On the Properties of the Deformed Consensus Protocol

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***Abstract*— This paper studies a generalization of the standard continuous-time consensus protocol, obtained by replacing the Laplacian matrix of the undirected communication graph with the so-called *deformed Laplacian*. The deformed Laplacian is a second-degree matrix polynomial in the real variable s which reduces to the standard Laplacian for s equal to unity. The stability properties of the ensuing *deformed consensus protocol* are studied in terms of parameter s for some special families of undirected graphs, and for graphs of arbitrary topology by leveraging the spectral theory of quadratic eigenvalue problems. Examples and simulation results are provided to illustrate our theoretical findings.**

I. Introduction

The last decade has witnessed a spurt of interest in multi-agent systems research, in both the control and robotics communities [1], [2]. Distributed control and consensus problems [3], had a large share in this research activity. Consensus theory originated from the work of Tsitsiklis [4], Jadbabaie *et al*. [5] and Olfati-Saber *et al*. [6], in which the consensus problem was first formulated in system-theoretical terms. A very rich literature emanated from these seminal contributions in recent years. In particular, numerous extensions to the prototypal consensus protocol in [6] have been proposed both in the continuous- and discrete-time domain: among them, we limit ourselves to mention here the cases of time-varying network topology [7], of networks with delayed [6] or quantized/noisy communication and link failure [8], [9], of random networks [10], of distributed average tracking [11], of finite-time convergence [12], and of nonlinear agreement [13].

This paper follows this vibrant line of research and presents an original extension to the basic continuous-time consensus protocol in [6], that exhibits a rich variety of behaviors and whose flexibility makes it ideal for a broad range of mobile robotic applications (e.g. for clustering or formation control). The new protocol, termed *deformed consensus protocol*, relies on the so-called *deformed Laplacian matrix*, a second-degree matrix polynomial in the real variable s, which extends the standard Laplacian matrix and reduces to it for s equal to unity: the deformed Laplacian is indeed an instance of a more general theory of deformed differential operators developed in mathematical physics in the last three decades (c.f. [14, Ch. 18]). Parameter s has a dramatic effect on the stability properties of the deformed consensus protocol, and it can be potentially used by an external supervisor to dynamically modify the behavior of the network and trigger different agents' responses. The stability properties of the proposed protocol are studied in terms of parameter s for some special families of undirected graphs for which the eigenvalues and eigenvectors of the deformed Laplacian can be computed in closed form. Our analysis is also extended to graphs of arbitrary topology by exploiting the spectral theory of quadratic eigenvalue problems [15].

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The rest of the paper is organized as follows. Sect. II presents some preliminaries on algebraic graph theory. The main theoretical results of the work are provided in Sect. III. Finally, in Sect. IV, the theory is illustrated via numerical simulations and in Sect. V the main contributions of the paper are summarized and possible future research directions are outlined.

II. Preliminaries

In this section, we briefly recall some basic notions of algebraic graph theory that will be used through the paper.

Let $\mathcal{G} = (V, E)$ be a graph[1] where $V = \{1, \ldots, n\}$ is the set of nodes, and E is the set of edges [16].

***Definition** 1 (Adjacency matrix $\mathbf{A}$):* The adjacency matrix $\mathbf{A} = [a_{ij}]$ of graph $\mathcal{G}$ is an $n \times n$ matrix defined as,

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

$\diamond$

***Definition** 2 (Laplacian matrix $\mathbf{L}$):* The Laplacian matrix of graph $\mathcal{G}$ is an $n \times n$ matrix defined as,

$$\mathbf{L} = \mathbf{D} - \mathbf{A},$$

where $\mathbf{D} = \mathrm{diag}(\mathbf{A}\mathbb{1})$ is the degree matrix and $\mathbb{1} = \mathbb{1}_n$ is a column vector of n ones. $\diamond$

Note that the Laplacian $\mathbf{L}$ is a symmetric positive semi-definite matrix.

***Property** 1 (Spectral properties of $\mathbf{L}$):* Let $\lambda_1(\mathbf{L}) \leq \lambda_2(\mathbf{L}) \leq \ldots \leq \lambda_n(\mathbf{L})$ be the ordered eigenvalues of the Laplacian $\mathbf{L}$. Then, we have that:

1) $\lambda_1(\mathbf{L}) = 0$ with corresponding eigenvector $\mathbb{1}$. The algebraic multiplicity of $\lambda_1(\mathbf{L})$ is equal to the number of connected components in $\mathcal{G}$.
2) $\lambda_2(\mathbf{L}) > 0$ if and only if the graph $\mathcal{G}$ is connected. $\lambda_2(\mathbf{L})$ is called the *algebraic connectivity* or *Fiedler value* of the graph $\mathcal{G}$. $\diamond$

***Definition** 3 (Bipartite graph):* A graph $\mathcal{G}$ is called bipartite if its node set V can be divided into two disjoint sets V_1 and V_2, such that every edge connects a node in V_1 to one in V_2. Equivalently, a graph is bipartite if and only if it does not contain cycles of odd length. $\diamond$

***Definition** 4 (Signless Laplacian matrix $\mathbf{Q}$ [17]):* The signless Laplacian matrix of graph $\mathcal{G}$ is defined as,

$$\mathbf{Q} = \mathbf{D} + \mathbf{A}.$$

$\diamond$

Note that as $\mathbf{L}$, the signless Laplacian $\mathbf{Q}$ is a symmetric positive semidefinite matrix (but it is not necessarily singular).

***Property** 2 (Spectral properties of $\mathbf{Q}$ [17]):* The signless Laplacian $\mathbf{Q}$ has the following properties:

1) Let $\mathcal{G}$ be a *regular graph* of degree κ (i.e., each node of the graph $\mathcal{G}$ has degree $\kappa \leq n-1$).

