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On the Properties of k-Fibonacci Numbers

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Abstract

In this paper, we obtain some new identities for k-Fibonacci numbers. Moreover the identities including generating functions for k-Fibonacci numbers have been obtained by Binet's Formula, also divisibility properties of these numbers have been investigated.

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1 Introduction

In recent years, Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book and the nature. Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature(e.g. see [1-5]). The well-known Fibonacci $\{F_n\}$ sequence is defined as

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \ \ for \ \ n \geq 2$$

where F_n denotes the nth Fibonacci number.

In [2], these general k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ were found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge(4TLE) partition. Many properties of these numbers are obtained directly from elementary matrix algebra. In [3], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [4], authors defined k-Fibonacci hyperbolic functions similar to hyperbolic functions and Fibonacci hyperbolic functions. They deduced some properties of k-Fibonacci hyperbolic functions related with the analogous identities for the k-Fibonacci numbers. Finally, authors studied 3-dimensional k-Fibonacci spirals with a geometric point of view in [5]. Several properties of these new k-Fibonacci hyperbolic functions are studied in an easy way. In the $19th$ century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [6]. m-extension of the Fibonacci and Lucas $p-$ numbers are defined in [7]. Afterwards, the continuous functions for the m-extension of the Fibonacci and Lucas p-numbers using the generalized Binet formulas. In [8], Stakhov and Rozin showed that the formulas are similar to the Binet formulas given for the classical Fibonacci numbers, also defined to be of generalized Fibonacci and Lucas numbers or Fibonacci and Lucas p-numbers.

In this paper, we obtain new identities for k-Fibonacci numbers. Moreover, the identities including generating functions for k-Fibonacci numbers
have been obtained by Binet's Formula, also divisibility properties of these numbers have been investigated.

2 The k-Fibonacci numbers and some properties

The k-Fibonacci numbers defined by Falco'n and Plaza for any real number k as follows. In this section, we give some properties of the k-Fibonacci numbers.

Definition 1 (Falco'n and Plaza) *For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by*

$$F_{k,0} = 0,\ F_{k,1} = 1,\ and\ F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad for\ n \geq 1 \tag{1}$$

The equation given by (1) is a second order difference equation with constant coefficients. Therefore, it has the characteristic equation;

$$r^2 = kr + 1 \tag{2}$$

The roots of the characteristic equation Eq.2 are $r_{1,2} = \frac{k\mp\sqrt{k^2+4}}{2}$. Note that, since $0 < k$, then

$$r_2 < 0 < r_1, |r_2| < r_1$$

$$r_1 + r_2 = k,\ r_1 r_2 = -1,\ r_1 - r_2 = \sqrt{k^2 + 4}. \tag{3}$$

Throughout this paper, $F_{k,n}$ denotes the n^{th} k-Fibonacci number and r_1, r_2 are the roots of the characteristic equation Eq.2.

Proposition 1 *For any integer $n \geq 1$, we have,*

$$r_1^{n+2} = kr_1^{n+1} + r_1^n$$

and

$$r_2^{n+2} = kr_2^{n+1} + r_2^n$$

where r_1, r_2 are the roots of the characteristic equation Eq.2.

Proof. Since r_1 and r_2 are the roots of the characteristic equation Eq.2, then

$$\begin{aligned} r_1^2 &= kr_1 + 1 \\ r_2^2 &= kr_2 + 1 \end{aligned} \tag{4}$$

Now, multiplying both sides of these equations by r_1^n and r_2^n respectively, we obtain the desired result. ■

Proposition 2 *The n^{th} k-Fibonacci number is given by*

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{5}$$

where r_1, r_2 are the roots of the characteristic equation Eq.2 and $r_1 > r_2$.

Proof. We use the principle of mathematical induction (PMI) on n. It is clear that the result is true for $n = 0$ and $n = 1$ by hypothesis. Assume that it is true for any i such that $0 \leq i \leq l + 1$, then

$$F_{k,i} = \frac{1}{r_1 - r_2}\left(r_1^i - r_2^i\right)$$

It follows from definition of the k-Fibonacci numbers and from Eq. 5

$$\begin{aligned} F_{k,l+2} &= kF_{k,l+1} + F_{k,l} \\ &= \frac{r_1^{l+2} - r_2^{l+2}}{r_1 - r_2} \end{aligned}$$

Thus, the formula is true for any positive integer n. Immediate consequences of Binet's Formula given by Eq. 5, can be seen in the following identities. ■

