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ON THE PROPERTIES OF A SPECIAL FUNCTION DEFINED BY AN INTEGRAL

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Abstract- The properties of a special function which is defined by an integral is presented. The numerical values of this function are tabulated correct to twenty decimal places. The curves of this function and its complementary are plotted. Some properties of this function are investigated.

Keywords- A function defined by integral, linear differential equation, power series solution, a function defined by a differential equation, tanh function.

1. INTRODUCTION

Two classes of nonelementary function are very important in applied mathematics. The first class consists of function arising in connection with linear differential equations. These functions are obtained by power series solutions of linear differential equations. Examples of these functions are Bessel, modified Bessel, Legendre, associated Legendre, Laguerre functions. The second class consists of function defined by integrals which cannot be calculated in terms of finitely many elementary functions. Examples are the Gamma function, the Beta function, the error function, the Exponential integral, the sine and cosine integrals, the Fresnel integral, the Dawson integral. These two class functions are tabulated [1]. Some suggestions have been proposed on the symbols of these functions [2]. The discussions on the properties of these functions may be found in [3].

In this paper, a new function defined by an integral is presented. This function arises due to the calculation of the volume flux for a flow over a plane wall bounded by two side walls perpendicular to the plane. This flow may be a simple but not being realistic model for a paint brush and has been discussed in [4]. The shear stress at the bottom wall cannot be calculated by the expression of the velocity which is in a series form. The reason is that there is a discontinuity in the expression of the velocity, therefore, term-by-term differentiation does not give a convergent series [4]. In order to overcome this difficulty the solution of the governing equation is obtained by the sine transform method. The expressions of the velocity, the volume flux and the stress at the bottom wall have been given in [5].

The expression of the volume flux for the flow over a plane wall bounded by two side walls perpendicular to the plane suggests a function which is in the following form

$$\int_0^x \frac{t - \tanh t}{t^{-3}} dt \tag{1}$$

The origin of this function is discussed and the numerical values are tabulated. The expression of this function for small and large values of the argument are given in a series form in terms of the Bernoulli numbers. The differential equation satisfied by this function is obtained. The curves of this function and its complementary are plotted.

2. THE ORIGIN OF *F(x)* FUNCTION

Assume that the fluid is over a plane wall and between two side walls perpendicular to the plane and that the side walls to be of infinite extent in the *x*- and *z*-directions as shown in Figure 1. The fluid is set in motion by a constant speed of the bottom wall in the absence of an imposed pressure gradient. The governing equation is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 , \tag{2}$$

where u is the velocity in the x- direction, and y and z are coordinates. It is convenient to take axes fixed in the plates. The boundary conditions become

$$\begin{aligned} u(y,0) &= U \quad \text{for} \quad -b < y < b , \\ u(y,\infty) &= 0 \quad \text{for} \quad -b \le y \le b , \\ u(\pm b, z) &= 0 \quad \text{for} \quad 0 < z \le \infty , \end{aligned} \tag{3}$$

where $2b$ is the distance between two side walls.

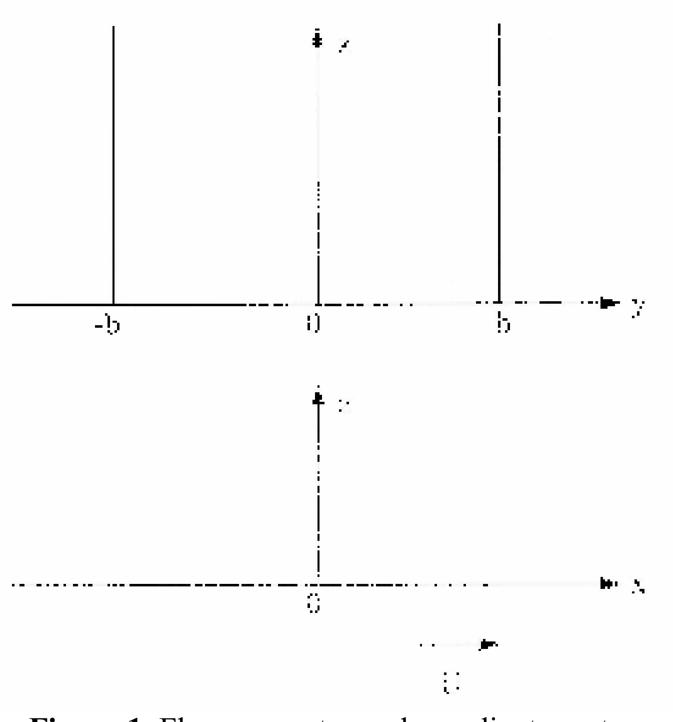

Figure 1. Flow geometry and coordinate system

The solution subject to the boundary conditions given by (3) is

$$\frac{u}{U} = \frac{4}{\pi}\sum_{n=0}\frac{(-1)^n}{2n+1}e^{-\lambda_n z}\cos\lambda_n y \quad . \tag{4}$$

