\xi x}\, d\xi \\ &= -2i\pi\mathrm{H} \cos\kappa x - 4k^2 \int_h^k \frac{\xi(2\xi^2 - k^2)\, \alpha\beta e^{-i\xi x}\, d\xi}{\mathrm{F}(\xi) f(\xi)} \quad . \quad . \quad . \quad (67), \end{aligned}$$

* The function under the integral sign in (66) is uniquely determined (by continuity) within and on the contour when once the values of the radicals $\sqrt{(\zeta^2 - h^2)}$ and $\sqrt{(\zeta^2 - k^2)}$ at some one point are assigned. The convention implied in the text is that the radicals are both positive at the point $(+\infty, 0)$.

It will be noticed that over the portion of the axis of ξ between $-k$ and $-h$ the function in (66) differs from that involved in the value of u_0 as given by (52). This is allowed for in the second member of (67). Corrections, or rather adjustments, of this kind occur repeatedly in the transformations of this paper.

† The symbol $\mathfrak{P}$ is used to distinguish the "principal value" of an integral (with respect to a real variable) to which it is prefixed.

where

$$\mathrm{H} = -\frac{\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)}{\mathrm{F}'(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68),$$

a numerical quantity depending only on the ratio $\lambda : \mu$.

To examine the value of v_0 we take the integral

$$\int \Psi(\zeta)\, d\zeta = \int \frac{k^2\sqrt{(\zeta^2 - h^2)}\, e^{i\zeta x}\, d\zeta}{(2\zeta^2 - k^2)^2 - 4\sqrt{(\zeta^2 - h^2)}\sqrt{(\zeta^2 - k^2)}\,\zeta^2} \quad . \quad . \quad . \quad (69)$$

round the same contour. Integrating along the axis of ξ we find

$$\int_{-\infty}^{-k} \Psi(\zeta)\, d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{-k^2\alpha e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi\,\frac{-k^2\alpha_1}{\mathrm{F}'(-\kappa)}\, e^{-i\kappa x},$$

$$\int_{-k}^{-h} \Psi(\zeta)\, d\zeta = \int_{-k}^{-h} \frac{-k^2\alpha e^{i\xi x}\, d\xi}{f(\xi)},$$

$$\int_{-h}^{\infty} \Psi(\zeta)\, d\zeta = \mathfrak{P}\int_{-h}^{\infty} \frac{k^2\alpha e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi\,\frac{k^2\alpha_1}{\mathrm{F}'(\kappa)}\, e^{i\kappa x},$$

and thence by addition, since the terms due to the infinitely distant parts of the contour vanish as before,

$$\begin{aligned}
\mathfrak{P}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} &= -2i\pi\mathrm{K}\cos\kappa x + \mathfrak{P}\int_{-\infty}^{-k} \frac{2k^2\alpha e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} \\
&\qquad + \int_{-k}^{-h} \left\{\frac{1}{\mathrm{F}(\xi)} + \frac{1}{f(\xi)}\right\} k^2\alpha e^{i\xi x}\, d\xi \\
&= -2i\pi\mathrm{K}\cos\kappa x + 2\mathfrak{P}\int_{k}^{\infty} \frac{k^2\alpha e^{-i\xi x}\, d\xi}{\mathrm{F}(\xi)} \\
&\qquad + 2k^2\int_{h}^{k} \frac{(2\xi^2 - k^2)^2\, \alpha e^{-i\xi x}\, d\xi}{\mathrm{F}(\xi) f(\xi)} \quad . \quad . \quad . \quad . \quad . \quad (70),
\end{aligned}$$

where

$$\mathrm{K} = -\frac{k^2\alpha_1}{\mathrm{F}'(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Hence if to the principal values of the expressions in (52) we add the system of free Rayleigh waves,

$$u_0 = i\,\frac{\mathrm{Q}}{\mu}\,\mathrm{H}\sin\kappa x, \qquad v_0 = -i\,\frac{\mathrm{Q}}{\mu}\,\mathrm{K}\cos\kappa x \quad . \quad . \quad . \quad . \quad . \quad (72),$$

which is evidently of the type (65), we obtain, on inserting the time-factor,

$$u_0 = -\frac{Q}{\mu}\,He^{i(pt-\kappa x)} - \frac{2Q}{\pi\mu}\int_h^k \frac{k^2\xi(2\xi^2-k^2)\sqrt{(\xi^2-h^2)}\sqrt{(k^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4+16\xi^4(\xi^2-h^2)(k^2-\xi^2)} \quad (73),$$

$$v_0 = -\frac{Q}{\pi\mu}\,\mathfrak{P}\int_k^\infty \frac{k^2\sqrt{(\xi^2-h^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^2-4\xi^2\sqrt{(\xi^2-h^2)}\sqrt{(\xi^2-k^2)}}$$

$$-\frac{Q}{\pi\mu}\int_h^k \frac{k^2(2\xi^2-k^2)^2\sqrt{(\xi^2-h^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4+16\xi^4(\xi^2-h^2)(k^2-\xi^2)} \quad . \quad . \quad . \quad (74).$$

This is for x positive; the corresponding results for x negative would be obtained by changing the sign of x in the exponentials, and reversing the sign of u_0.

The solution thus found is made up of waves travelling outwards, right and left, from the origin, and so satisfies all the conditions of the question.

The first term in u_0 gives, on each side, a train of waves travelling unchanged with the velocity c^{-1}. The second term gives an aggregate of waves travelling with velocities ranging from b^{-1} to a^{-1}. As x is increased, this term diminishes indefinitely, owing to the more and more rapid fluctuations in the value of $e^{i\xi x}$.

On the other hand, the part of v_0 which corresponds to the first term of u_0 remains embedded in the first definite integral in (74). To disentangle it we must have recourse to another treatment of the integral $\int\Psi(\zeta)\,d\zeta$. One way of doing this is to take the integral round the pair of contours shown in fig. 2, where a consistent scheme of

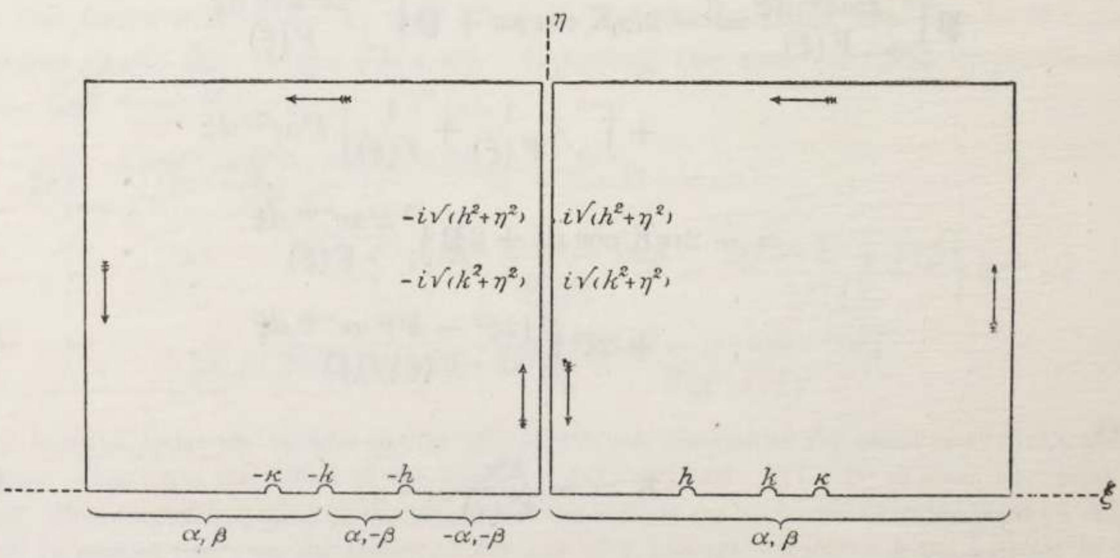

Fig. 2.

values to be attributed to the radicals $\sqrt{(\zeta^2-h^2)}$ and $\sqrt{(\zeta^2-k^2)}$ is indicated. For the only parts of the left-hand contour which need be taken into account we find

$$\int_{-\infty}^{-k} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi\frac{k^2\alpha_1}{\mathrm{F}'(-\kappa)}e^{-i\kappa x},$$

$$\int_{-k}^{-h} \Psi(\zeta)\,d\zeta = \int_{-k}^{-h} \frac{k^2\alpha e^{i\xi x}\,d\xi}{f(\xi)},$$

$$\int_{-h}^{0} \Psi(\zeta)\,d\zeta = \int_{-h}^{0} \frac{-k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)},$$

$$\int_{0}^{i\infty} \Psi(\zeta)\,d\zeta = \int_{0}^{\infty} \frac{-i\sqrt{(h^2+\eta^2)}\,k^2 e^{-\eta x} i\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}}.$$

Similarly, in the right-hand contour,

$$\int_{i\infty}^{0} \Psi(\zeta)\,d\zeta = \int_{\infty}^{0} \frac{i\sqrt{(h^2+\eta^2)}\,k^2 e^{-\eta x} i\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}},$$

$$\int_{0}^{\infty} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{0}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi\frac{k^2\alpha_1}{\mathrm{F}'(\kappa)}e^{i\kappa x}.$$

We infer, by addition,

$$\mathfrak{P}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} = 2\pi\mathrm{K}\sin\kappa x + 2\int_0^h \frac{k^2\alpha e^{-i\xi x}\,d\xi}{\mathrm{F}(\xi)}$$
$$+ \int_h^k \left\{\frac{1}{\mathrm{F}(\xi)} - \frac{1}{f(\xi)}\right\} k^2\alpha e^{-i\xi x} d\xi - 2\int_0^\infty \frac{k^2\sqrt{(h^2+\eta^2)}\,e^{-\eta x}\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}} \quad (75).$$

If we multiply this by $-\mathrm{Q}/2\pi\mu$, and add in the term due to the free Rayleigh waves represented by (72), we obtain, as an equivalent form of (74),

$$v_0 = -\frac{i\mathrm{Q}}{\mu}\mathrm{K}e^{i(pt-\kappa x)} - \frac{i\mathrm{Q}}{\pi\mu}\int_0^h \frac{k^2\sqrt{(h^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^2 + 4\xi^2\sqrt{(h^2-\xi^2)}\sqrt{(k^2-\xi^2)}}$$
$$- \frac{4i\mathrm{Q}}{\pi\mu}\int_h^k \frac{k^2\xi^2(\xi^2-h^2)\sqrt{(k^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4 + 16\xi^4(\xi^2-h^2)(k^2-\xi^2)}$$
$$+ \frac{\mathrm{Q}}{\pi\mu}e^{ipt}\int_0^\infty \frac{k^2\sqrt{(h^2+\eta^2)}\,e^{-\eta x}\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}} \quad (76).*$$

It is evident that all terms after the first diminish indefinitely as x is increased.