[1] All graphs in this paper are finite and undirected, with no loops and multiple edges.

Then, $p_{\mathbf{L}}(\lambda) = (-1)^n\, p_{\mathbf{Q}}(2\kappa - \lambda)$ where $p_{\mathbf{L}}(\lambda)$ denotes the characteristic polynomial of the Laplacian $\mathbf{L}$. If $\mathcal{G}$ is a *bipartite graph*, then $p_{\mathbf{L}}(\lambda) = p_{\mathbf{Q}}(\lambda)$.

2) The least eigenvalue of $\mathbf{Q}$ of a connected graph is equal to 0 if and only if the graph is *bipartite*. In this case, 0 is a simple eigenvalue.
3) In any graph, the multiplicity of the eigenvalue 0 of $\mathbf{Q}$ is equal to the number of bipartite components of $\mathcal{G}$. $\diamond$

III. Deformed consensus protocol

A. Problem formulation

It is well-known [6], that if the undirected communication graph $\mathcal{G}$ is connected, each component of the state $\mathbf{x} \triangleq [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ of the linear time-invariant system,

$$\dot{\mathbf{x}}(t) \,=\, -\mathbf{L}\,\mathbf{x}(t), \tag{1}$$

asymptotically converges to the average of the initial states $x_1(0), \ldots, x_n(0)$,

$$\lim_{t\to\infty}\, x_i(t) \,=\, \frac{1}{n}\sum_{i\,=\,1}^{n} x_i(0) \,=\, \frac{1}{n}\,\mathbf{x}_0^T\,\mathbb{1},$$

where $\mathbf{x}_0 \triangleq [x_1(0),\, \ldots,\, x_n(0)]^T$, i.e., *average consensus* is achieved. The converge rate of the consensus protocol (1) is dictated by the algebraic connectivity $\lambda_2(\mathbf{L})$.

Let us now consider the following generalization of the Laplacian $\mathbf{L}$.

***Definition** 5 (Deformed Laplacian $\boldsymbol{\Delta}(s)$):* The *deformed Laplacian* of the graph $\mathcal{G}$ is an $n \times n$ matrix defined as,

$$\boldsymbol{\Delta}(s) \,=\, (\mathbf{D} - \mathbf{I}_n)\, s^2 \,-\, \mathbf{A}\, s \,+\, \mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and s is a real parameter. $\diamond$

Note that $\boldsymbol{\Delta}(s)$ is a symmetric matrix (but *not* positive semidefinite as $\mathbf{L}$, in general), and that:

$$\boldsymbol{\Delta}(1) = \mathbf{L}, \qquad \boldsymbol{\Delta}(-1) = \mathbf{Q}.$$

Inspired by (1), in this paper we will study the stability properties of the following linear system,

$$\dot{\mathbf{x}}(t) \,=\, -\boldsymbol{\Delta}(s)\,\mathbf{x}(t), \tag{2}$$

in terms of the real parameter s, assuming that the graph $\mathcal{G}$ is *connected*. We will refer to (2), as the *deformed consensus protocol*.

Note that since $\boldsymbol{\Delta}(1) = \mathbf{L}$, we will always achieve average consensus for $s = 1$. Moreover, since $\boldsymbol{\Delta}(s)$ is real symmetric, all the eigenvalues of $\boldsymbol{\Delta}(s)$ (which are nonlinear functions of s) are *real*, and the deformed Laplacian admits the spectral decomposition $\boldsymbol{\Delta}(s) \,=\, \mathbf{U}(s)\,\boldsymbol{\Lambda}(s)\,\mathbf{U}^T(s)$, where $\mathbf{U}(s) = [\mathbf{u}_1(s)\; \mathbf{u}_2(s)\, \ldots\, \mathbf{u}_n(s)]$ is the matrix consisting of normalized and mutually orthogonal eigenvectors of $\boldsymbol{\Delta}(s)$ and $\boldsymbol{\Lambda}(s) \,=\, \mathrm{diag}(\lambda_1(\boldsymbol{\Delta}(s)), \ldots, \lambda_n(\boldsymbol{\Delta}(s)))$. The solution of (2), can thus be written as,

$$\mathbf{x}(t) \,=\, \sum_{i=1}^{n} e^{-\lambda_i(\boldsymbol{\Delta}(s))t}\,(\mathbf{u}_i^T(s)\,\mathbf{x}_0)\,\mathbf{u}_i(s). \tag{3}$$

In Sect. III-B, we will focus on some special families of graphs for which the eigenvalues and eigenvectors of $\boldsymbol{\Delta}(s)$ can be computed in closed form, and thus the stability properties of system (2) can be easily deduced from (3). In Sect. III-C, we will address, instead, the more challenging case of undirected graphs with arbitrary topology.

B. Stability conditions for special families of graphs

This section presents a sequence of nine propositions which provide stability conditions for system (2), in the case of path, cycle, full m-ary tree, wheel, m-cube (or hypercube), Petersen, complete, complete bipartite and star graphs (see [16] for a precise definition of these graphs). In the next two propositions, $\mathbb{k} \triangleq [-1, 1, -1, 1, \ldots, (-1)^{n-1}, (-1)^n]^T \in \mathbb{R}^n$, and hereafter we will use $\lambda_i(s)$ as a shorthand for $\lambda_i(-\boldsymbol{\Delta}(s))$, $i \in \{1, \ldots, n\}$.

***Proposition** 1 (Path graph P_n):* For a path graph P_n with $n \geq 2$ nodes (we number the nodes from 1 to n in the natural order from left to right), we have that:

- For $|s| < 1$, system (2) is asymptotically stable.
- For $|s| > 1$, system (2) is unstable.
- For $s = -1$, system (2) is marginally stable. In this case, it is possible to identify two groups of $n/2$ nodes (if n is even), or one group of $\lfloor n/2 \rfloor$ nodes and one of $\lfloor n/2 \rfloor + 1$ nodes (if n is odd). The states associated to the nodes in one group asymptotically converge to $\frac{1}{n}\,\mathbf{x}_0^T\mathbb{k}$ and the states associated to the nodes in the other group converge to $-\frac{1}{n}\,\mathbf{x}_0^T\mathbb{k}$.