Proposition 3 *For any integer $n \geq 0$, we write,*

$$\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = F_{k,2n}.$$

Proof. By using Eq.5,

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} &= \sum_{i=0}^{n} \binom{n}{i} k^i \left(\frac{r_1^i - r_2^i}{r_1 - r_2}\right) \\
&= \frac{1}{r_1 - r_2}\left[\sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i\right]
\end{aligned}$$

and by summing up the geometric partial sums $\sum_{i=0}^{n} r_j^i$ for $j = 1, 2$ we obtain

$$\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = \frac{1}{r_1 - r_2}\left[(1 + kr_1)^n - (1 + kr_2)^n\right],$$

where r_1, r_2 are the roots of the characteristic equation of $F_{k,n}$. Now by using Eq.4, we get

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} &= \frac{1}{r_1 - r_2}\left(r_1^{2n} - r_2^{2n}\right) \\
&= F_{k,2n}.
\end{aligned}$$

■

Proposition 4 *For arbitrary integers $n, m \geq 1$, we have*

$$\sum_{i=1}^{n} F_{k,mi} = \frac{F_{k,mn+m} - (-1)^m F_{k,mn} - F_{k,m}}{r_1^m + r_2^m - (-1)^m - 1}.$$

Proof. By using of Binet's Formula given in Eq.5 and taking into account that $r_1 - r_2 = k$, $r_1 r_2 = -1$ it is obtained

$$\begin{aligned}
\sum_{i=1}^{n} F_{k,mi} &= \sum_{i=1}^{n} \frac{r_1^{mi} - r_2^{mi}}{r_1 - r_2} \\
&= \frac{F_{k,mn+m} - (-1)^m F_{k,mn} - F_{k,m}}{r_1^m + r_2^m - (-1)^m - 1}
\end{aligned}$$

as desired. ■

For $k=1$, we get

$$\sum_{i=1}^{n} F_{mi} = \frac{F_{mn+m} - (-1)^m F_{mn} - F_m}{r_1^m + r_2^m - (-1)^m - 1}.$$

In a similar way , the following identities can be proved.

i) If $j > m$, then $\sum_{i=0}^{n} F_{k,mi+j} = \frac{F_{k,mn+m+j} - (-1)^m F_{k,mn+j} + (-1)^m F_{k,j-m} - F_{k,j}}{r_1^m + r_2^m - (-1)^m - 1}$

ii) $\sum_{i=0}^{n} F_{k,i+j} = -\frac{1}{k}\left[-F_{k,n+j} - F_{k,n+j+1} + F_{k,j} + F_{k,j-1}\right]$

iii) $F_{k,-n} = (-1)^{n+1} F_{k,n} \quad n \geq 1$

Proposition 5 *The following equalities are valid for all $a, b, c \in N$:*

i) $F_{k,a+b-1} = F_{k,a}F_{k,b} + F_{k,a-1}F_{k,b-1}$,

ii) $F_{k,a+b-2} = \frac{1}{k}\left(F_{k,a}F_{k,b} - F_{k,a-2}F_{k,b-2}\right)$,

iii) $F_{k,a+b+c-3} = \frac{1}{k}\left(F_{k,a}F_{k,b}F_{k,c} + kF_{k,a-1}F_{k,b-1}F_{k,c-1} - F_{k,a-2}F_{k,b-2}F_{k,c-2}\right)$

Proof. i) It is clear from Eq.5

ii) From the definition of the k-Fibonacci sequence,

$$\begin{aligned} F_{k,a+b-2} &= F_{k,a}F_{k,b-1} + F_{k,a-1}F_{k,b-2} \\ &= F_{k,a}\left(\frac{F_{k,b} - F_{k,b-2}}{k}\right) + \left(\frac{F_{k,a} - F_{k,a-2}}{k}\right) F_{k,b-2} \\ &= \frac{1}{k}\left(F_{k,a}F_{k,b} - F_{k,a-2}F_{k,b-2}\right) \end{aligned}$$

is obtained.

iii) Using the definition of the k-Fibonacci sequence, we have

$$\begin{aligned} F_{k,a+b+c-3} &= F_{k,a-1}F_{k,b+c-3} + F_{k,a-1}F_{k,b+c-2} \\ &= F_{k,a-1}\left(F_{k,b-2}F_{k,c-2} + F_{k,b-1}F_{k,c-1}\right) + F_{k,a}\left(\frac{F_{k,b}F_{k,c} + F_{k,b-2}F_{k,c-2}}{k}\right). \end{aligned}$$