* From this we can deduce, by the same method as in Art. 4, an expression for the mean rate W at which a vertical pressure Q cos pt does work in generating waves, viz.,

$$\mathrm{W} = \tfrac{1}{2}\mathrm{K}\frac{p\mathrm{Q}^2}{\mu} + \frac{p\mathrm{Q}^2}{2\pi\mu}\int_0^h \frac{k^2\sqrt{(h^2-\xi^2)}\,d\xi}{(2\xi^2-k^2)^2 + 4\xi^2\sqrt{(h^2-\xi^2)}\sqrt{(k^2-\xi^2)}}$$
$$+ \frac{2p\mathrm{Q}^2}{\pi\mu}\int_h^k \frac{k^2\xi^2(\xi^2-h^2)\sqrt{(k^2-\xi^2)}\,d\xi}{(2\xi^2-k^2)^4 + 16\xi^4(\xi^2-h^2)(k^2-\xi^2)}.$$

If in (73) and (76) we regard only the terms which are sensible at a great distance from the origin, we have, for x positive,

$$u_0 = -\frac{Q}{\mu} \mathrm{H} e^{ip(t-cx)}, \qquad v_0 = -i\frac{Q}{\mu} \mathrm{K} e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad . \quad (77);$$

and similarly for x negative we should find

$$u_0 = \frac{Q}{\mu} \mathrm{H} e^{ip(t+cx)}, \quad v_0 = -i\frac{Q}{\mu} \mathrm{K} e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (78).$$

These formulæ represent a system of free Rayleigh waves, except for the discontinuity at the origin, where the extraneous force is applied. The vibrations are elliptic, with horizontal and vertical axes in the ratio of the two numbers H and K, which are defined by (68) and (71), respectively. To calculate these, we have, since $\mathrm{F}(\kappa) = 0$,

$$f(\kappa) = 2(2\kappa^2 - k^2)^2 = 8\alpha_1\beta_1\kappa^2,$$

and therefore

$$\mathrm{H} = \frac{k^2(2\kappa^2 - k^2)^3}{-\kappa \mathrm{F}'(\kappa) f(\kappa)}, \quad \mathrm{K} = \frac{2k^2\alpha_1(2\kappa^2 - k^2)^2}{-\mathrm{F}'(\kappa) f(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad (79),$$

where, by differentiation of (62),

$$-\mathrm{F}'(\kappa) f(\kappa) = 16k^6\kappa\left\{1 - \left(6 - 4\frac{h^2}{k^2}\right)\frac{\kappa^2}{k^2} + 6\left(1 - \frac{h^2}{k^2}\right)\frac{\kappa^4}{k^4}\right\} \quad . \quad . \quad (80).$$

In the case of incompressibility I find

$$\mathrm{H} = \cdot 05921, \quad \mathrm{K} = \cdot 10890;$$

whilst on POISSON'S hypothesis

$$\mathrm{H} = \cdot 12500, \quad \mathrm{K} = \cdot 18349,$$

so that the amplitudes are, for the same value of μ and for the same applied force, about double what they are in the case of incompressibility.

A similar treatment applies to the formulæ (55), which represent the effect of a concentrated *horizontal* force $\mathrm{P}e^{ipt}$. Taking account only of the more important terms, I find, for x positive,

$$u_0 = -\frac{i\mathrm{P}}{\mu} \mathrm{H}' e^{ip(t-cx)}, \quad v_0 = \frac{\mathrm{P}}{\mu} \mathrm{K}' e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81),$$

and, for x negative,

$$u_0 = -\frac{i\mathrm{P}}{\mu} \mathrm{H}' e^{ip(t+cx)}, \quad v_0 = -\frac{\mathrm{P}}{\mu} \mathrm{K}' e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (82),$$

where

$$\left.\begin{aligned} \mathrm{H}' &= -\frac{k^2\beta_1}{\mathrm{F}'(\kappa)} = \frac{2k^2\beta_1(2\kappa^2 - k^2)^2}{-\mathrm{F}'(\kappa) f(\kappa)} \\ \mathrm{K}' &= -\frac{\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)}{\mathrm{F}'(\kappa)} = \frac{k^2(2\kappa^2 - k^2)^3}{-\kappa \mathrm{F}'(\kappa) f(\kappa)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (83).$$

The ratio of H′ to K′ is, of course, equal to that of H to K; K′ is, moreover, identical with H, in conformity with the principle of reciprocity already referred to. It appears, therefore, from the numerical values of H, K above given, that for $\lambda = \infty$

$$H' = \cdot 03219, \quad K' = \cdot 05921;$$

and for $\lambda = \mu$

$$H' = \cdot 08516, \quad K' = \cdot 12500.$$

Again, in the case of the internal source (56) I find, for large positive values of x,

$$u_0 = -8\kappa H' e^{-a_1 f} e^{ip(t-cx)}, \quad v_0 = 8i\kappa K' e^{-a_1 f} e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad (84),$$

and, for large negative values,

$$u_0 = 8\kappa H' e^{-a_1 f} e^{ip(t+cx)}, \quad v_0 = 8i\kappa K' e^{-a_1 f} e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (85).$$

The factor $e^{-a_1 f}$ indicates how the surface effect (at a sufficient distance) varies with the depth of the source.

8. If in any of the preceding cases we wish to examine more closely the nature and magnitude of the residual disturbance, so far as it is manifested at the surface, it is more convenient to use the system of contours shown in fig. 3. With this system we

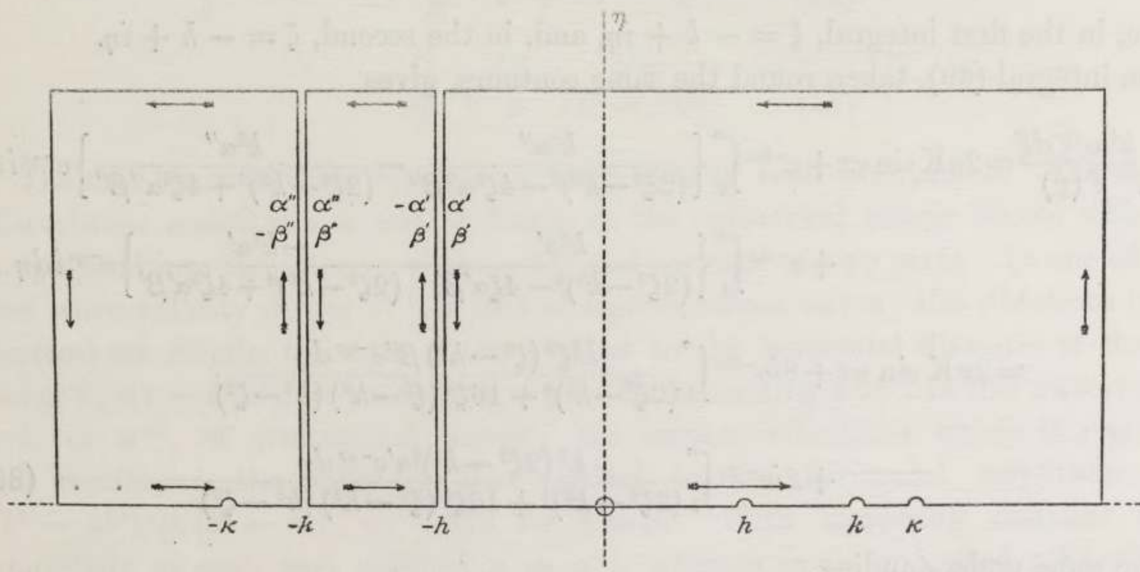

Fig. 3.

can so adjust matters that the radicals $\sqrt{(\zeta^2 - h^2)}$ and $\sqrt{(\zeta^2 - k^2)}$ shall assume in all parts of the axis of ξ exactly the values α, β with which we are concerned in formulæ such as (52). It is convenient, for brevity, to denote by $\pm\alpha'$, β' the values assumed by the same radicals on the two sides of the lines $\xi = -h$, and by α'', $\pm\beta''$ their values on the two sides of the line $\xi = -k$, these values being supposed determined in accordance with the requirements of continuity. Thus, with the allocation shown in the figure, we shall have, for small values of η,

$$\left.\begin{array}{ll} \alpha' = -\sqrt{(2h\eta)}e^{-\frac{1}{4}i\pi}, & \beta' = i\sqrt{(k^2-h^2)} \\ \alpha'' = \sqrt{(k^2-h^2)}, & \beta'' = -\sqrt{(2k\eta)}\,e^{-\frac{1}{4}i\pi} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (86),$$

approximately.

Taking the integral (66) round the several contours, in the directions shown by the arrows, we find

$$\mathfrak{P}\int_{-\infty}^{\infty} \frac{\xi(2\xi^2-k^2-2\alpha\beta)\,e^{i\xi x}\,d\xi}{F(\xi)} = -2i\pi H\cos\kappa x$$

$$+\,e^{-ikx}\int_0^\infty \left\{\frac{2\zeta^2-k^2-2\alpha''\beta''}{(2\zeta^2-k^2)^2-4\zeta^2\alpha''\beta''} - \frac{2\zeta^2-k^2+2\alpha''\beta''}{(2\zeta^2-k^2)^2+4\zeta^2\alpha''\beta''}\right\}\zeta e^{-\eta x}i\,d\eta$$

$$+\,e^{-ihx}\int_0^\infty \left\{\frac{2\zeta^2-k^2-2\alpha'\beta'}{(2\zeta^2-k^2)^2-4\zeta^2\alpha'\beta'} - \frac{2\zeta^2-k^2+2\alpha'\beta'}{(2\zeta^2-k^2)^2+4\zeta^2\alpha'\beta'}\right\}\zeta e^{-\eta x}i\,d\eta\,,$$

$$= -2i\pi H\cos\kappa x + 4ie^{-ikx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)\,\alpha''\beta''\zeta e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}$$

$$+\,4ie^{-ihx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)\,\alpha'\beta'\zeta e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \quad . \quad . \quad (87),$$

where, in the first integral, $\zeta = -k + i\eta$, and, in the second, $\zeta = -h + i\eta$.

The integral (69), taken round the same contours, gives

$$\mathfrak{P}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}d\xi}{F(\xi)} = 2\pi K\sin\kappa x + e^{-ikx}\int_0^\infty \left\{\frac{k^2\alpha''}{(2\zeta^2-k^2)^2-4\zeta^2\alpha''\beta''} - \frac{k^2\alpha''}{(2\zeta^2-k^2)^2+4\zeta^2\alpha''\beta''}\right\}e^{-\eta x}i\,d\eta$$

$$+\,e^{-ihx}\int_0^\infty \left\{\frac{k^2\alpha'}{(2\zeta^2-k^2)^2-4\zeta\alpha'\beta'} - \frac{-k^2\alpha'}{(2\zeta^2-k^2)^2+4\zeta^2\alpha'\beta'}\right\}e^{-\eta x}i\,d\eta$$

$$= 2\pi K\sin\kappa x + 8ie^{-ikx}\int_0^\infty \frac{k^2\zeta^2(\zeta^2-h^2)\,\beta'' e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}$$

$$+\,2ie^{-ihx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)^2\alpha' e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \quad . \quad . \quad . \quad (88),$$

on the same understanding.

The definite integrals in these results can all be expanded in asymptotic forms by means of the formula

$$\int_0^\infty \eta^{\frac{1}{2}}\chi(\eta)\,e^{-\eta x}\,d\eta = \frac{\Pi(\frac{1}{2})}{x^{\frac{3}{2}}}\chi(0) + \frac{\Pi(\frac{3}{2})}{x^{\frac{5}{2}}}\frac{\chi'(0)}{1\,!} + \frac{\Pi(\frac{5}{2})}{x^{\frac{7}{2}}}\frac{\chi''(0)}{2\,!} + \ldots \quad . \quad (89);$$

and when hx, and therefore also kx, is sufficiently large, the first terms in the expansions will give an adequate approximation.