Proof: In this case, $-\boldsymbol{\Delta}(s)$ is a tridiagonal matrix,

$$-\boldsymbol{\Delta}(s) = \begin{bmatrix} -1 & s & & & \\ s & -(s^2+1) & s & & \\ & & \ddots & & \\ & & s & -(s^2+1) & s \\ & & & s & -1 \end{bmatrix}.$$

The Sturm sequence of $-\boldsymbol{\Delta}(s)$ [18, Sect. 8.5.2], is given by $1, -1, 1, -1, 1, \ldots, -1, 1, s^2 - 1$, if n is odd, and $1, -1, 1, -1, 1, \ldots, 1, -1, 1 - s^2$, if n is even. Therefore by Theorem 8.5.1 of [18], if $|s| < 1$, $s \neq 0$, all the eigenvalues of $-\boldsymbol{\Delta}(s)$ are strictly negative and system (2) is asymptotically stable. On the other hand, the Sturm sequence of $\boldsymbol{\Delta}(s)$, for all n, is given by $1,\; 1, \ldots,\; 1,\; 1 - s^2$, from which we deduce that for $|s| > 1$, $\boldsymbol{\Delta}(s)$ has a negative eigenvalue, and hence system (2) is unstable. We also note that $\boldsymbol{\Delta}(0) = \mathbf{I}_n$, and hence the system is asymptotically stable for $s = 0$. Finally, for $s = -1$, system (2) is marginally stable and the unit-norm eigenvector associated to the zero eigenvalue of $-\boldsymbol{\Delta}(-1)$ is $\frac{1}{\sqrt{n}}\,\mathbb{k}$. ■

***Proposition** 2 (Cycle graph C_n):* For a cycle graph C_n with $n > 2$ nodes, we have that:

- If n is even:
  - For all $s \in \mathbb{R} \setminus \{-1, 1\}$, system (2) is asymptotically stable.
  - For $s = -1$, system (2) is marginally stable. In this case, the states associated to $n/2$ nodes asymptotically converge to $\frac{1}{n}\,\mathbf{x}_0^T\,\mathbb{k}$ and the states associated to the other $n/2$ nodes converge to $-\frac{1}{n}\,\mathbf{x}_0^T\,\mathbb{k}$.
- If n is odd, system (2) is asymptotically stable for all $s \in \mathbb{R} \setminus \{1\}$.

Proof: In this case, $-\boldsymbol{\Delta}(s)$ is a *circulant matrix* [19],

$$-\boldsymbol{\Delta}(s) = \mathrm{circ}[-(s^2+1),\, s,\, 0,\, \ldots,\, 0,\, s].$$

It is well-known that circulant matrices are diagonalizable by the Fourier matrix and hence their eigenvalues can be computed in closed form. The eigenvalues of a general $n \times n$ circulant matrix $\mathbf{C} = \mathrm{circ}[c_1,\, c_2, \ldots,\, c_n]$, in fact, are given by:

$$\lambda_i(\mathbf{C}) = \rho_{\mathbf{C}}(\omega^{i-1}),\; i \in \{1, \ldots, n\},$$

where $\omega \triangleq e^{2\pi j/n}$, $j = \sqrt{-1}$, and the polynomial $\rho_{\mathbf{C}}(\xi) = c_n\xi^{n-1} + \ldots + c_3\xi^2 + c_2\xi + c_1$ is called the circulant's representer [19, Th. 3.2.2]. By applying this result to matrix $-\mathbf{\Delta}(s)$, for $i \in \{1, \ldots, n\}$ we have that,

$$\lambda_i(-\mathbf{\Delta}(s)) = -\,s^2 \,+\, 2\,\cos(2\,\pi\,(i-1)/n)\,s \,-\, 1. \quad (4)$$

Observe now that the coordinates of the vertex of the parabola (4) are $[\cos(2\pi(i-\ 1)/n),\ -\sin^2(2\pi(i-1)/n)]$, $i \in \{1, \ldots, n\}$. If n is even, then $\lambda_i(-\mathbf{\Delta}(s)) < 0, \forall\, s \in \mathbb{R}$ and $\forall\, i \neq \{1, n/2+1\}$. For $i = 1$, $\lambda_1(-\mathbf{\Delta}(s)) \le 0$ and $\lambda_1(-\mathbf{\Delta}(s)) = 0$ only for $s = 1$. For $i = n/2+1$, $\lambda_{n/2+1}(-\mathbf{\Delta}(s)) \le 0$ and $\lambda_{n/2+1}(-\mathbf{\Delta}(s)) = 0$ only for $s = -1$. The unit-norm eigenvector associated to $\lambda_{n/2+1}(-\mathbf{\Delta}(-1))$ is $\frac{1}{\sqrt{n}}\,\Bbbk$. On the other hand, if n is odd, then $\lambda_i(-\mathbf{\Delta}(s)) < 0$, $\forall\, s \in \mathbb{R}$ and $\forall\, i \neq 1$. For $i = 1$, $\lambda_1(-\mathbf{\Delta}(s)) \le 0$ and $\lambda_1(-\mathbf{\Delta}(s)) = 0$ only for $s = 1$. ■

Note that $\det(-\mathbf{\Delta}(s)) = (-1)^n\,(s^{2n} - 2s^n + 1)$ for the cycle graph C_n, and that the $2n$ roots of $(-1)^n(s^{2n}-2s^n+1)$ are equally spaced on the unit circle.