The volume flux is

$$Q = \int_{-b}^{b}\int_{0}^{\infty} u\, dy\, dz \,.$$

Inserting (4) into the expression for Q, the volume flux becomes

$$\frac{Q}{Ub^2} = \frac{32}{\pi^3}\sum_{n=0}\frac{1}{(2n+1)^3}\,. \tag{5}$$

The summation in the right-hand side of (5) is tabulated correct to twenty decimal places [1].

The shear stress at the bottom wall cannot be calculated by (4) [5]. The reason is that there is a discontinuity in $u(y,z)$ at $y = \pm b$ and therefore term-by-term differentiation does not give a convergent series. In order to overcome this difficulty, a convenient expression for velocity must be found. This can be realized by applying the sine transform to (2). The sine transform of $u(y,z)$ is [6]

$$\bar{u} = \int_0^\infty u \sin\lambda z\, dz\,,$$

and the boundary condition becomes $\bar{u}(\pm b) = 0$. The application of the sine transform to (2) gives $\bar{u}'' - \lambda^2\bar{u} = -\lambda U$, where a prime denotes differentiation with respect to y. The solution subject to the boundary condition is

$$\bar{u} = \frac{U}{\lambda}\left(1 - \frac{\cosh\lambda y}{\cosh\lambda b}\right).$$

The inverse of $\bar{u}(y,z)$ is [6]

$$u(y,z) = \frac{2}{\pi}\int_0^\infty \bar{u}\sin\lambda z\, d\lambda\,.$$

Inserting of $\bar{u}$ into this integral one obtains

$$\frac{u}{U} = 1 - \frac{2}{\pi}\int_0^\infty \frac{\cosh\lambda y}{\cosh\lambda b}\frac{\sin\lambda z}{\lambda}d\lambda\,. \tag{6}$$

This expression of the velocity gives a correct result for the stress at $z = 0$. The shear stress can be calculated by (6). The shear stress at $z = 0$ is in the following form

$$\frac{\left(\sigma_{xy}\right)_{z=0}}{\rho U^2} = \frac{1}{R\cos\frac{\pi y}{2b}},$$

where $R = Ub/\nu$. The effect of the side walls on the flow is minimum at $y = 0$ and this effect increases near $y = \pm b$.

The volume flux can also be calculated by inserting (6) into the expression of the volume flux. This gives

$$Q = \frac{4Ub}{\pi}\int_0^\infty \left(1 - \frac{\tanh \lambda b}{\lambda b}\right)\frac{1}{\lambda^2}d\lambda \quad \text{or} \quad \frac{Q}{Ub^2} = \frac{4}{\pi}F(\infty),$$

where

$$F(\infty) = \int_0^\infty \frac{t - \tanh t}{t^3}dt. \tag{7}$$

By using the expression given by (5) one obtains

$$F(\infty) = \frac{8}{\pi}\sum_0 \frac{1}{(2n+1)^3}. \tag{8}$$

Since the summation on the right-hand side and $1/\pi^2$ are known [1], $F(\infty)$ becomes $0.85255\ 67976\ 34934\ 85979$ correct to twenty decimal places. The integration in (7) suggests a function that can be defined as

$$F(x) = \int_0^x \frac{t - \tanh t}{t^3}dt. \tag{9}$$

This integral can be tabulated by using a numerical integration method. The values of $F(x)$ correct to twenty decimal places are given in Table 1 in the Appendix at the end of this paper.

3. SOME PROPERTIES OF *F(x)* FUNCTION

The function $F(x)$ is defined by an integral which is given in (9). The complementary function is defined as

$$f(x) = \int_x^\infty \frac{t - \tanh t}{t^3}dt. \tag{10}$$

The integral in (9) can be used for small values of x. The integration in (10) can be used for large values of x. It is clearly seen that $F(0)=0$ and if one puts $-x$ for the upper limit of the integral in (9), then, one finds $F(-x)=-F(x)$. This shows that $F(x)$ is an odd function of x. However, there is no a physical meaning for negative values of the argument. The variation of $F(x)$ and its complementary function $f(x)$ with x are illustrated in Figure 2.