Thus, taking account of (86), the last members of (87) and (88) are equivalent to

$$- 2i\pi H \cos \kappa x + 2\surd(2\pi) \sqrt{\left(1 - \frac{h^2}{k^2}\right)} \cdot \frac{ie^{-i(kx+\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$+ 2\surd(2\pi) \frac{h^3k^2 \surd(k^2 - h^2)}{(k^2 - 2h^2)^3} \cdot \frac{e^{-i(hx+\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c.,$$

and

$$2\pi K \sin \kappa x - 4\surd(2\pi) \left(1 - \frac{h^2}{k^2}\right) \cdot \frac{ie^{-i(kx+\frac{3}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$- \surd(2\pi) \frac{h^2k^2}{(k^2 - 2h^2)^2} \cdot \frac{ie^{i-(hx+\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c.,$$

respectively. Substituting in (52), and adding in the system (72) as before, we have, for large positive values of x,

$$u_0 = - \frac{Q}{\mu} He^{i(pt-\kappa x)} + \frac{Q}{\mu} \sqrt{\frac{2}{\pi}} \sqrt{\left(1 - \frac{h^2}{k^2}\right)} \cdot \frac{e^{i(pt-kx-\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$- \frac{Q}{\mu} \sqrt{\frac{2}{\pi}} \cdot \frac{h^3k^2 \surd(k^2 - h^2)}{(k^2 - 2h^2)^3} \cdot \frac{ie^{i(pt-hx-\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c. \quad . \quad (90),$$

$$v_0 = - \frac{iQ}{\mu} Ke^{i(pt-\kappa x)} + \frac{2Q}{\mu} \sqrt{\frac{2}{\pi}} \cdot \left(1 - \frac{h^2}{k^2}\right) \cdot \frac{ie^{i(pt-kx-\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$+ \frac{Q}{2\mu} \sqrt{\frac{2}{\pi}} \cdot \frac{h^2k^2}{(k^2 - 2h^2)^2} \cdot \frac{ie^{i(pt-hx-\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c. \quad . \quad . \quad . \quad (91).$$

The first terms in these expressions have already been interpreted. The residual disturbance constitutes a sort of fringe to the cylindrical elastic waves which are propagated into the interior of the solid, and consists of two parts. In one of these the wave-velocity p/k, or b^{-1}, is that of equivoluminal waves; the vibrations (at the surface) are elliptic, the ratio of the vertical to the horizontal diameter of the orbit being $2\surd(1 - h^2/k^2)$, or 1·633 for $\lambda = \mu$. The remaining part has the wave-velocity p/h, or a^{-1}, of irrotational waves; the surface vibrations which it represents are rectilinear, the ratio of the vertical to the horizontal amplitude being $(k^2 - 2h^2)/2h(k^2 - h^2)^{\frac{1}{2}}$, or ·3535 for $\lambda = \mu$. With increasing distance x the amplitude of each part diminishes as $x^{-\frac{3}{2}}$, whereas in an unlimited solid the law is $x^{-\frac{1}{2}}$, as appears from (42).

Similar results will obviously hold in the case of the other problems considered in Art. 5.

9. It has been assumed, up to this stage, that the primary disturbance varies as a simple-harmonic function of the time. It is proposed now to generalize the law of variation, and in particular to examine the effect of a single impulse of short duration. From this the general case can be inferred by superposition.

It is to be noticed, in all our formulæ, that if we write

$$\xi = p\theta, \quad h = pa, \quad k = pb, \quad \kappa = pc,$$

the symbol p which determines the frequency will disappear, except in the exponentials; this greatly facilitates the desired generalization by means of FOURIER's theorem. Thus, in the case of a concentrated vertical pressure $Q(t)$ acting on the surface, the formulæ (73) and (74) lead to

$$u_0 = -\frac{H}{\mu} Q(t - cx) - \frac{2}{\pi\mu}\int_a^b \frac{b^2\theta(2\theta^2 - b^2)\sqrt{(\theta^2 - a^2)}\sqrt{(b^2 - \theta^2)}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} \cdot Q(t - \theta x)\, d\theta \quad . \quad (92),$$

$$v_0 = -\frac{1}{\pi\mu}\int_a^b \frac{b^2(2\theta^2 - b^2)^2\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} \cdot Q(t - \theta x)\, d\theta$$

$$- \frac{1}{\pi\mu}\,\mathfrak{P}\int_b^\infty \frac{b^2\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^2 - 4\theta^2\sqrt{(\theta^2 - a^2)}\sqrt{(\theta^2 - b^2)}} \cdot Q(t - \theta x)\, d\theta \quad . \quad (93).$$

The definite integrals represent aggregates of waves, of the same general type, travelling with slownesses ranging from a to b, and from b to ∞, respectively.

If we suppose that $Q(t)$ vanishes for all but small values of t, it appears from (92) that the horizontal disturbance at a distance x begins (as we should expect) after a time ax, which is the time a wave of expansion would take to travel the distance; it lasts till a time bx, which is the time distortional waves would take to travel the distance; and then, for a while, ceases.* Finally, about the time cx, comes a solitary wave of short duration (the same as that of the primary impulse) represented by the first term of (92). This wave is of unchanging type, whereas the duration of the preliminary disturbance varies directly as x, and its amplitude (as will be seen immediately) varies inversely as x.

If we put

$$\bar{Q} = \int Q(t)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94),$$

the integration extending over the short range for which Q is sensible, the preliminary horizontal disturbance will be given by

$$u_0 = \frac{2\bar{Q}}{\pi\mu b x} \cdot U\left(\frac{t}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

provided

$$U(\theta) = -\frac{b^3\theta(2\theta^2 - b^2)\sqrt{(\theta^2 - a^2)}\sqrt{(b^2 - \theta^2)}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} \quad . \quad . \quad . \quad . \quad (96),$$

where $a < \theta < b$. The following table gives the values of $U(\theta)$ for a series of values of θ/a, on the hypothesis of $\lambda = \mu$, or $b/a = 1{\cdot}7321$.

* This temporary cessation of the horizontal motion is special to the case of a normal impulse. If the impulse be tangential, the contrast between the horizontal and vertical motions, in this respect, is reversed.

θ/a.	U (θ).	θ/a.	U (θ).	θ/a.	U (θ).	θ/a.	U (θ).
1·000	0	1·025	+ ·62777	1·10	+ ·22789	1·550	− ·15122
1·001	+ ·31247	1·030	+ ·59351	1·15	+ ·10295	1·600	− ·15842
1·002	+ ·42080	1·035	+ ·55806	1·20	+ ·02722	1·625	− ·15927
1·003	+ ·49148	1·040	+ ·52308	1·25	− ·02311	1·650	− ·15681
1·004	+ ·54191	1·050	+ ·45741	1·30	− ·05905	1·675	− ·14845
1·005	+ ·57926	1·060	+ ·39889	1·35	− ·08622	1·700	− ·12795
1·010	+ ·66493	1·070	+ ·34746	1·40	− ·10771	1·725	− ·07021
1·015	+ ·67536	1·080	+ ·30238	1·45	− ·12527	b/a	0
1·020	+ ·65744	1·090	+ ·26279	1·50	− ·13975	—	—

The function has a maximum value + ·67643 when $\theta/a = 1{\cdot}01368$; it changes sign when $\theta/a = 1{\cdot}22474$; and it has a minimum value − ·159319 when $\theta/a = 1{\cdot}62076$.* The graph of this function is shown in the upper part of fig. 4. If the scales be

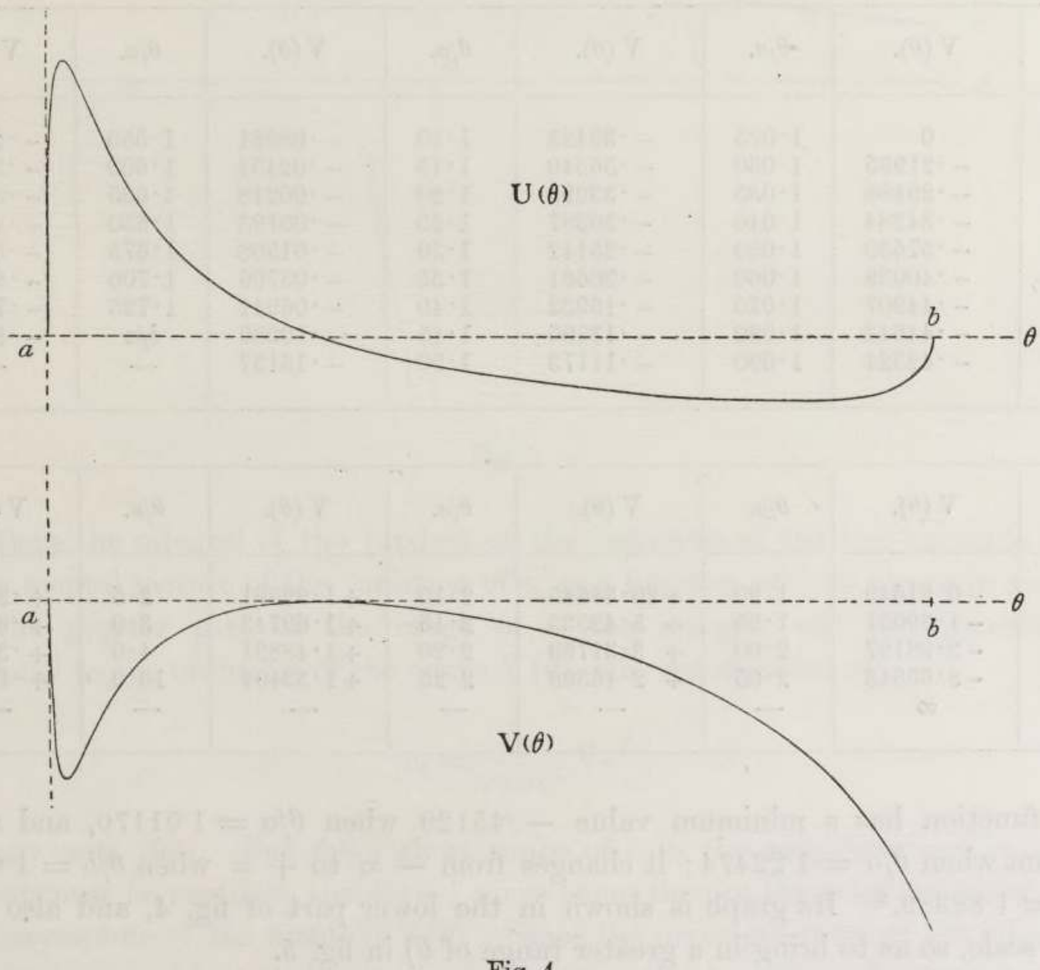

Fig. 4.

properly chosen, the curve will represent the variation of u_0 with t, during the "preliminary" disturbance, at any assigned point x. For this purpose the horizontal scale must vary directly, and the vertical scale inversely, as x.

* The calculations were made almost entirely by Mr. H. J. WOODALL, to whom I am much indebted.

The interpretation of the expression (93) for the vertical displacement v_0 is not quite so simple. For a given value of x, the most important part is that corresponding to $t = cx$, or $\theta = c$, nearly, when the integrand in the second term changes sign by passing through infinity. This is the epoch of the main shock; the minor disturbance which sets in when $t = ax$ leads up continuously to this, and only dies out gradually after it.

As a first step we may tabulate the function V (θ) defined by

$$\mathrm{V}(\theta) = -\frac{b^3(2\theta^2 - b^2)^2\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)}, \text{ for } a < \theta < b,$$

$$= -\frac{b^3\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^2 - 4\theta^2\sqrt{(\theta^2 - a^2)}\sqrt{(\theta^2 - b^2)}}, \text{ for } \theta > b \quad . \quad . \quad (97).$$

θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).
1·000	0	1·025	− ·39425	1·10	− ·08981	1·550	− ·22781
1·001	− ·21995	1·030	− ·36340	1·15	− ·02454	1·600	− ·31645
1·002	− ·29488	1·035	− ·33293	1·20	− ·00218	1·625	− ·37299
1·003	− ·34284	1·040	− ·30387	1·25	− ·00193	1·650	− ·44110
1·004	− ·37630	1·050	− ·25142	1·30	− ·01508	1·675	− ·52493
1·005	− ·40039	1·060	− ·20681	1·35	− ·03796	1·700	− ·63087
1·010	− ·44907	1·070	− ·16932	1·40	− ·06941	1·725	− ·76935
1·015	− ·44543	1·080	− ·13795	1·45	− ·10989	b/a	− ·81649
1·020	− ·42324	1·090	− ·11173	1·50	− ·16137	—	—

θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).
b/a	−0·81649	1·90	+20·38685	2·10	+1·99591	2·5	+ ·91464
1·75	−1·39031	1·95	+ 5·42335	2·15	+1·69743	3·0	+ ·60196
1·80	−2·98197	2·00	+ 3·31759	2·20	+1·48891	4·0	+ ·38179
1·85	−8·65843	2·05	+ 2·46398	2·25	+1·33404	10·0	+ ·13292
c/a	∞	—	—	—	—	—	—

The function has a minimum value − ·45120 when $\theta/a = 1·01170$, and a zero maximum when $\theta/a = 1·22474$; it changes from $-\infty$ to $+\infty$ when $\theta/b = 1·08767$, or $\theta/a = 1·88389$.* Its graph is shown in the lower part of fig. 4, and also (on a smaller scale, so as to bring in a greater range of θ) in fig. 5.