A full *m-ary tree* is a rooted tree in which every node other than the leaves has m children. The *depth* δ of a node is the length of the path from the root to the node. The set of all nodes at a given depth is called a *level* of the tree: by definition, the root node is at depth zero. The number of nodes of a full m-ary tree is $n = \sum_{i=0}^{\delta} m^i$.

***Proposition** 3 (Full m-ary tree):* For a full m-ary tree with $m \ge 2$, we have that:

- For $|s| < 1$, system (2) is asymptotically stable.
- For $|s| > 1$, system (2) is unstable.
- For $s = -1$, system (2) is marginally stable. In this case, the states associated to the nodes in the even levels of the tree asymptotically converge to $\frac{1}{\sqrt{n}}\,\mathbf{x}_0^T\mathbf{u}_1$, while the states associated to the nodes in the odd levels of the tree converge to $-\frac{1}{\sqrt{n}}\,\mathbf{x}_0^T\mathbf{u}_1$, where $\mathbf{u}_1$ is the unit-norm eigenvector associated to the zero eigenvalue of $-\mathbf{\Delta}(-1)$.

Proof: The stability properties of (2) are determined in this case by only one of the eigenvalues of $-\mathbf{\Delta}(s)$ (in fact, the other $n-1$ are negative for all s). This eigenvalue is negative for $|s| < 1$ and positive for $|s| > 1$. ■

***Proposition** 4 (Wheel graph W_n):* Consider a wheel graph W_n with $n > 3$ nodes where node 1 is the center of the wheel, and let μ be the non-unitary root of

$$-\frac{n}{2}\,s^2 \,+\, s \,+\, \frac{\sqrt{(n-4)^2\,s^2 + 4\,(n-4)\,s + 4n}}{2}\ s \,-\, 1.$$

μ monotonically decreases from $1/2$ (for $n = 4$) to 0 (for $n = \infty$). We have that:

- For $s > 1$ and for $s < \mu$, system (2) is asymptotically stable.
- For $s \in (\mu, 1)$, system (2) is unstable.
- For $s = \mu$, system (2) is marginally stable. In this case the state associated to node 1 asymptotically converges to $\mathbf{x}_0^T[\alpha^2,\, \alpha\beta,\, \ldots, \alpha\beta]^T$, and the states associated to the other $n-1$ nodes converge to $\mathbf{x}_0^T[\alpha\beta,\, \beta^2,\, \ldots, \beta^2]^T$, where $[\alpha,\, \beta,\, \ldots, \beta]^T$, $\alpha,\, \beta \in \mathbb{R}$, is the unit-norm eigenvector associated to the zero eigenvalue of $-\mathbf{\Delta}(\mu)$.

Proof: The eigenvalues of matrix $-\mathbf{\Delta}(s)$ are:

$$\lambda_{1,2}(s) = -\frac{n}{2}s^2 + s \pm \frac{\sqrt{(n-4)^2 s^2 + 4(n-4)s + 4n}}{2}\ s - 1,$$

$$\lambda_{i+1}(s) = -2s^2 + 2\cos\left(\tfrac{2\pi(i-1)}{n-1}\right)s - 1,\ i \in \{2, \ldots, n-1\}.$$

Note that the coordinates of the vertex of the parabola $\lambda_{i+1}(s)$, $i \in \{2, \ldots, n-1\}$, are $\left[\frac{1}{2}\cos\left(\frac{2\,\pi(i-1)}{n-1}\right),\ -\frac{1}{2}\left(1 + \sin^2\left(\frac{2\,\pi(i-1)}{n-1}\right)\right)\right]$, therefore $\lambda_{i+1}(s) < 0$, $\forall\, s \in \mathbb{R}$, $\forall\, i \in \{2, \ldots, n-1\}$. We also have that $\lambda_2(s) < 0$, $\forall\, s \in \mathbb{R}$. Finally, it is easy to verify that $\lambda_1(s)$ has always two roots, $s = \mu$ and $s = 1$. $\lambda_1(s) > 0$ for $s \in (\mu,\, 1)$ and $\lambda_1(s) < 0$ for $s > 1$ and $s < \mu$. ■

***Proposition** 5 (m-cube Q_m):* For the m-cube (or hypercube) graph Q_m with $n = 2^m > 4$ nodes, we have that:

- For $|s| > 1$ and for $|s| < \frac{1}{m-1}$, system (2) is asymptotically stable.
- For $s \in \left(-1, -\frac{1}{m-1}\right)$ and for $s \in \left(\frac{1}{m-1},\, 1\right)$, system (2) is unstable.
- For $s = \frac{1}{m-1}$, average consensus is achieved. The convergence rate to $\frac{1}{n}\,\mathbf{x}_0^T\mathbb{1}_n$ is slower for $s = \frac{1}{m-1}$ than for $s = 1$.
- For $s \in \{-1, -\frac{1}{m-1}\}$, system (2) is marginally stable. In this case, the states associated to $n/2$ nodes asymptotically converge to $\frac{1}{\sqrt{n}}\,\mathbf{x}_0^T\mathbf{u}_1$, while the states associated to the other $n/2$ nodes converge to $-\frac{1}{\sqrt{n}}\,\mathbf{x}_0^T\mathbf{u}_1$, where $\mathbf{u}_1$ is the unit-norm eigenvector associated to the zero eigenvalue of $-\mathbf{\Delta}(-1)$ or $-\mathbf{\Delta}(-\frac{1}{m-1})$.