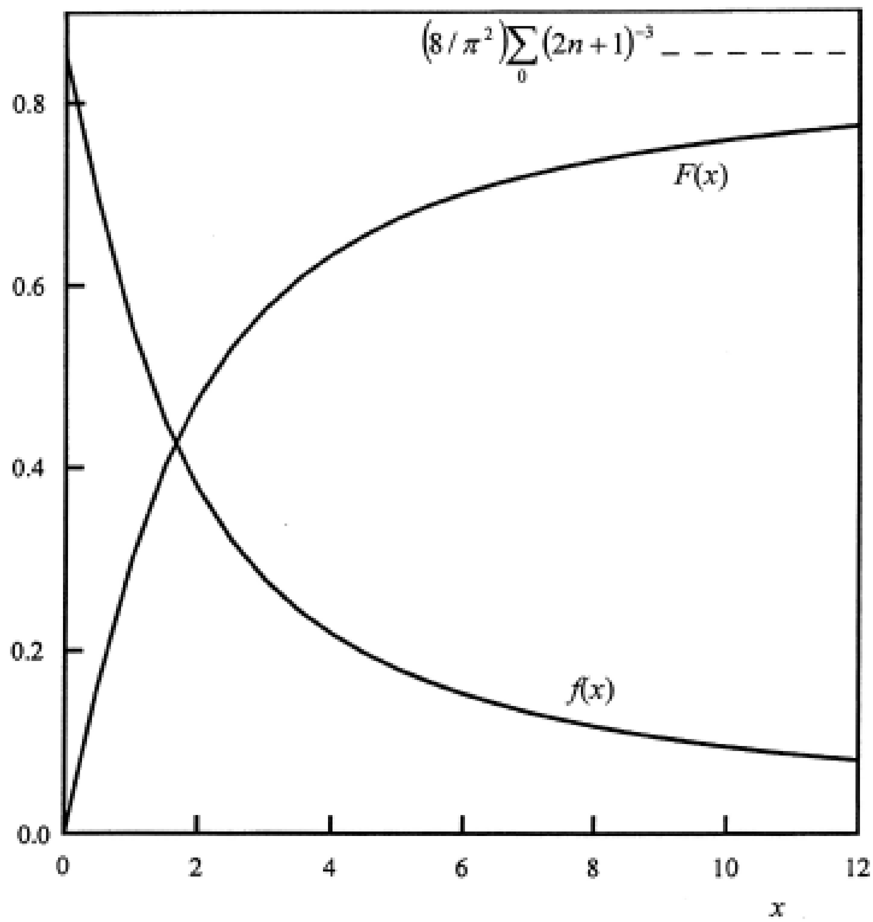

Figure 2. The variations of $F(x)$ and its complementary function $f(x)$ with x

For small values of x, $\tanh x$ is in the following form

$$\tanh x = \sum_{n=1} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!} B_{2n}\, x^{2n-1},$$
$$= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots,$$

where B_n are the Bernoulli numbers and the generation function is [1]

$$\frac{x}{e^x - 1} = \sum_{n=0} \frac{B_n x^n}{n!},$$

and some of them are given in Table 2 in the Appendix at the end of this paper. Inserting the expression of $\tanh x$ into the integral for $F(x)$, one finds

$$F(x) = \sum_{n=2} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}\frac{B_{2n}x^{2n-3}}{(2n-3)}, \tag{11}$$
$$= \frac{x}{3} - \frac{2}{45}x^3 + \frac{17}{1575}x^5 - \frac{62}{19845}x^7 + \frac{2764}{1403325}x^9 - \cdots.$$

This series can be used for $x < 1$.

For large values of x, (9) is written in the following form

$$F(x) = F(\infty) - \frac{1}{x} + \int_x^\infty \frac{\tanh t}{t}\,dt. \tag{12}$$

For large values of x, $\tanh x$ is written in the following form

$$\tanh x = 1 + 2\sum_{n=1}(-1)^n e^{-2nx}.$$

Inserting this expression into the integral in (12) one obtains

$$F(x) = F(\infty) - \left(\frac{1}{x} - \frac{1}{2x^2}\right)$$
$$+ 2\sum_{n=1}(-1)^n e^{-2nx}\left(\frac{1}{2nx^3} - \frac{3}{4n^2x^4} + \frac{3}{2n^3x^5} - \frac{15}{4n^4x^6} + \cdots\right). \tag{13}$$

For large values of x, $F(x)$ can also be written in the following form

$$F(x) = F(\infty) - \left(\frac{1}{x} - \frac{1}{2x^2}\right) + 2\sum_{n=1}(-1)^n\left[e^{-2nx}\left(\frac{1}{2x^2} - \frac{1}{nx}\right) + 2n^2 Ei(2nx)\right], \tag{14}$$

where

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$$

is a tabulated function [1]. The series given by (14) can be used for $x > 2$. The expression of $F(x)$ given by (14) seems to be more convenient than that of given by (13).