It is postulated that the function Q (t) is sensible only for values of t lying within a short range on each side of 0; the function Q ($t - \theta x$) will therefore be sensible only for values of θ in the neighbourhood of t/x. We will suppose that for given values of x and t its graph (as a function of θ) has some such form as that of the

* As in the case of U (θ), the calculations are due chiefly to Mr. WOODALL.

dotted curve in fig. 5. If x be constant, the effect of increasing t will be to cause this graph to travel uniformly from left to right; and if we imagine that in each of

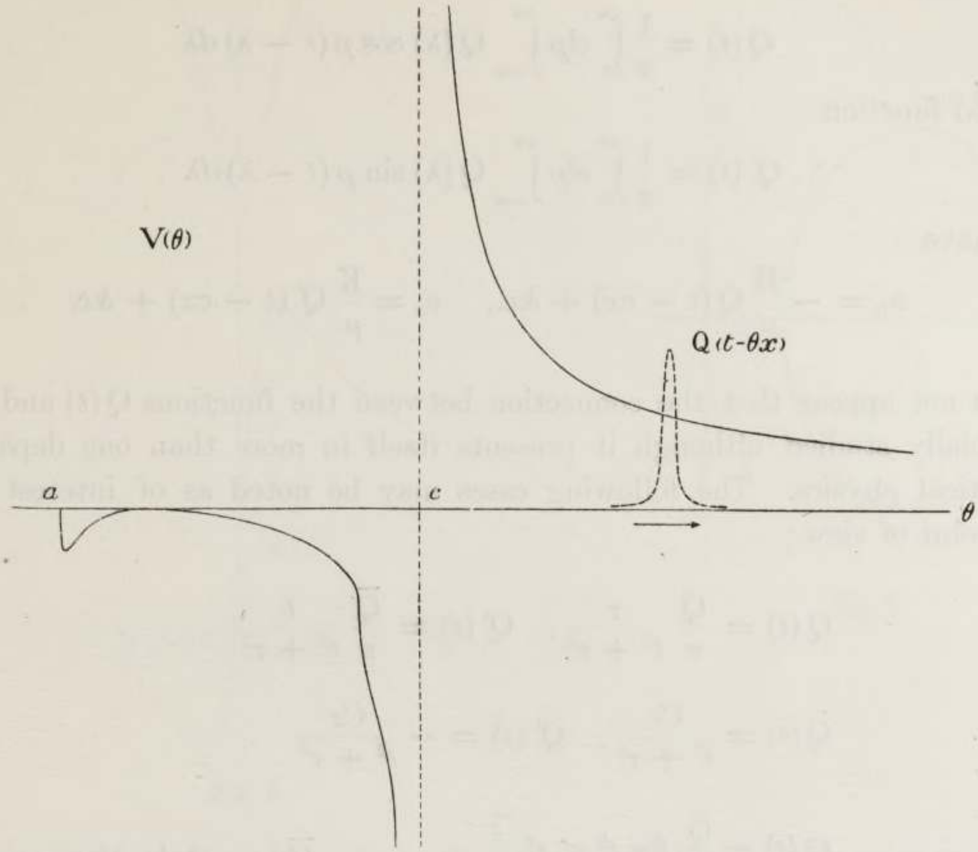

Fig. 5.

its positions the integral of the product of the ordinates of the two curves is taken, we get a mental picture of the variation of v_0 as a function of t, on a certain scale.

For the greater part of the range of t, the integral will be approximately proportional to the ordinates of the curve $\mathrm{V}(\theta)$, viz., we shall have

$$v_0 = \frac{\overline{\mathrm{Q}}}{\pi\mu b x} \cdot \mathrm{V}\left(\frac{t}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

in analogy with (95). But for a short range of t, in the neighbourhood or cx, the statement must be modified, the dotted curve being then in the neighbourhood of the vertical asymptote of the function $\mathrm{V}(\theta)$. Since the *principal value* of the integral is to be taken, it is evident that as t approaches the critical epoch and passes it, v_0 will sink to a relatively low minimum, and then passing through zero will attain a correspondingly high maximum, after which it will decrease asymptotically to zero, the later stages coming again under the formula (98).

Although the above argument gives perhaps the best view of the whole course of the disturbance, we are not dependent upon it for a knowledge of what takes place about the critical epoch cx. We may proceed, instead, by generalizing the expressions (77). This introduces, in addition to the given function $Q(t)$, whose Fourier expression is

$$Q(t) = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^\infty Q(\lambda)\cos p(t-\lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (99),$$

the related function

$$Q'(t) = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^\infty Q(\lambda)\sin p(t-\lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (100);$$

viz., we have

$$u_0 = -\frac{H}{\mu} Q(t-cx) + \&c., \quad v_0 = \frac{K}{\mu} Q'(t-cx) + \&c. \quad . \quad . \quad (101).$$

It does not appear that the connection between the functions $Q(t)$ and $Q'(t)$ has been specially studied, although it presents itself in more than one department of mathematical physics. The following cases may be noted as of interest from our present point of view:

$$Q(t) = \frac{\bar{Q}}{\pi}\frac{\tau}{t^2+\tau^2}, \quad Q'(t) = \frac{\bar{Q}}{\pi}\frac{t}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (102);$$

$$Q(t) = \frac{Ct}{t^2+\tau^2}, \quad Q'(t) = -\frac{C\tau}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103);$$

$$\left.\begin{aligned} Q(t) &= \frac{\bar{Q}}{2\tau} \text{ for } t^2 < \tau^2, \\ &= 0 \text{ for } t^2 > \tau^2, \end{aligned}\right\} \quad Q'(t) = \frac{\bar{Q}}{4\pi\tau}\log\left(\frac{t+\tau}{t-\tau}\right)^2 \quad . \quad . \quad . \quad (104).$$

It is evident, generally, that if Q be an odd function, Q' will be an even function, and *vice versâ*.

The values of u_0 and v_0, as given by (101), are represented graphically in fig. 6, for the case where $Q(t)$ and $Q'(t)$ have the forms given in (102).* Moreover, writing

$$H\bar{Q}/2\pi\mu\tau = f, \quad K\bar{Q}/2\pi\mu\tau = g, \quad t - cx = \tau \tan\chi,$$

we have

$$u_0 = -(1+\cos 2\chi)\,.\,f, \quad v_0 = \sin 2\chi\,.\,g \quad . \quad . \quad . \quad . \quad . \quad (105);$$

the orbit of a surface-particle is therefore an ellipse with horizontal and vertical semi-axes f and g. And if from the equilibrium position O we project any other position P of the particle on to a vertical straight line, the law of P's motion is that the projection (R) describes this line with constant velocity. See fig. 7, where the positive direction of y is supposed to be downwards.

* The relation between the scales of the ordinates in the graphs of u_0 and v_0 depends upon the ratio of the elastic constants λ, μ. The figures are constructed on the hypothesis of $\lambda = \mu$.

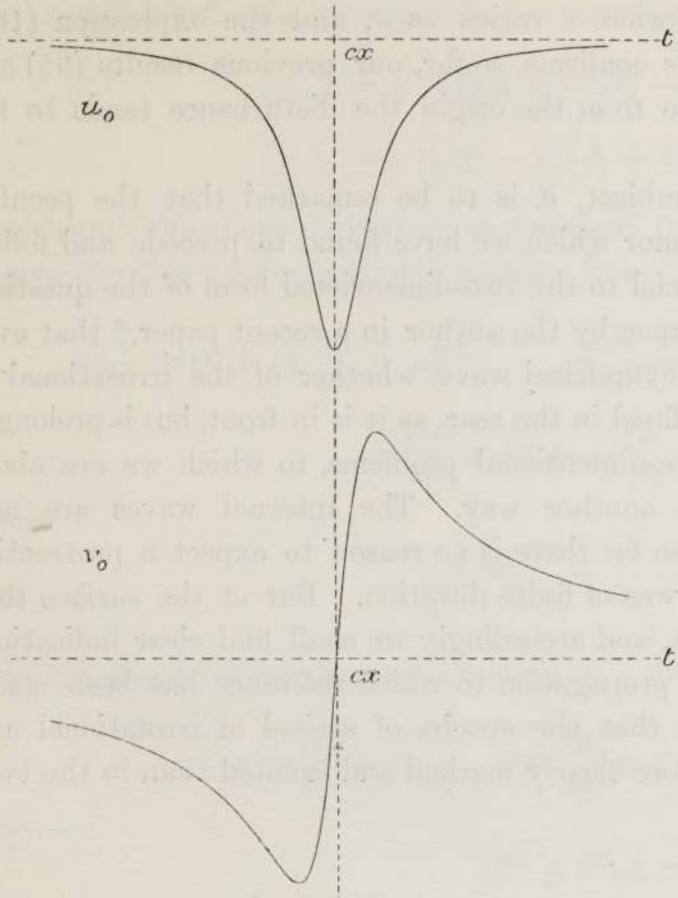

Fig. 6.

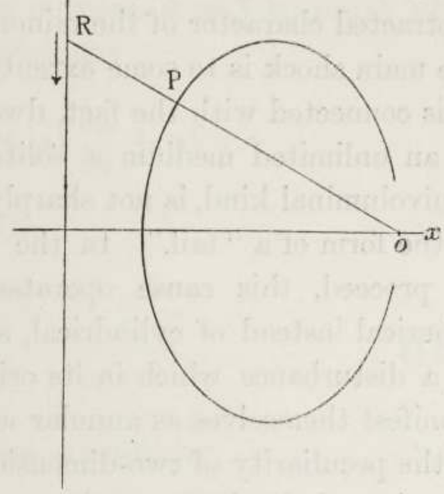

Fig. 7.

A similar treatment would apply to the formulæ (81), and (with some modification) to (84).

It remains to justify these approximations by showing that the residual disturbance tends with increasing x to the limit 0. For this purpose we have recourse to the formulæ of Art. 8. As a sufficient example, take the second term in the last member of (88). If we multiply by e^{ipt}, take the real part, and substitute $\eta = p\phi$, $k = pb$, the corresponding term in the value of v_0, as given by (52), assumes the form*

$$\frac{Q}{\mu}\cos p(t-bx)\int_0^\infty F(\phi)\,e^{-px\phi}\,d\phi + \frac{Q}{\mu}\sin p(t-bx)\int_0^\infty f(\phi)\,e^{-px\phi}\,d\phi,$$

where the functions $F(\phi)$ and $f(\phi)$, which do not involve p, are of the order ϕ^{-1} when ϕ is large. If we generalize this expression by FOURIER's Theorem (see equation (99)), we obtain, in the case of an impulse $\bar{Q}$ of short duration,

$$\frac{\bar{Q}}{\pi\mu}\int_0^\infty F(\phi)\,d\phi\int_0^\infty e^{-x\phi p}\cos p(t-bx)\,dp + \frac{\bar{Q}}{\pi\mu}\int_0^\infty f(\phi)\,d\phi\int_0^\infty e^{-x\phi p}\sin p(t-bx)\,dp$$

$$= \frac{\bar{Q}}{\pi\mu}\int_0^\infty F(\phi)\,\frac{x\phi\,d\phi}{x^2\phi^2+(t-bx)^2} + \frac{\bar{Q}}{\pi\mu}\int_0^\infty f(\phi)\,\frac{(t-bx)\,d\phi}{x^2\phi^2+(t-bx)^2} \quad . \quad . \quad (106).$$

* The symbols ϕ, F, f are here used temporarily in new senses.