Proof: In this case, the stability properties of (2) are only determined by the following two eigenvalues of $-\mathbf{\Delta}(s)$ (in fact, the other $n-2$ are negative for all s):

$$\lambda_{1,2}(s) \,=\, -((m-1)\,s^2 \,\mp\, m\,s \,+\, 1).$$

We have that $\lambda_1(s) < 0$ for $s > 1$ and for $s < 1/(m-1)$. Instead, $\lambda_2(s) < 0$ for $s > -1/(m-1)$ and for $s < -1$. For $s \in \{-1, \pm 1/(m-1)\}$, system (2) is marginally stable. Finally, note that $\mathbf{\Delta}\left(\frac{1}{m-1}\right) = \frac{1}{m-1}\,\mathbf{\Delta}(1)$, hence the convergence rate to $\frac{1}{n}\,\mathbf{x}_0^T\mathbb{1}_n$ is slower for $s = 1/(m-1)$ than for $s = 1$. ■

***Proposition** 6 (Petersen graph):* For the Petersen graph, we have that:

- For $s > 1$ and for $s < 1/2$, system (2) is asymptotically stable.
- For $s \in (1/2,\, 1)$, system (2) is unstable.
- For $s = 1/2$, average consensus is achieved. The convergence rate to $\frac{1}{10}\,\mathbf{x}_0^T\mathbb{1}_{10}$ is slower for $s = 1/2$ than for $s = 1$.

Proof: The ten eigenvalues of $-\mathbf{\Delta}(s)$ are:

$$\begin{aligned}
\lambda_1(s) &= -(2\,s^2 - 3\,s + 1),\\
\lambda_2(s) &= \ \ldots\ = \lambda_5(s) = -(2\,s^2 - s + 1),\\
\lambda_6(s) &= \ \ldots\ = \lambda_{10}(s) = -(2\,s^2 + 2\,s + 1).
\end{aligned}$$

We have that $\lambda_2(s) < 0$ and $\lambda_6(s) < 0$, $\forall\, s \in \mathbb{R}$. Moreover, $\lambda_1(s) < 0$ for $s > 1$ and for $s < 1/2$, and the unit-norm eigenvector associated to $\lambda_1(1/2)$ is $\frac{1}{\sqrt{10}}\,\mathbb{1}_{10}$. Finally, $\mathbf{\Delta}(1/2) = \frac{1}{2}\,\mathbf{\Delta}(1)$, hence the convergence rate to $\frac{1}{10}\,\mathbf{x}_0^T\mathbb{1}_{10}$ is slower for $s = 1/2$ than for $s = 1$. ■

***Proposition** 7 (Complete graph K_n):* For the complete graph K_n with $n > 2$ nodes, we have that:

- For $s > 1$ and for $s < \frac{1}{n-2}$, system (2) is asymptotically stable.
- For $s \in \left(\frac{1}{n-2},\, 1\right)$, system (2) is unstable.
- For $s = \frac{1}{n-2}$, average consensus is achieved. The convergence rate to $\frac{1}{n}\,\mathbf{x}_0^T\mathbb{1}$ is slower for $s = \frac{1}{n-2}$ than for $s = 1$.

Graph name	Asymptotic stability for :	Marginal stability for :
Path graph P_n, $n \geq 2$	$\lvert s \rvert < 1$	$s = -1$ (2 groups of nodes)
Cycle graph C_n, $n > 2$, n even	$\forall\, s \in \mathbb{R} \setminus \{-1,\, 1\}$	$s = -1$ (2 groups of nodes)
Cycle graph C_n, $n > 2$, n odd	$\forall\, s \in \mathbb{R} \setminus \{1\}$	
Full m-ary tree, $m \geq 2$	$\lvert s \rvert < 1$	$s = -1$ (2 groups of nodes)
Wheel graph W_n, $n > 3$	$s > 1$ and $s < \mu$	$s = \mu$ (2 groups of nodes)
m-cube, Q_m, $n = 2^m > 4$	$\lvert s \rvert > 1$ and $\lvert s \rvert < \frac{1}{m-1}$	$s \in \{-1, -\frac{1}{m-1}\}$ (2 groups of nodes) $s = \frac{1}{m-1}$ (average consensus)
Petersen graph	$s > 1$ and $s < 1/2$	$s = 1/2$ (average consensus)
Complete graph K_n, $n > 2$	$s > 1$ and $s < \frac{1}{n-2}$	$s = \frac{1}{n-2}$ (average consensus)
Complete bipartite graph $K_{m,n}$, $m, n \geq 2$	$\lvert s \rvert > 1$ and $\lvert s \rvert < \frac{1}{\sqrt{(m-1)(n-1)}}$	$s \in \{-1, -\frac{1}{\sqrt{(m-1)(n-1)}}\}$ (2 groups of nodes) $s = \frac{1}{\sqrt{(m-1)(n-1)}}$ (if $m = n$, average consensus) (if $m \neq n$, 2 groups of nodes)
Star graph $K_{1,n}$, $n \geq 3$	$\lvert s \rvert < 1$	$s = -1$ (2 groups of nodes)

TABLE I

SUMMARY OF THE STABILITY PROPERTIES OF THE DEFORMED CONSENSUS PROTOCOL (2), FOR SOME SPECIAL FAMILIES OF GRAPHS.