The function $F(x)$ satisfies the differential equation

$$xF'' + 3F' = \frac{\tanh^2 x}{x^2}, \tag{15}$$

with the conditions

$$F(0) = 0 \qquad \text{and} \qquad F'(0) = \frac{1}{3}. \tag{16}$$

where a prime denotes differentiation with respect to *x*. The solution of (15) subject to the condition given by (16) can be found by integrating (15). One of the solution subject to conditions (16) is $F' = (x - \tanh x + B)/x^3$ with $B = 0$, and finally one finds

$$F(x) = \int_0^x \frac{t - \tanh t}{t^3} dt .$$

4. CONCLUSIONS

A new special function which is defined by an integral is presented. This function arises due to the calculation of the volume flux for a flow over a plane wall between two adjoining plates perpendicular to the plane. Some properties of this function are given. The numerical values of this function are tabulated correct to twenty decimal places in the Appendix at the end of this paper.

5. REFERENCES

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APPENDIX

Table 1. The values of $F(x)$

x	*F(x)*	*x*	*F(x)*
0.0	0.00000 00000 00000 00000	3.6	0.61334 76799 29588 60727
0.1	0.03328 89965 14072 22354	3.7	0.61880 07383 84325 03308
0.2	0.06631 45255 87055 07039	3.8	0.62401 82467 02771 17324
0.3	0.09882 55641 28911 64988	3.9	0.62901 45475 78029 99218
0.4	0.13059 45423 38231 64504	4.0	0.63380 29007 89254 94157
0.5	0.16142 57813 48969 78792	4.1	0.63839 55781 58877 48582
0.6	0.19116 07260 16506 82672	4.2	0.64280 39478 46398 49348
0.7	0.21967 98948 03140 26318	4.3	0.64703 85509 86461 90225
0.8	0.24690 19366 01465 36603	4.4	0.65110 91713 40187 30934
0.9	0.27278 04790 46060 39786	4.5	0.65502 48985 83362 43600
1.0	0.29729 95575 94302 65472	4.6	0.65879 41858 14552 40211
1.1	0.32046 83655 08557 28465	4.7	0.66242 49018 19129 49687
1.2	0.34231 59220 64992 22852	4.8	0.66592 43785 83616 86054
1.3	0.36288 60786 36263 82022	4.9	0.66929 94545 10084 10195
1.4	0.38223 31126 50422 31311	5.0	0.67255 65137 41205 06425
1.5	0.40041 80213 62870 63529	6.0	0.69977 89968 22778 55398
1.6	0.41750 55284 89534 42764	7.0	0.71990 37343 95076 04481
1.7	0.43356 17544 86565 36497	8.0	0.73536 92974 48511 20549
1.8	0.44865 24682 64359 56073	9.0	0.74761 85260 12066 24236
1.9	0.46284 18259 22861 71598	10.0	0.75755 67976 33199 82989
2.0	0.47619 15031 58184 14843	11.0	0.76577 99381 30687 45421
2.1	0.48876 01364 61969 58540	12.0	0.77269 56865 23858 33829
2.2	0.50060 30000 93198 02506	13.0	0.77859 23005 93608 17755
2.3	0.51177 18584 15300 81767	14.0	0.78367 92466 14740 86534
2.4	0.52231 49451 34951 69141	15.0	0.78811 23531 91089 40978
2.5	0.53227 70314 97159 88040	16.0	0.79200 99226 35029 00670
2.6	0.54169 95543 53498 70232	17.0	0.79546 33720 29448 39007
2.7	0.55062 07822 34895 63378	18.0	0.79854 44519 56013 20151
2.8	0.55907 60032 97123 48053	19.0	0.80131 02602 38864 64671
2.9	0.56709 77234 70571 52420	20.0	0.80380 67976 35002 29029
3.0	0.57471 58665 67321 76479	30.0	0.81977 90198 57756 80771
3.1	0.58195 79706 78911 52921	40.0	0.82786 92932 80619 20064
3.2	0.58884 93771 18417 56069	50.0	0.83275 67921 88825 36770
3.3	0.59541 34095 70568 76348	100.0	0.84260 67867 60006 77623
3.4	0.60167 15421 31772 20790	∞	0.85255 67976 34934 85979
3.5	0.60764 35556 46953 27061		

Table 2. Some of the Bernoulli numbers [1].

n	B_n	n	B_n	n	B_n	n	B_n
0	1	4	-1/30	8	-1/30	12	-691/2730
1	-1/2	5	0	9	0	13	0
2	1/6	6	1/42	10	-5/56	14	7/6
3	0	7	0	11	0	15	0