For any particular phase of the motion, t varies as x, and the expression (106) therefore varies inversely as x. This confirms, so far, our previous results (95) and (98). Hence with increasing distance from the origin the disturbance tends to the limiting form represented by (101).

Before leaving this part of the subject, it is to be remarked that the peculiar protracted character of the minor tremor which we have found to precede and follow the main shock is to some extent special to the two-dimensional form of the question. It is connected with the fact, dwelt upon by the author in a recent paper,* that even in an unlimited medium a solitary cylindrical wave, whether of the irrotational or equivoluminal kind, is not sharply defined in the rear, as it is in front, but is prolonged in the form of a "tail." In the three-dimensional problems, to which we are about to proceed, this cause operates in another way. The internal waves are now spherical instead of cylindrical, and so far there is no reason to expect a protraction of a disturbance which in its origin was of finite duration. But at the surface they manifest themselves as annular waves, and accordingly we shall find clear indications of the peculiarity of two-dimensional propagation to which reference has been made. On the whole, however, it appears that the epochs of arrival of irrotational and equivoluminal waves are relatively more clearly marked and isolated than in the two-dimensional cases.

PART II.

Three-Dimensional Problems.

10. Assuming symmetry about the axis of z, we write

$$\varpi = \sqrt{(x^2 + y^2)}, \qquad u = \frac{x}{\varpi}\, q, \qquad v = \frac{y}{\varpi}\, q \quad . \quad . \quad . \quad . \quad (107),$$

so that q denotes displacement perpendicular to that axis.

A typical solution of the elastic equations, convenient for our purposes, is derived at once from Art. 3, if we imagine an infinite number of two-dimensional vibration-types of the kind specified by (25) and (30) to be arranged uniformly in all azimuths about the axis of z, and take the mean. In this way we obtain from (33), with the necessary change of notation,

$$\left.\begin{aligned} q_0 &= (i\xi\mathrm{A} - \beta\mathrm{B}) \,.\, \frac{1}{\pi}\int_0^\pi e^{i\xi\varpi\cos\omega}\cos\omega\, d\omega = -(\xi\mathrm{A} + i\beta\mathrm{B})\,\mathrm{J}_1(\xi\varpi) \\ w_0 &= (-\alpha\mathrm{A} - i\xi\mathrm{B}) \,.\, \frac{1}{\pi}\int_0^\pi e^{i\xi\varpi\cos\omega}\, d\omega = -(\alpha\mathrm{A} + i\xi\mathrm{B})\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} . \quad (108).$$

* Cited on p. 37 *post*.

Also, from (40), for the corresponding stresses at the plane $z = 0$, we have

$$\left.\begin{aligned}[p_{z\varpi}]_0 &= \mu\,\{2\xi\alpha\mathrm{A} + i\,(2\xi^2 - k^2)\,\mathrm{B}\}\,\mathrm{J}_1(\xi\varpi)\\ [p_{zz}]_0 &= \mu\,\{(2\xi^2 - k^2)\,\mathrm{A} + 2i\xi\beta\mathrm{B}\}\,\mathrm{J}_0(\xi\varpi)\end{aligned}\right\} \quad . \quad . \quad . \quad (109).$$

Although the above derivation is sufficient for our purpose, it may be worth while to give the direct investigation,* starting from the equations

$$\rho\,\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\,\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u, \qquad \rho\,\frac{\partial^2 v}{\partial t^2} = (\lambda + \mu)\,\frac{\partial\Delta}{\partial y} + \mu\nabla^2 v,$$

$$\rho\,\frac{\partial^2 w}{\partial t^2} = (\lambda + \mu)\,\frac{\partial\Delta}{\partial z} + \mu\nabla^2 w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110),$$

where

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111).$$

In the case of simple-harmonic motion (e^{ipt}) these are satisfied by

$$u = \frac{\partial\phi}{\partial x} + u', \qquad v = \frac{\partial\phi}{\partial y} + v', \qquad w = \frac{\partial\phi}{\partial z} + w' \quad . \quad . \quad . \quad . \quad (112),$$

provided

$$(\nabla^2 + h^2)\,\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113),$$

and

$$\left.\begin{aligned}(\nabla^2 + k^2)\,u' = 0, \quad (\nabla^2 + k^2)\,v' = 0, \quad (\nabla^2 + k^2)\,w' = 0\\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\end{aligned}\right\} \quad . \quad . \quad (114),$$

where h^2, k^2 are defined as before by (28). A particular solution of (114) is

$$u' = \frac{\partial^2\chi}{\partial x\,\partial z}, \quad v' = \frac{\partial^2\chi}{\partial y\,\partial z}, \quad w' = \frac{\partial^2\chi}{\partial z^2} + k^2\chi \quad . \quad . \quad . \quad . \quad . \quad (115).$$

provided

$$(\nabla^2 + k^2)\,\chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

On the hypothesis of symmetry about Oz we have

$$\nabla^2 = \frac{\partial^2}{\partial\varpi^2} + \frac{1}{\varpi}\,\frac{\partial}{\partial\varpi} + \frac{\partial^2}{\partial z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117),$$

and the formulæ (112), (115) are equivalent to

$$q = \frac{\partial\phi}{\partial\varpi} + \frac{\partial^2\chi}{\partial\varpi\,\partial z}, \quad w = \frac{\partial\phi}{\partial z} + \frac{\partial^2\chi}{\partial z^2} + k^2\chi \quad . \quad . \quad . \quad . \quad . \quad (118).$$

* *Cf.* 'Proc. Lond. Math. Soc.,' vol. 34, p. 276, for the corresponding statical investigation.

If we take, as the typical solution of (113) and (116),

$$\phi = \mathrm{A}e^{-\alpha z}\,\mathrm{J}_0(\xi\varpi), \quad \chi = \mathrm{B}e^{-\beta z}\,\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad (119),$$

where α, β have the same meanings and are subject to the same convention as in Art. 3, we have, from (118),

$$\left.\begin{aligned} q &= (-\,\xi\mathrm{A}e^{-\alpha z} + \xi\beta\mathrm{B}e^{-\beta z})\,\mathrm{J}_1(\xi\varpi) \\ w &= (-\,\alpha\mathrm{A}e^{-\alpha z} + \xi^2\mathrm{B}e^{-\beta z})\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (120);$$

and thence for the stresses in the plane $z = 0$

$$\left.\begin{aligned} [p_{z\varpi}]_0 &= \mu\left[\frac{\partial q}{\partial z} + \frac{\partial w}{\partial \varpi}\right]_0 = \mu\,\{2\xi\alpha\mathrm{A} - (2\xi^2 - k^2)\,\xi\mathrm{B}\}\,\mathrm{J}_1(\xi\varpi) \\ [p_{zz}]_0 &= \left[\lambda\Delta + 2\mu\,\frac{\partial w}{\partial z}\right]_0 = \mu\,\{(2\xi^2 - k^2)\,\mathrm{A} - 2\xi^2\beta\mathrm{B}\}\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} \quad . \quad (121).$$

The formulæ differ from (108) and (109) only in the substitution of $i\xi\mathrm{B}$ for B. The notation of (119) is adopted as the basis of the subsequent calculations.

If we are to assume, in place of (119),

$$\phi = \mathrm{A}'e^{\alpha z}\mathrm{J}_0(\xi\varpi), \quad \chi = \mathrm{B}'e^{\beta z}\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122),$$

the corresponding forms of (120) and (121) would be obtained by affixing accents to A and B, and changing the signs of α and β where they occur explicitly.

11. As in Art. 4, we begin by applying the preceding formulæ to the solution of a known problem, viz., where a given periodic force acts at a point in an unlimited solid.

Let us suppose, in the first place, that an extraneous force of amount $\mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi)\,e^{ipt}$, per unit area, acts parallel to z on an infinitely thin stratum coincident with the plane $z = 0$. The formulæ (119) will then apply for $z > 0$, and (122) for $z < 0$. The normal stress will be discontinuous, viz.:

$$[p_{zz}]_{z=+0} - [p_{zz}]_{z=-0} = -\,\mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123),$$

whilst $p_{z\varpi}$ is continuous. Hence

$$\left.\begin{aligned} (2\xi^2 - k^2)(\mathrm{A} - \mathrm{A}') - 2\xi^2\beta(\mathrm{B} + \mathrm{B}') &= -\,\frac{\mathrm{Z}}{\mu} \\ 2\alpha(\mathrm{A} + \mathrm{A}') - (2\xi^2 - k^2)(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (124).$$

Also, the continuity of q and ϖ requires

$$\left.\begin{aligned} \mathrm{A} - \mathrm{A}' - \beta(\mathrm{B} + \mathrm{B}') &= 0 \\ \alpha(\mathrm{A} + \mathrm{A}') - \xi^2(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125).$$

We infer

$$\mathrm{A} = -\,\mathrm{A}' = \frac{\mathrm{Z}}{2k^2\mu}, \quad \mathrm{B} = \mathrm{B}' = \frac{\mathrm{Z}}{2k^2\mu\beta} \quad . \quad . \quad . \quad . \quad . \quad (126),$$

and therefore, for $z > 0$,

$$\phi = \frac{Z}{2k^2\mu} e^{-\alpha z} J_0(\xi\varpi), \quad \chi = \frac{Z}{2k^2\mu} \frac{e^{-\beta z}}{\beta} J_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad (127).$$

To pass to the case of a concentrated force Re^{ipt}, acting parallel to z at the origin, we have recourse to the formula (20), where we suppose $f(\lambda)$ to vanish for all but infinitesimal values of λ, and to become infinite for these in such a way that

$$\int_0^\infty f(\lambda)\, 2\pi\lambda\, d\lambda = R.$$

We therefore write $Z = R\xi\, d\xi/2\pi$, and integrate with respect to ξ from 0 to ∞.* We thus find, for $z > 0$,

$$\phi = \frac{R}{4\pi p^2\rho} \int_0^\infty e^{-\alpha z} J_0(\xi\varpi)\, \xi\, d\xi, \qquad \chi = \frac{R}{4\pi p^2\rho} \int_0^\infty \frac{e^{-\beta z}}{\beta} J_0(\xi\varpi)\, \xi\, d\xi \quad . \quad (128),$$

which are equivalent, by (18), to

$$\phi = -\frac{R}{4\pi p^2\rho} \cdot \frac{\partial}{\partial z} \frac{e^{-ihr}}{r}, \qquad \chi = \frac{R}{4\pi p^2\rho} \cdot \frac{e^{-ikr}}{r} \quad . \quad . \quad . \quad . \quad (129).$$

This will be found to agree with the known solution of the problem.† If we retain only the terms which are most important at a great distance r, we find, from (118),

$$\left.\begin{aligned} q &= \frac{R}{4\pi} \left\{ \frac{1}{\lambda + 2\mu} \frac{z\varpi}{r^3} e^{-ihr} - \frac{1}{\mu} \frac{z\varpi}{r^3} e^{-ikr} \right\} \\ w &= \frac{R}{4\pi} \left\{ \frac{1}{\lambda + 2\mu} \frac{z^2}{r^3} e^{-ihr} + \frac{1}{\mu} \frac{\varpi^2}{r^3} e^{-ikr} \right\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (130).$$

Inserting the time-factor, the radial displacement is

$$\frac{zw + \varpi q}{r} = \frac{R}{4\pi(\lambda + 2\mu)} \cdot \frac{z}{r^2} e^{ip(t - ar)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (131),$$

and the transverse displacement in the meridian plane is

$$\frac{\varpi w - zq}{r} = \frac{R}{4\pi\mu} \cdot \frac{\varpi}{r^2} e^{ip(t - br)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132).$$

Returning to the exact formulæ (128), the expression for the velocity parallel to z at the plane $z = 0$ is found to be

$$\frac{\partial w}{\partial t} = \frac{iRe^{ipt}}{4\pi p\rho} \int_0^\infty \left(-\alpha + \frac{\xi^2}{\beta} \right) J_0(\xi\varpi)\, \xi\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (133),$$

* A more rigorous procedure would be to suppose in the first instance that the force R is uniformly distributed over a circular area of radius a, using the formula (22). If in the end we make $a = 0$, we obtain the results in the text.