Proof: The eigenvalues of $-\mathbf{\Delta}(s)$ are:

$$\lambda_1(s) \;=\; -((n-2)\, s^2 - (n-1)\, s + 1),$$
$$\lambda_2(s) \;=\; \ldots \;=\; \lambda_n(s) \;=\; -((n-2)\, s^2 + s + 1).$$

We have that $\lambda_2(s) < 0$, $\forall\, s \in \mathbb{R}$. Moreover, $\lambda_1(s) < 0$ for $s > 1$ and for $s < \frac{1}{n-2}$, and the unit-norm eigenvector associated to $\lambda_1\big(\frac{1}{n-2}\big)$ is $\frac{1}{\sqrt{n}}\,\mathbb{1}$. Finally, $\mathbf{\Delta}\big(\frac{1}{n-2}\big) = \frac{1}{n-2}\,\mathbf{\Delta}(1)$, hence, the convergence rate to $\frac{1}{n}\,\mathbf{x}_0^T\mathbb{1}$ is slower for $s = \frac{1}{n-2}$ than for $s = 1$. ■

***Proposition** 8 (Complete bipartite graph $K_{m,n}$):* For the complete bipartite graph $K_{m,n} = (V_1 \cup V_2, E)$, where $\mathrm{Card}(V_1) = m$, $\mathrm{Card}(V_2) = n$ with $m, n \geq 2$, we have that:

- For $\lvert s \rvert > 1$ and $\lvert s \rvert < [(m-1)(n-1)]^{-1/2}$, system (2) is asymptotically stable.
- For $s \in (-1, -[(m-1)(n-1)]^{-1/2})$ and $s \in ([(m-1)(n-1)]^{-1/2},\, 1)$, system (2) is unstable.
- For $s \in \{-1, \pm[(m-1)(n-1)]^{-1/2}\}$ system (2) is marginally stable. In particular, with $\mathbf{x}_0 \in \mathbb{R}^{m+n}$:
  - If $m \neq n$: for $s = -1$, the states associated to the nodes in V_1 asymptotically converge to $\frac{1}{m+n}\,\mathbf{x}_0^T[\mathbb{1}_m^T, -\mathbb{1}_n^T]^T$ and the states associated to the nodes in V_2 converge to $-\frac{1}{m+n}\,\mathbf{x}_0^T[\mathbb{1}_m^T, -\mathbb{1}_n^T]^T$. For $s = \pm[(m-1)(n-1)]^{-1/2}$ the states of the nodes in V_1 and V_2 converge to two different values (not further specified herein).
  - If $m = n$: for $s = 1/(n-1)$ average consensus is achieved, and the convergence rate to $\frac{1}{2n}\,\mathbf{x}_0^T\mathbb{1}_{2n}$ is slower for $s = 1/(n-1)$ than for $s = 1$. For $s \in \{-1, -1/(n-1)\}$ the states associated to the nodes in V_1 asymptotically converge to $\frac{1}{2n}\,\mathbf{x}_0^T[\mathbb{1}^T, -\mathbb{1}^T]^T$ and the states associated to the nodes in V_2 converge to $-\frac{1}{2n}\,\mathbf{x}_0^T[\mathbb{1}^T, -\mathbb{1}^T]^T$.

Proof: In this case, only two of the $m+n$ eigenvalues of $-\mathbf{\Delta}(s)$ determine the stability properties of system (2), (the others are negative for all s):

$$\lambda_{1,2}(s) = -\frac{n+m-2}{2}\, s^2 \pm \frac{\sqrt{(n-m)^2 s^2 + 4mn}}{2}\, s \,-\, 1.$$

A systematic study of the roots of $\lambda_1(s)$, $\lambda_2(s)$ easily leads to the result. Note that if $m = n$, $\mathbf{\Delta}\big(\frac{1}{n-1}\big) = \frac{1}{n-1}\,\mathbf{\Delta}(1)$, hence, the rate of convergence to $\frac{1}{2n}\,\mathbf{x}_0^T\mathbb{1}_{2n}$ is slower for $s = 1/(n-1)$ than for $s = 1$. ■

From Prop. 8, we can easily deduce the following result:

***Proposition** 9 (Star graph $K_{1,n}$):* For the star graph $K_{1,n}$ with $n \geq 3$ where node 1 is the center of the star, we have:

- For $\lvert s \rvert < 1$, system (2) is asymptotically stable.
- For $\lvert s \rvert > 1$, system (2) is unstable.
- For $s = -1$, system (2) is marginally stable. In this case, the state associated to node 1 asymptotically converges to $\frac{1}{n+1}\,\mathbf{x}_0^T[1, -\mathbb{1}^T]^T$ and the states associated to the other n nodes converge to $-\frac{1}{n+1}\,\mathbf{x}_0^T[1, -\mathbb{1}^T]^T$. ■

For the reader's convenience, all the results found in this section are summarized in Table I.

***Remark** 1:* Note that the path, cycle (with n even), full m-ary tree, m-cube, complete bipartite and star graphs are all *bipartite* graphs. Then, because of Property 2.2, for $s = -1$ the state of system (2) asymptotically converges to $(\mathbf{u}_1\mathbf{u}_1^T)\,\mathbf{x}_0$ with these graphs, where $\mathbf{u}_1$ is the unit-norm eigenvector associated to the zero eigenvalue of the signless Laplacian $\mathbf{Q}$. ◇

***Remark** 2 (Discrete-time deformed consensus protocol):* Note that the stability properties in terms of parameter s, of protocol (2) and its discrete-time version,

$$\mathbf{x}(k+1) \;=\; \mathbb{P}(s)\,\mathbf{x}(k), \quad k \in \{0,\, 1,\, 2, \ldots\},$$

where $\mathbb{P}(s) = \mathbf{I}_n - \epsilon\,\mathbf{\Delta}(s)$ is the *deformed Perron matrix*, $0 < \epsilon < 1/d_{\max}$ is the step-size and $d_{\max} = \max_i\big(\sum_{j \neq i} a_{ij}\big)$ is the maximum degree of $\mathcal{G}$, are the same. ◇