Now, after some algebra

$$F_{k,a+b+c-3} = \frac{1}{k}\left(F_{k,a}F_{k,b}F_{k,c} + kF_{k,a-1}F_{k,b-1}F_{k,c-1} - F_{k,a-2}F_{k,b-2}F_{k,c-2}\right).$$

■

Proposition 6 *For any integer $n \geq 1$,*

$$r_1^n = r_1 F_{k,n} + F_{k,n-1}.$$

Proof. By using the Eq.4 and taking $r_1 + r_2 = k$ and $r_1 r_2 = -1$, we can write,

$$\begin{aligned} r_1^2 &= r_1(k - r_2) = r_1 k - r_1 r_2 = r_1 k + 1 \\ r_1^3 &= r_1(r_1 k + 1) = r_1^2 k + r_1 = r_1\left(k^2 + 1\right) + k \\ &\vdots \end{aligned}$$

Continuing in this manner, we get

$$r_1^n = r_1 F_{k,n} + F_{k,n-1}$$

and proof is completed. Similarly, $r_2^n = r_2 F_{k,n} + F_{k,n-1}$. ■

2.1 Divisibility Properties of the k-Fibonacci Numbers

In this section, we investigate the divisibility properties of the k-Fibonacci numbers.

Proposition 7 *Any two consecutive k-Fibonacci numbers are relatively prime, that is* $(F_{k,n}, F_{k,n-1}) = 1$.

Proof. Using the Euclidean algorithm with $F_{k,n}$ as the original dividend and $F_{k,n-1}$ as the original divisor. This yields following system of linear equations:

$$\begin{aligned} F_{k,n} &= kF_{k,n-1} + F_{k,n-2} \\ F_{k,n-1} &= kF_{k,n-2} + F_{k,n-3} \\ &\vdots \\ F_{k,3} &= kF_{k,2} + F_{k,1} \\ F_{k,2} &= kF_{k,1} + 0. \end{aligned}$$

Thus it follows by the Euclidean algorithm that $(F_{k,n}, F_{k,n-1}) = F_{k,1} = 1$. ■

Proposition 8 *For $n, m \in \mathbb{Z}^+$, then $F_{k,m} \mid F_{k,mn}$.*

Proof. (By induction). The given statement is clearly true when $n = 1$. Now assume that it is true for all integers through l where $l \geq 1$: $F_{k,m} \mid F_{k,mi}$ for every i, where $1 \leq i \leq l$.

To show that $F_{k,m} \mid F_{k,m(l+1)}$, we invoke Eq.6:

$$F_{k,r+s} = F_{k,r-1}F_{k,s} + F_{k,r}F_{k,s+1} \tag{6}$$

$$F_{k,m(l+1)} = F_{k,ml+m} = F_{k,ml-1}F_{k,m} + F_{k,ml}F_{k,m+1}$$

Since $F_{k,m} \mid F_{k,ml}$, by the induction hypothesis, it follows that $F_{k,m} \mid F_{k,m(l+1)}$. Thus, by the strong version of the PMI, the result is true for all integer $n \geq 1$. ■

Proposition 9 *For $q, n \in \mathbb{Z}^+$, $(F_{k,qn-1}, F_{k,n}) = 1$.*

Proof. Let $d = (F_{k,qn-1}, F_{k,n})$. Then $d \mid F_{k,qn-1}$ and $d \mid F_{k,n}$. Since $F_{k,n} \mid F_{k,qn}$ by proposition 10, $d \mid F_{k,qn}$.Thus $d \mid F_{k,qn-1}$ and $d \mid F_{k,qn}$. But $(F_{k,qn-1}, F_{k,n}) = 1$ by divisibility algorithm. Therefore $d \mid 1$, so $d = 1$. Thus $(F_{k,qn-1}, F_{k,n}) = 1$. ■

Proposition 10 *Let $m = qn + r$ for $m, n, q, r \in Z^+$. Then*

$$(F_{k,m}, F_{k,n}) = (F_{k,n}, F_{k,r}).$$

Proof.

$$\begin{aligned}
(F_{k,m}, F_{k,n}) &= (F_{k,qn+r}, F_{k,n}) \\
&= (F_{k,qn+1}F_{k,r} + F_{k,qn}F_{k,r+1}, F_{k,n}) \quad \text{by Eq.6} \\
&= (F_{k,qn-1}F_{k,r}, F_{k,n}) \\
&= (F_{k,r}, F_{k,n}) \quad \text{by proposition 11}
\end{aligned}$$

■

The next theorem shows that the greatest common divisor (gcd) of two k-Fibonacci numbers is always k-Fibonacci number . The proof is easy from the Euclidean algorithm proposition 10 and this proposition.