† STOKES, 'Camb. Trans.,' vol. 9 (1849); 'Mathematical and Physical Papers,' vol. 2, p. 278.

or, taking the real part,

$$\frac{\partial w}{\partial t} = \frac{\mathrm{R}}{4\pi p\rho}\left\{\int_0^k \frac{\xi^3}{\sqrt{(k^2-\xi^2)}}\,\mathrm{J}_0(\xi\varpi)\,d\xi + \int_0^h \xi\sqrt{(h^2-\xi^2)}\,\mathrm{J}_0(\xi\varpi)\,d\xi\right\}\cos pt$$
$$+ \text{ terms in } \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134).$$

The terms in $\cos pt$ remain finite when we put $\varpi = 0$;* and the mean rate W at which a force $\mathrm{R}\cos pt$ does work in generating waves is thus found to be

$$\mathrm{W} = \frac{\mathrm{R}^2}{8\pi p\rho}\left\{\int_0^k \frac{\xi^3\,d\xi}{\sqrt{(k^2-\xi^2)}} + \int_0^h \xi\sqrt{(h^2-\xi^2)}\,d\xi\right\}$$
$$= \frac{\mathrm{R}^2}{24\pi p\rho}\cdot(2k^3+h^3) = \frac{p^2\mathrm{R}^2}{24\pi\rho}(a^3+2b^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (135),$$

a and b denoting as before the two elastic wave-slownesses. The result (135) can be deduced, as a particular case, from formulæ given by Lord KELVIN.†

12. Proceeding to the case of a semi-infinite solid occupying (say) the region $z > 0$, we begin with the special distribution of surface-stress :

$$[p_{zz}]_0 = \mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi), \quad [p_{z\varpi}] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136).$$

The coefficients A, B in (119) are now determined by

$$\left.\begin{aligned}(2\xi^2-k^2)\,\mathrm{A} - 2\xi^2\beta\mathrm{B} &= \frac{\mathrm{Z}}{\mu}\\ 2\alpha\mathrm{A} - (2\xi^2-k^2)\,\mathrm{B} &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137),$$

whence

$$\mathrm{A} = \frac{2\xi^2-k^2}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Z}}{\mu}, \quad \mathrm{B} = \frac{2\alpha}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Z}}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138),$$

the function $\mathrm{F}(\xi)$ having the same meaning as in Art. 5. The corresponding surface-displacements are

$$\left.\begin{aligned}q_0 &= -\frac{\xi(2\xi^2-k^2-2\alpha\beta)}{\mathrm{F}(\xi)}\cdot\mathrm{J}_1(\xi\varpi)\cdot\frac{\mathrm{Z}}{\mu}\\ w_0 &= \frac{k^2\alpha}{\mathrm{F}(\xi)}\cdot\mathrm{J}_0(\xi\varpi)\cdot\frac{\mathrm{Z}}{\mu}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (139).$$

This result might have been deduced immediately from (51) in the manner indicated at the beginning of Art. 10.

* The terms in $\sin pt$ become infinite. If the force R be distributed over a circular area, the awkwardness is avoided. A factor

$$\left\{\frac{\mathrm{J}_1(\xi a)}{\frac{1}{2}\xi a}\right\}^2$$

is thus introduced under the integral signs in the first line of (135), where a denotes (for the moment) the radius of the circle. Finally, we can make a infinitely small.

† 'Phil. Mag.,' Aug. 1899, pp. 234, 235.

If we put $Z = 0$ in (137) we get a system of free annular surface-waves, in which

$$\left.\begin{aligned} q_0 &= -\kappa\,(2\kappa^2 - k^2 - 2\alpha_1\beta_1) \,.\, J_1(\kappa\varpi) \,.\, Ce^{ipt} \\ w_0 &= k^2\alpha_1 \,.\, J_0(\kappa\varpi) \,.\, Ce^{ipt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (140),$$

where κ is the positive root of $F(\xi) = 0$, and α_1, β_1 are the corresponding values of α, β. These are of the nature of "standing" waves.

To pass to the case of a concentrated vertical pressure Re^{ipt} at O,* we put in accordance with (20), $Z = -R\xi d\xi/2\pi$, and integrate from 0 to ∞.† The formulæ (139) become

$$\left.\begin{aligned} q_0 &= \frac{R}{2\pi\mu}\int_0^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)}\, J_1(\xi\varpi)\, d\xi \\ w_0 &= -\frac{R}{2\pi\mu}\int_0^\infty \frac{k^2\xi\alpha}{F(\xi)}\, J_0(\xi\varpi)\, d\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (141).$$

Again, the case of an internal source of the type

$$\phi = \frac{e^{-ikr}}{r}, \quad \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142),$$

where r denotes distance from the point $(0, 0, f)$, can be solved by a process similar to that of Art. 5. First, superposing an equal source at $(0, 0, -f)$, distance from which is denoted by r', we have

$$\phi = \frac{e^{-ikr}}{r} + \frac{e^{-ikr'}}{r'}, \quad \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143);$$

and therefore, by (18), in the neighbourhood of the plane $z = 0$,

$$\phi = \int_0^\infty \frac{e^{-\alpha(z+f)}}{\alpha} J_0(\xi\varpi)\, \xi d\xi + \int_0^\infty \frac{e^{\alpha(z-f)}}{\alpha} J_0(\xi\varpi)\, \xi d\xi$$

$$= 2\int_0^\infty \frac{\cosh \alpha z}{\alpha}\, e^{-\alpha f} J_0(\xi\varpi)\, \xi d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144).$$

This makes

$$q_0 = -2\int_0^\infty \frac{e^{-\alpha f}}{\alpha} J_1(\xi\varpi)\, \xi^2\, d\xi, \quad w_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (145),$$

* This may be regarded as the kinetic analogue of BOUSSINESQ's well-known statical problem.

† It might appear at first sight that a simpler procedure would be possible, and that the effect of a pressure concentrated at a point might be inferred by superposing *lines* of pressure (through O) uniformly in all azimuths, and using the results of § 7. It is easily seen, however, that such a distribution of lines of pressure is equivalent to a pressure-intensity varying inversely as the distance (ϖ) from O. This is not an adequate representation of a localized pressure, since it makes the total pressure on a circular area having its centre at O increase indefinitely with the radius of the circle.

and

$$[p_{z\varpi}]_0 = 0, \quad [p_{zz}]_0 = 2\mu \int_0^\infty \frac{(2\xi^2 - k^2)}{\alpha} e^{-\alpha f} J_0(\xi\varpi)\, \xi\, d\xi \quad . \quad . \quad . \quad (146).$$

The additions to (143) which are required in order to annul the stresses on the plane $z = 0$ are accordingly found by writing

$$Z = -2\mu \cdot \frac{2\xi^2 - k^2}{\alpha} e^{-\alpha f}\, \xi\, d\xi$$

in (139), and then integrating with respect to ξ from 0 to ∞. In this way we obtain, finally,

$$\left.\begin{aligned} q_0 &= 4 \int_0^\infty \frac{k^2\xi^2\beta}{F(\xi)} e^{-\alpha f} J_1(\xi\varpi)\, d\xi \\ w_0 &= -2 \int_0^\infty \frac{k^2\xi(2\xi^2 - k^2)}{F(\xi)} e^{-\alpha f} J_0(\xi\varpi)\, d\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (147).$$

In a similar manner, with the help of Art. 11, we might calculate the effect of a periodic vertical force, acting at an internal point.

13. For the sake of comparison with our previous two-dimensional formulæ, it is convenient to write, from (2) and (6),

$$\left.\begin{aligned} J_0(\xi\varpi) &= -\frac{i}{\pi} \int_0^\infty (e^{i\xi\varpi \cosh u} - e^{-i\xi\varpi \cosh u})\, du \\ J_1(\xi\varpi) &= -\frac{1}{\pi} \int_0^\infty (e^{i\xi\varpi \cosh u} + e^{-i\xi\varpi \cosh u}) \cosh u\, du \end{aligned}\right\} \quad . \quad . \quad . \quad (148).$$

The formulæ (141) are thus equivalent to

$$\left.\begin{aligned} q_0 &= -\frac{R}{2\pi^2\mu} \int_0^\infty \cosh u\, du \int_{-\infty}^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} e^{i\xi\varpi \cosh u}\, d\xi \\ w_0 &= \frac{iR}{2\pi^2\mu} \int_0^\infty du \int_{-\infty}^\infty \frac{k^2\xi\alpha}{F(\xi)} e^{i\xi\varpi \cosh u}\, d\xi \end{aligned}\right\} . \quad . \quad (149).$$

These results are closely comparable with (52), and our previous methods of treatment will apply. It is, however, unnecessary to go through all the details of the work, since the definite integrals with respect to ξ which appear in (149) can be derived from those in (52) by performing the operation $-i\partial/\partial x$ upon the latter, and then replacing x by $\varpi \cosh u$.

Thus, from (67) and (70) we derive

$$\mathfrak{P} \int_{-\infty}^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} e^{i\xi\varpi \cosh u}\, d\xi = 2\pi\kappa H \sin(\kappa\varpi \cosh u)$$
$$+ 4k^2 \int_h^k \frac{\xi^2(2\xi^2 - k^2)\, \alpha\beta}{F(\xi) f(\xi)} e^{-i\xi\varpi \cosh u}\, d\xi \quad . \quad (150),$$

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{k^2\xi\alpha}{F(\xi)}e^{i\xi\varpi\cosh u}\,d\xi = 2\pi\kappa K\sin(\kappa\varpi\cosh u) - 2k^2\mathfrak{P}\int_k^{\infty}\frac{\xi\alpha}{F(\xi)}e^{-i\xi\varpi\cosh u}\,d\xi$$
$$- 2k^2\int_h^k\frac{\xi(2\xi^2-k^2)^2\alpha}{F(\xi)f(\xi)}e^{-i\xi\varpi\cosh u}\,d\xi \quad . \quad . \quad . \quad . \quad . \quad (151),$$

where H and K are the numerical quantities defined by (68) and (71). Substituting in (149) we have

$$\mathfrak{P}q_0 = -\frac{\kappa R}{2\mu}\,.\,H\,.\,K_1(\kappa\varpi) + \frac{ik^2R}{\pi\mu}\int_h^k\frac{\xi^2(2\xi^2-k^2)\alpha\beta}{F(\xi)f(\xi)}D_1(\xi\varpi)\,d\xi \quad . \quad (152),$$

$$\mathfrak{P}w_0 = \frac{i\kappa R}{2\mu}\,.\,K\,.\,J_0(\kappa\varpi) - \frac{ik^2R}{2\pi\mu}\mathfrak{P}\int_k^{\infty}\frac{\xi\alpha}{F(\xi)}D_0(\xi\varpi)\,d\xi$$
$$-\frac{ik^2R}{2\pi\mu}\int_h^k\frac{\xi(2\xi^2-k^2)^2\alpha}{F(\xi)f(\xi)}D_0(\xi\varpi)\,d\xi \quad . \quad . \quad (153),$$

where the notation of the various BESSEL'S Functions is as in Art. 2.