C. Stability conditions for graphs of arbitrary topology

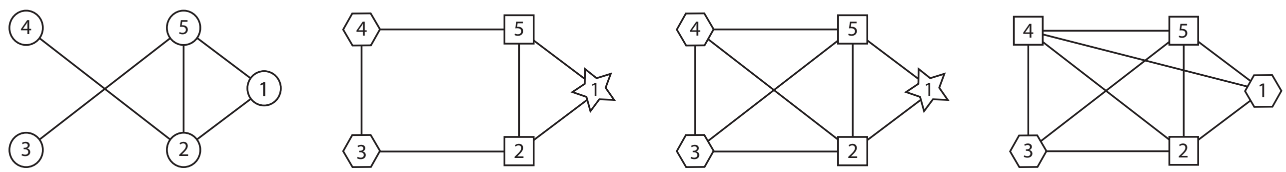
In order to extend the analysis of the previous section to arbitrary graphs, we briefly review here the spectral theory of *quadratic eigenvalues problems* (QEPs) [15, Sect. 3]. Let

$$\mathbf{P}(\lambda) \;=\; \mathbf{B}_2\,\lambda^2 + \mathbf{B}_1\,\lambda + \mathbf{B}_0, \quad \mathbf{B}_2,\, \mathbf{B}_1,\, \mathbf{B}_0 \in \mathbb{C}^{n \times n},$$

be an $n \times n$ *matrix polynomial* of degree 2.

***Definition** 6 (Spectrum of $\mathbf{P}(\lambda)$, [15]):* The *spectrum* of $\mathbf{P}(\lambda)$ is defined as $\Sigma(\mathbf{P}) = \{\lambda \in \mathbb{C} : \det(\mathbf{P}(\lambda)) = 0\}$, i.e., it is the set of eigenvalues of $\mathbf{P}(\lambda)$. ◇

***Definition** 7 (Regular $\mathbf{P}(\lambda)$, [15]):* The matrix $\mathbf{P}(\lambda)$ is called *regular* when $\det(\mathbf{P}(\lambda))$ is not identically zero for all values of λ, and *nonregular* otherwise. ◇



(a) (b) (c) (d)

Fig. 1. *Example 1*: In (b)-(d), different shapes are used to identify distinct groups of nodes: the states associated to the nodes in these groups asymptotically converge to the same value when system (2) is marginally stable.

Note that $\det(\mathbf{P}(\lambda)) = \det(\mathbf{B}_2)\lambda^{2n} +$ lower-order terms, so when $\mathbf{B}_2$ is nonsingular, $\mathbf{P}(\lambda)$ is regular and has $2n$ *finite* eigenvalues [15]. When $\mathbf{B}_2$ is singular, the degree of $\det(\mathbf{P}(\lambda))$ is $r < 2n$ and $\mathbf{P}(\lambda)$ has r finite eigenvalues and $2n - r$ infinite eigenvalues.

***Problem** 1 (Quadratic eigenvalue problem (QEP), [15]):* The QEP consists of finding scalars λ and nonzero vectors $\mathbf{z}$, $\mathbf{y}$, satisfying,

$$\mathbf{P}(\lambda)\,\mathbf{z} = \mathbf{0}, \quad \mathbf{y}^*\,\mathbf{P}(\lambda) = \mathbf{0},$$

where $\mathbf{z}, \mathbf{y} \in \mathbb{C}^n$ are respectively the right and left eigenvector corresponding to the eigenvalue $\lambda \in \mathbb{C}$, and $\mathbf{y}^*$ is the conjugate transpose of $\mathbf{y}$. ◇

A QEP has $2n$ eigenvalues (finite or infinite) with up to $2n$ right and $2n$ left eigenvectors. Note that a regular $\mathbf{P}(\lambda)$ may possess two distinct eigenvalues having the same eigenvector. In general, if a regular $\mathbf{P}(\lambda)$ has $2n$ distinct eigenvalues, then there exists a set of n linearly independent eigenvectors.

***Property** 3 (Spectral properties of $\mathbf{P}(\lambda)$, [15]):* If matrices $\mathbf{B}_2$, $\mathbf{B}_1$, $\mathbf{B}_0$ are *real symmetric*, the eigenvalues of $\mathbf{P}(\lambda)$ are either real or occur in complex conjugate pairs, and the sets of left and right eigenvectors coincide. ◇

By leveraging the previous facts, we deduce the following property of the deformed consensus protocol:

***Proposition** 10:* The *finite real eigenvalues* λ of the QEP,

$$((\mathbf{I}_n - \mathbf{D})\,\lambda^2 + \mathbf{A}\,\lambda - \mathbf{I}_n)\,\mathbf{z} \;=\; \mathbf{0}, \tag{5}$$

are the values of s for which system (2) is marginally stable. Moreover, if λ is one of these eigenvalues and $\overline{\mathbf{z}} = \mathbf{z}/\|\mathbf{z}\|$ is the associated unit-norm eigenvector, we have that:

$$\lim_{t\to\infty} \mathbf{x}(t) \;=\; (\overline{\mathbf{z}}\,\overline{\mathbf{z}}^T)\,\mathbf{x}_0. \qquad \blacksquare$$

***Remark** 3 (Computation of the eigenvalues of the QEP):* The eigenvalues of the QEP (5) can be easily computed by converting it to a standard *generalized eigenvalue problem*[2] of size $2n$, by defining the new vector $\mathbf{w} = \lambda\,\mathbf{z}$. In terms of $\mathbf{z}$ and $\mathbf{w}$, problem (5) then becomes [15, Sect. 3.4]:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_n & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n - \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}. \qquad \diamond$$

The following proposition elucidates the relationship between the topology of the communication graph $\mathcal{G}$, and the properties of the QEP (5).

***Proposition** 11:* If the graph $\mathcal{G}$ has a node with degree equal to one (i.e. only one edge is incident to that node), the matrix $(\mathbf{I}_n - \mathbf{D})\lambda^2 + \mathbf{A}\,\lambda - \mathbf{I}_n$ is nonregular, and the QEP (5) admits at least two infinite eigenvalues. ■

[2]This construction is called "linearization" in the literature [15, Sect. 3.4] and it is not unique, in general (c.f. Matlab's function `polyeig`).