Proposition 11 *Let m and n be positive integers. Then,*

$$(F_{k,m}, F_{k,n}) = F_{k,(m,n)}.$$

Proof. Suppose $m \geq n$. Applying the Euclidean algorithm with the same dividend and the divisor m, we get the following sequence of equations:

$$\begin{aligned}
m &= q_0 n + r_1 \qquad 0 \leq r_1 < n \\
n &= q_1 r_1 + r_2 \qquad 0 \leq r_2 < r_1 \\
r_1 &= q_2 r_2 + r_3 \qquad 0 \leq r_3 < r_2 \\
&\vdots \\
r_{n-2} &= q_{n-1} r_{n-1} + r_n \qquad 0 \leq r_n < r_{n-1} \\
r_{n-1} &= q_n r_n + 0
\end{aligned}$$

By proposition 12, we can write $(F_{k,m}, F_{k,n}) = (F_{k,n}, F_{k,r_1}) = (F_{k,r_1}, F_{k,r_2}) = \cdots = \left(F_{k,r_{n-1}}, F_{k,r_n}\right)$. But $r_n \mid r_{n-1}$, so $F_{k,r_n} \mid F_{k,r_{n-1}}$, by proposition 10. Therefore $\left(F_{k,r_{n-1}}, F_{k,r_n}\right) = F_{k,r_n}$. Thus $(F_{k,m}, F_{k,n}) = F_{k,r_n}$. But by Euclidean algorithm, $r_n = (m, n)$; Therefore $(F_{k,m}, F_{k,n}) = F_{k,(m,n)}$. ■

Proposition 12 *For $m, n \in Z^+$, If $(m, n) = 1$, then $F_{k,m}F_{k,n} \mid F_{k,mn}$.*

Proof. By Proposition 10, $F_{k,m} \mid F_{k,mn}$ and $F_{k,n} \mid F_{k,mn}$. Therefore, $[F_{k,m}, F_{k,n}] \mid F_{k,mn}$. But $(F_{k,m}, F_{k,n}) = F_{k,(m,n)} = F_{k,1} = 1$, so $[F_{k,m}, F_{k,n}] = F_{k,m}F_{k,n}$. Thus $F_{k,m}F_{k,n} \mid F_{k,mn}$. ■

2.2 Generating Functions for k-Fibonacci Sequences

In this section, some properties of the generating functions for the k-Fibonacci sequences are given.

Proposition 13 *Let m and n be integers and $F_{k,n}$ is the nth k-Fibonacci number. The following equalities are valid:*

i) $\sum_{n=0}^{\infty} F_{k,n}x^n = \frac{x}{1-kx-x^2}$

ii) $\sum_{n=0}^{\infty} F_{k,n+1}x^n = \frac{1}{1-kx-x^2}$

iii) $\sum_{n=0}^{\infty} F_{k,mn}x^n = \frac{F_{k,m}x}{1-\left(r_1^m+r_2^m\right)+(-1)^m x^2}$

iv) $\sum_{n=0}^{\infty} F_{k,mn+r}x^n = \frac{F_{k,r}-(-1)^m F_{k,r-m}x}{1-\left(r_1^m+r_2^m\right)+(-1)^m x^2}$

v) $\sum_{n=0}^{\infty} F_{k,2n}x^n = \frac{xk}{1-(k^2+2)x+x^2}$

vi) $\sum_{n=0}^{\infty} F_{k,2n+1}x^n = \frac{(1-x)k}{1-(k^2+2)x+x^2}$

Proof. By using of Binet's Formula given in Eq.5, i), ii), iii), iv), v) and vi) can be shown.

Let $k = 1$ in v) and vi), then we obtain following equalities which we known from Fibonacci sequence

$\sum_{n=0}^{\infty} F_{2n} = \frac{x}{1-3x+x^2}$, $\sum_{n=0}^{\infty} F_{2n+1} = \frac{1-x}{1-3x+x^2}$. ■

3 Conclusions

This paper presents the results of some properties of k-Fibonacci numbers by using the Binet's Formula. The conclusions arising from the work are as follows:

1. Have been obtained some new identities for k-Fibonacci numbers which like classical Fibonacci numbers .

2. Divisibility properties of k-Fibonacci numbers have been investigated.

3. The identities including generating functions for the k-Fibonacci numbers have been obtained. Also, this identities has wonderful application to graph theory.

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