Superposing the system of free waves in which

$$q_0 = \frac{i\kappa R}{2\mu}\,.\,H\,.\,J_1(\kappa\varpi), \quad w_0 = -\frac{i\kappa R}{2\mu}\,.\,K\,.\,J_0(\kappa\varpi) \quad . \quad . \quad . \quad . \quad (154),$$

we obtain, finally, on inserting the time-factor,

$$q_0 = -\frac{\kappa R}{2\mu}\,.\,H\,.\,D_1(\kappa\varpi)\,e^{ipt} + \frac{ik^2R}{\pi\mu}\int_h^k\frac{\xi^2(2\xi^2-k^2)\alpha\beta}{F(\xi)f(\xi)}D_1(\xi\varpi)\,e^{ipt}\,d\xi \quad . \quad (155),$$

$$w_0 = -\frac{ik^2R}{2\pi\mu}\mathfrak{P}\int_k^{\infty}\frac{\xi\alpha}{F(\xi)}D_0(\xi\varpi)\,e^{ipt}\,d\xi - \frac{ik^2R}{2\pi\mu}\int_h^k\frac{\xi(2\xi^2-k^2)^2\alpha}{F(\xi)f(\xi)}D_0(\xi\varpi)\,e^{ipt}\,d\xi \quad . \quad (156).$$

Since these expressions are made up entirely of diverging waves, they constitute the complete solution of the problem where a periodic normal force Re^{ipt} is applied to the surface at the origin.

An alternative form of (156), which puts in evidence that part of the vertical disturbance which is most important at a great distance from the origin, is obtained from (75). Attending only to the "singular" term, we find

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{k^2\xi\alpha}{F(\xi)}e^{i\xi\varpi\cosh u}\,d\xi = -2i\pi\kappa K\,.\,\cos(\kappa\varpi\cosh u) + \&c. \quad . \quad . \quad (157),$$

and therefore, from (149),

$$\mathfrak{P}w_0 = \frac{\kappa R}{2\mu}\,.\,K\,.\,K_0(\kappa\varpi) + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158).$$

Adding in the system (154) we have altogether

$$q_0 = -\frac{\kappa R}{2\mu}\,.\,H\,.\,D_1(\kappa\varpi)\,e^{ipt} + \&c., \quad w_0 = \frac{\kappa R}{2\mu}\,.\,K\,.\,D_0(\kappa\varpi)\,e^{ipt} + \&c. \quad (159).$$

Hence, by (7), we have, at a great distance ϖ,

$$q_0 = -\frac{i\kappa R}{2\mu} H \cdot \sqrt{\frac{2}{\pi\kappa\varpi}} \cdot e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)}, \quad w_0 = \frac{\kappa R}{2\mu} K \cdot \sqrt{\frac{2}{\pi\kappa\varpi}} \cdot e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)}. \quad (160).$$

This may be compared with (77). The vibrations are elliptic, with the same ratio of horizontal and vertical diameters as in the case of two dimensions; but the amplitude diminishes with increasing distance according to the usual law $\varpi^{-\frac{1}{2}}$ of annular divergence.

In the same manner we obtain, in the case of an internal source of the type (142),

$$\left.\begin{aligned} q_0 &= -\frac{4\pi k^2\kappa\beta_1}{F'(\kappa)} e^{-\alpha_1 f} D_1(\kappa\varpi) e^{ipt} + \&c., \\ w_0 &= \frac{2\pi k^2\kappa(2\kappa^2 - k^2)}{F'(\kappa)} e^{-\alpha_1 f} D_0(\kappa\varpi) e^{ipt} + \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (161),$$

where the factor $e^{-\alpha_1 f}$ shows the effect of the depth of the source.

The expressions for the residual disturbance might be derived from the formulæ of Art. 8 by the same artifice. Without attempting to give the complete results, which would be somewhat complicated, it may be sufficient to ascertain their general form, and order of magnitude, when $h\varpi$ and $k\varpi$ are large. To take, for example, the parts due to the distortional waves, if we perform the operation $-i\partial/\partial x$ on the second terms of the unnumbered expressions which occur between equations (89) and (90), above, and then replace x by $\varpi \cosh u$, the more important part of the result in each case is

$$e^{-ik\varpi\cosh u}/(k\varpi\cosh u)^{3/2},$$

multiplied by a constant factor. This result is to be substituted for the definite integrals with respect to ξ which occur in (149); the corresponding terms in q_0 and w_0 are therefore of the types

$$\frac{1}{(k\varpi)^{\frac{3}{2}}}\int_0^\infty \frac{e^{-ik\varpi\cosh u}\,du}{(\cosh u)^{\frac{1}{2}}}, \quad \text{and} \quad \frac{1}{(k\varpi)^{\frac{3}{2}}}\int_0^\infty \frac{e^{-ik\varpi\cosh u}\,du}{(\cosh u)^{\frac{3}{2}}},$$

respectively. By the method by which the asymptotic expansion (7) of the function $D_0(\zeta)$ is obtained, it may be shown, again, that these terms are ultimately comparable with

$$e^{ip(t-b\varpi)}/(k\varpi)^2,$$

where the time-factor has been restored. In the same way, the terms in q_0 and w_0 which correspond to the expansional waves are ultimately comparable with

$$e^{ip(t-a\varpi)}/(h\varpi)^2.$$

The attenuation with increasing distance is much more rapid than in the case of the annular Rayleigh waves, so that the latter ultimately predominate.* It is also much more rapid than in the case of elastic waves diverging from a centre in an unlimited medium, where the amplitude varies inversely as the distance.

14. The generalization of the preceding results, so as to apply to an arbitrary time-variation of the source, follows much the same course as in Art. 9. The full interpretation is however more difficult, so far at least as regards the minor tremors.

The main part of the disturbance, in the case of a local vertical pressure applied to the surface, is obtained by generalizing the formulæ (159). These may be written

$$q_0 = \frac{\mathrm{H}}{\pi}\frac{\mathrm{R}}{\mu}\frac{\partial}{\partial\varpi}\int_0^\infty e^{ip(t-c\varpi\cosh u)}\,du + \&c., \quad w_0 = -\frac{i\mathrm{K}}{\pi}\frac{\mathrm{R}c}{\mu}\frac{\partial}{\partial t}\int_0^\infty e^{ip(t-c\varpi\cosh u)}\,du + \&c. \quad (162).$$

Hence, corresponding to an arbitrary pressure $\mathrm{R}(t)$, we have

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}(t-c\varpi\cosh u)\,du + \&c., \quad w_0 = \frac{\mathrm{K}c}{\pi\mu}\frac{\partial}{\partial t}\int_0^\infty \mathrm{R}'(t-c\varpi\cosh u)\,du + \&c. \quad (163),$$

where, in analogy with (100),

$$\mathrm{R}'(t) = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^\infty \mathrm{R}(\lambda)\sin p(t-\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (164).$$

The character of the function of t represented by the first definite integral in (163) has been examined by the author† for various simple forms of $\mathrm{R}(t)$, and a similar treatment applies to the second integral. For example, if we take

$$\mathrm{R}(t) = \frac{\mathrm{R}}{\pi}\frac{\tau}{t^2+\tau^2}, \quad \mathrm{R}'(t) = \frac{\mathrm{R}}{\pi}\frac{t}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad (165),$$

it is found, on putting

$$t - c\varpi = \tau\tan\chi,\text{§}$$

that for values of ϖ large compared with τ/c, and for moderate values of χ,

$$\int_0^\infty \mathrm{R}(t-c\varpi\cosh u)\,du = \frac{\mathrm{R}}{2\tau}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}\cos\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\chi\right)\sqrt{(\cos\chi)} \quad . \quad (166)‡,$$

$$\int_0^\infty \mathrm{R}'(t-c\varpi\cosh u)\,du = -\frac{\mathrm{R}}{2\tau}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}\sin\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\chi\right)\sqrt{(\cos\chi)} . \quad (167),$$

approximately. Substituting in (163), we have, ignoring the residual terms,

$$\left.\begin{aligned} q_0 &= -f\sin\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi \\ w_0 &= g\cos\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (168),$$

* *Cf.* the footnote on p. 2 *ante*.

† "On Wave-Propagation in Two Dimensions," 'Proc. Lond. Math. Soc.,' vol. 35, p. 141 (1902).

‡ *Cf.* Equation (36) of the paper cited. It may be noticed that the functions on the right hand of (166) and (167) are interchanged, with a change of sign, when we reverse the sign of χ.

§ The symbol χ is no longer required in the sense of equations (115), &c.

where

$$f = \mathrm{H}\,\frac{\mathrm{R}c}{4\pi\mu\tau^2}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}, \qquad g = \mathrm{K}\,\frac{\mathrm{R}c}{4\pi\mu\tau^2}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}.$$

The following numerical table is derived from one given on p. 155 of the paper referred to :—

$2\chi/\pi$.	$(t-c\varpi)/\tau$.	q_0/f.	w_0/g.
− ·9	− 6·314	− ·014	− ·060
− ·8	− 3·078	− ·078	− ·153
− ·7	− 1·963	− ·199	− ·233
− ·6	− 1·376	− ·365	− ·265*
− ·5	− 1·000	− ·549	− ·228
− ·4	− ·727	− ·719	− ·114
− ·3	− ·510	− ·838	+ ·066
− ·2	− ·325	− ·882*	+ ·287
− ·1	− ·158	− ·837	+ ·513
0	0	− ·707	+ ·707
+ ·1	+ ·158	− ·513	+ ·837
+ ·2	+ ·325	− ·287	+ ·882*
+ ·3	+ ·510	− ·066	+ ·838
+ ·4	+ ·727	+ ·114	+ ·719
+ ·5	+ 1·000	+ ·228	+ ·549
+ ·6	+ 1·376	+ ·265*	+ ·365
+ ·7	+ 1·963	+ ·233	+ ·199
+ ·8	+ 3·078	+ ·153	+ ·078
+ ·9	+ 6·314	+ ·060	+ ·014

* Extremes.

The graphs of q_0 and w_0 as functions of t, in the neighbourhood of the critical epoch $c\varpi$, are shown in fig. 8, which may be compared with fig. 6.† The corresponding orbit of a surface particle is traced in fig. 9, where the positive direction of z is downwards; it may be derived by a homogeneous strain from a portion of the curve whose polar equation is

$$r^{\frac{2}{3}} = a^{\frac{2}{3}} \cos \tfrac{2}{3}\left(\theta - \tfrac{3}{4}\pi\right).$$

The amplitude of this part of the disturbance diminishes, with increasing distance from the source, according to the law $\varpi^{-\frac{1}{2}}$.

Complete expressions for the disturbance are obtained by generalizing (155) and (156). They may be written

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t - c\varpi\cosh u)\,du - \frac{2}{\pi^2 b\mu}\int_a^b \mathrm{U}\,(\theta)\,.\,\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}(t-\theta\varpi\cosh u)\,du\,.\,d\theta \quad (169),$$

$$w_0 = \frac{1}{\pi^2 b\mu}\,\mathfrak{P}\int_a^\infty \theta\mathrm{V}(\theta)\,.\,\frac{\partial}{\partial t}\int_0^\infty \mathrm{R}\,(t-\theta\varpi\cosh u)\,du\,.\,d\theta \quad . \quad . \quad . \quad (170),$$

where U (θ) and V (θ) are the functions defined and tabulated in Art. 9.

† See the footnote on p. 26 *ante*.

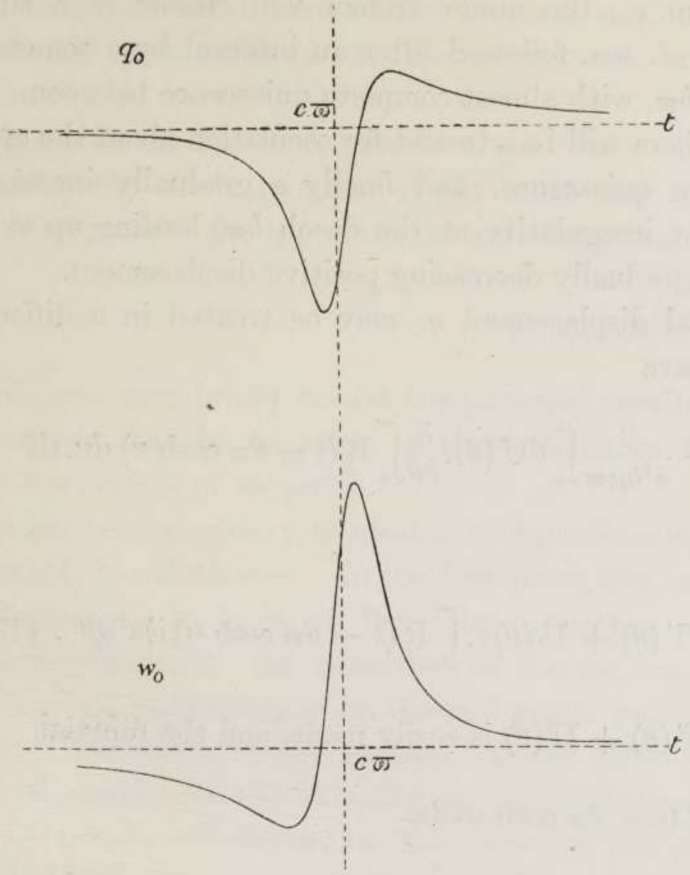

Fig. 8.

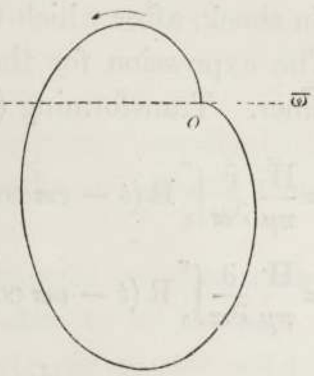

Fig. 9.

The method applied in that Article to obtain a general view of the whole progress of the vertical displacement at any point might be employed again here, the upper and lower curves in fig. 4 being combined with auxiliary movable graphs of

$$-\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t-\theta\varpi\cosh u)\,du \quad \text{and} \quad \theta\,\frac{\partial}{\partial t}\int_0^\infty \mathrm{R}\,(t-\theta\varpi\cosh u)\,du,$$

considered as functions of θ. In the case of a primary impulse of the type (165), both graphs would have somewhat the form of the *lower* curve in fig. 8, the functions being practically (except for a constant factor) of the type

$$\frac{\theta}{\sqrt{\varpi}}\sin\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi, \qquad \text{where} \qquad \chi=\tan^{-1}\frac{t-\theta\varpi}{\tau},$$

in the more important part of the range. Both graphs, if drawn to the scale of fig. 4 or 5, would be excessively contracted horizontally when we are concerned with values of ϖ large compared with τ/c_0. Owing to the compensation between positive and negative ordinates in the auxiliary graphs, it is plain that the disturbance expressed by the θ-integrals in (169) and (170) will be relatively very small except when t/ϖ has values θ for which the gradient of U (θ) or V (θ) is considerable. As regards the horizontal displacement q_0, the minor tremor will consist of a single to-and-fro oscillation about the epoch $a\varpi$, followed after an interval by a somewhat similar oscillation about the epoch $b\varpi$, with almost complete quiescence between. As regards the vertical displacement, there will be a to-and-fro oscillation about the epoch $a\varpi$, then a period of comparative quiescence, and finally a gradually increasing negative displacement (with a slight irregularity at the epoch $b\varpi$) leading up to the main shock, after which there is a gradually decreasing positive displacement.

The expression for the horizontal displacement q_0 may be treated in a different manner. Transforming (169) we have

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t - c\varpi\cosh u)\,du - \frac{2}{\pi^2 b\mu\varpi}\int_a^b \theta\mathrm{U}\,(\theta).\frac{\partial}{\partial\theta}\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du.d\theta$$

$$= \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t - c\varpi\cosh u)\,du$$

$$+ \frac{2}{\pi^2 b\mu\varpi}\int_a^b \{\theta\mathrm{U}'(\theta) + \mathrm{U}\,(\theta)\}.\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du.d\theta \quad . \quad (171).$$

A rough sketch of the graph of $\theta\mathrm{U}'(\theta) + \mathrm{U}\,(\theta)$ is easily made, and the function

$$\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du$$

is, in such a case as (165), one-signed, but its integral with respect to θ does not converge when the lower limit is large and negative. The method therefore fails to

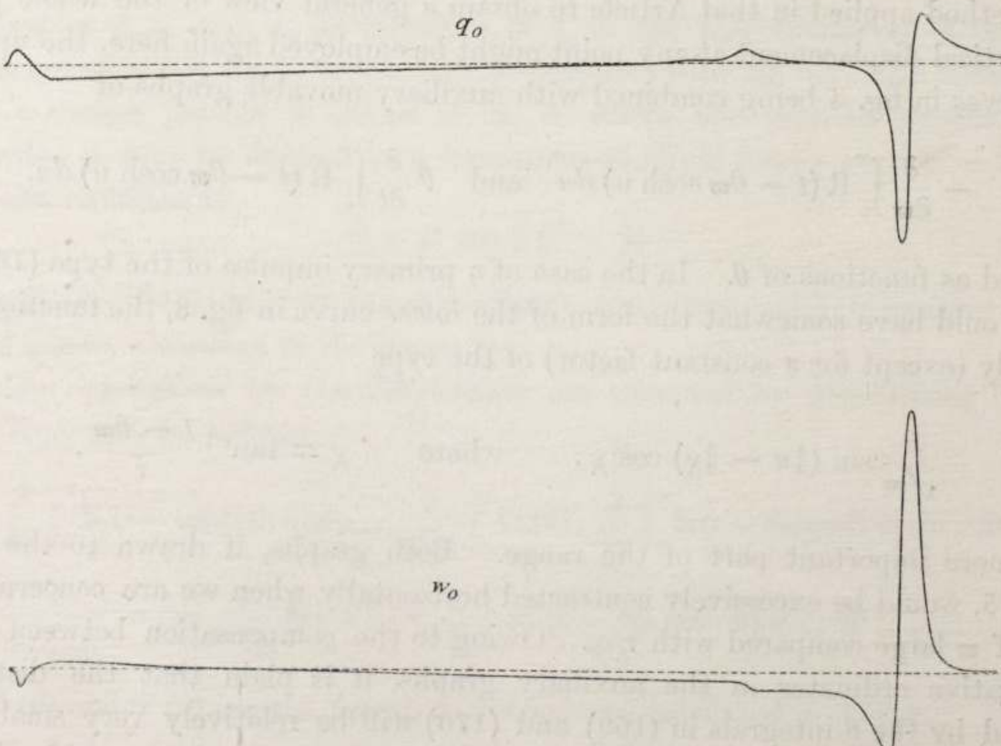

Fig. 10.

give us a convenient view of the progress of q_0 as a function of t. The difficulty is due to the peculiarities of annular propagation to which reference has already been made.

In fig. 10 an attempt, based on the former method, is made to represent (very roughly) the whole progress of the horizontal and vertical displacements due to a single impulse of the type (165) at a distance large compared with τ/c.

Summary.

We may now briefly review the principal results of the foregoing investigation, so far as they may be expected to throw light on the propagation of seismic tremors over the surface of the earth.

It has been necessary to idealize this problem in various ways in order to render it amenable to calculation. In the first place, the material is taken to be compact and homogeneous, to have, in fact, the properties of the "isotropic elastic solid" of theory. Moreover, the curvature of the surface is neglected. Again, instead of a disturbance originating at an internal point, we study chiefly the case of an impulse applied vertically to the surface. Under these conditions the disturbance spreads over the surface in the form of a symmetrical annular wave-system. The initial form of this system will depend on the history of the primitive impulse, but if this be of limited duration, the system gradually develops a characteristic form, marked by three salient features travelling with the velocities proper to irrotational, equi-voluminal, and Rayleigh waves, respectively. As the wave-system, thus established, passes any point of the surface, the *horizontal* displacement shows first of all a single well-marked oscillation followed by a period of comparative quiescence, and then another oscillation corresponding to the epoch of arrival of equivoluminal waves. The whole of this stage constitutes what we have called the "minor tremor"; it is, of course, more and more protracted the greater the distance from the source, and its amplitude continually diminishes, not only absolutely but also relatively to that of the "main shock," which we identify with the arrival of the Rayleigh wave. It may be remarked that the history of the minor tremor depends chiefly on the *time-integral* of the primitive impulse; the main shock, on the other hand, follows the time-scale of the primitive impulse, and is affected by every feature of the latter.*

Similar statements apply to the *vertical* displacement, except that the minor tremor leads up more gradually to the main shock, being interrupted, however, by a sort of jerk at the epoch of arrival of equivoluminal waves.

The history of the horizontal and vertical displacements, about the epoch of the main shock, in the case of a typical impulse of the type (165), is shown in fig. 8;

* Observational evidence in favour of the existence of the three critical epochs in an earthquake disturbance has been collected and discussed by R. D. Oldham, "On the Propagation of Earthquake Motion to Great Distances," 'Phil. Trans.,' A, 1900, vol. 194, p. 135.

whilst fig. 9 shows the corresponding orbit of a surface-particle. In fig. 10 a sketch is attempted of the whole progress of the disturbance.

These results are of a fairly definite character, but they are based, as has been said, on purely ideal assumptions, and it remains to inquire how far they are likely to be modified by the actual conditions of the earth. The substitution of an *internal* source for a surface impulse will clearly not affect the general character of the results at a distance great compared with the depth of the source, although differences of detail in the wave-profile at the critical epochs will occur, and we can no longer assume that the disturbance is the same in all vertical planes through the source. Again, the chief qualitative difference introduced by the *curvature* of the earth will be that the minor tremor, whose main features are evidently associated with the outcrop of spherical elastic waves at the surface, will be propagated directly through the earth, so that the first two epochs will (at distances comparable with the radius) be accelerated relatively to the main shock,* which being due to the Rayleigh waves will travel, with the velocity proper to these, over the *surface*.†

It is a more difficult matter to estimate the nature and extent of the modifications produced by heterogeneity. It is, perhaps, possible to exaggerate these, for the qualitative effect of a *gradual* charge of elastic properties would not be serious, and even considerable discontinuities would have little influence if their scale were small compared with the wave-length‡ of the primitive impulse. A covering of loose material over the solid rock probably causes only local, though highly irregular, modifications, with some dissipation of energy.

It must be acknowledged that our theoretical curves differ widely in two respects from the records of seismographs. In the first place, they show nothing corresponding to the long successions of to-and-fro vibrations which are characteristic of the latter. It would appear that such indications, so far as they are real and not instrumental, are to be ascribed to a *succession* of primitive shocks, in itself probable enough. Again, the theory gives vertical and horizontal movements of the same order of magnitude, and in the case of the Rayleigh waves, at all events, where a definite comparison can be made, the vertical amplitude is distinctly the greater. The observations, on the other hand, make out the vertical motion to be relatively small. The difficulty must occur on almost any conceivable theory, and appears indeed to be clearly recognised by seismologists, who are accordingly themselves disposed to question the competence of their instruments in this respect.

* *Cf.* R. D. OLDHAM, *loc. cit.*

† The theory of free Rayleigh waves on a spherical surface is known; see Professor BROMWICH, *loc. cit.*

‡ This term is used in the same general sense in which in hydrodynamics we speak of the "length" of a solitary wave travelling along a canal. There is no question, in the present connection, of anything analogous to "oscillatory waves."