For example, with path graphs $\mathrm{rank}(\mathbf{I}_n - \mathbf{D}) = n - 2$, with star graphs $\mathrm{rank}(\mathbf{I}_n - \mathbf{D}) = 1$ and with full m-ary tree graphs $\mathrm{rank}(\mathbf{I}_n - \mathbf{D}) = n - m^{\delta}$: in the first two cases $\det((\mathbf{I}_n - \mathbf{D})\lambda^2 + \mathbf{A}\,\lambda - \mathbf{I}_n) = (-1)^{n-1}(\lambda^2 - 1)$, while in the last $\det((\mathbf{I}_n - \mathbf{D})\lambda^2 + \mathbf{A}\,\lambda - \mathbf{I}_n) = \lambda^2 - 1$. Hence, in all cases, the QEP (5) admits $2n - 2$ infinite eigenvalues.

The next proposition shows how to determine the s-stability interval of the deformed consensus protocol for graphs with arbitrary topology.

***Proposition** 12 (Stability interval of (2) for arbitrary $\mathcal{G}$):* Let $q(s) \triangleq \det((\mathbf{I}_n - \mathbf{D})\,s^2 + \mathbf{A}\,s - \mathbf{I}_n)$, then:

- If n is *even*, system (2) is asymptotically stable for all s such that $q(s) > 0$, and unstable for all s such that $q(s) < 0$.
- If n is *odd*, system (2) is asymptotically stable for all s such that $q(s) < 0$, and unstable for all s such that $q(s) > 0$.

Proof: The result follows from the observation that:

$$q(s) \;=\; (s-1)\textstyle\prod_{k=1}^{\ell}(s-\zeta_k)(s-\zeta_k^*)\prod_{j=1}^{r-2\ell-1}(s-\eta_j),$$

where $r \leq 2n$ is the degree of $q(s)$, (ζ_k, ζ_k^*) are the ℓ pairs of complex-conjugate roots of $q(s)$, and η_j the $r - 2\ell - 1$ non-unitary real roots of $q(s)$, (c.f. Prop. 10). ■

The following example illustrates the rich variety of behaviors exhibited by the deformed consensus protocol on four generic (nonbipartite) graphs with five nodes.

***Example** 1:* Consider the four graphs reported in Fig. 1. Owing to Prop. 10 and Prop. 12, we have that:

- With the graph in Fig. 1(a), system (2) is asymptotically stable $\forall\, s \in \mathbb{R} \setminus \{1\}$.
- With the graph in Fig. 1(b), system (2) is asymptotically stable for $s < 0.7022$ and for $s > 1$. For $s = 0.7022$, the system is marginally stable and three groups of nodes can be identified: $\{1\}$, $\{2, 5\}$, $\{3, 4\}$ (different shapes are used in Fig. 1(b) to identify these groups).
- With the graph in Fig. 1(c), system (2) is asymptotically stable for $s < 0.4396$ and for $s > 1$. For $s = 0.4396$, the system is marginally stable and three groups of nodes can be identified: $\{1\}$, $\{2, 5\}$, $\{3, 4\}$.
- With the graph in Fig. 1(d), system (2) is asymptotically stable for $s < 0.3804$ and for $s > 1$. For $s = 0.3804$, the system is marginally stable and two groups of nodes can be identified: $\{1, 3\}$, $\{2, 4, 5\}$.

From Figs. 1(b)-1(d), we notice that nodes in the same group have the same edge degree, and that an increase in the algebraic connectivity of the graph leads to a reduction of the s-stability interval of the deformed consensus protocol. ◇

***Remark** 4:* Note that parameter s in the deformed Laplacian can be regarded as an "exogenous input" and it can be exploited to *dynamically* modify the behavior of system (2). This may be useful when the nodes of the graph are mobile robots and a human supervisor is interested in changing the collective behavior of the team over time, e.g., by switching from a marginally- to an asymptotically-stable equilibrium point of system (2) or vice versa (c.f. Sect. IV). $\diamond$

IV. Simulation results

In order to illustrate the theory presented in Sect. III, let us consider a team of n vehicles modeled as single integrators,

$$\dot{\mathbf{p}}_i(t) = \boldsymbol{\nu}_i(t), \quad i \in \{1, \ldots, n\},$$

where $\mathbf{p}_i(t) = [p_{ix}(t),\, p_{iy}(t)]^T \in \mathbb{R}^2$ and $\boldsymbol{\nu}_i(t) \in \mathbb{R}^2$ denote respectively the position and the input of agent i at time t. Let the control input of vehicle i be of the form,

$$\boldsymbol{\nu}_i(t) = (s^2-1)\,\mathbf{p}_i(t) + s \sum_{j \in \mathcal{N}(i)} (\mathbf{p}_j(t) - s\,\mathbf{p}_i(t)), \quad (6)$$

where $\mathcal{N}(i)$ denotes the set of nodes adjacent to node i in the communication graph. Then, the collective dynamics of the group of agents adopting control (6), can be written as:

$$\dot{\mathbf{p}}(t) = (-\boldsymbol{\Delta}(s) \otimes \mathbf{I}_2)\,\mathbf{p}(t),$$

where $\mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T]^T \in \mathbb{R}^{2n}$ and "$\otimes$" denotes the Kronecker product. Fig. 2(a) shows the trajectory of $n = 6$ vehicles implementing the control law (6), when the communication topology is the path graph P_6 (the initial position of the agents is marked with a circle). In our simulation, $s = -1$ for $t \in [0, 50)$ sec, and $s = 0$ for $t \in [50, 100]$ sec. The time evolution of the x-, y-coordinates of the vehicles is reported in Fig. 2(b). As it is evident in Figs. 2(a) and 2(b), the vehicles first cluster in two distinct groups and then rendezvous at the origin (recall Prop. 1).

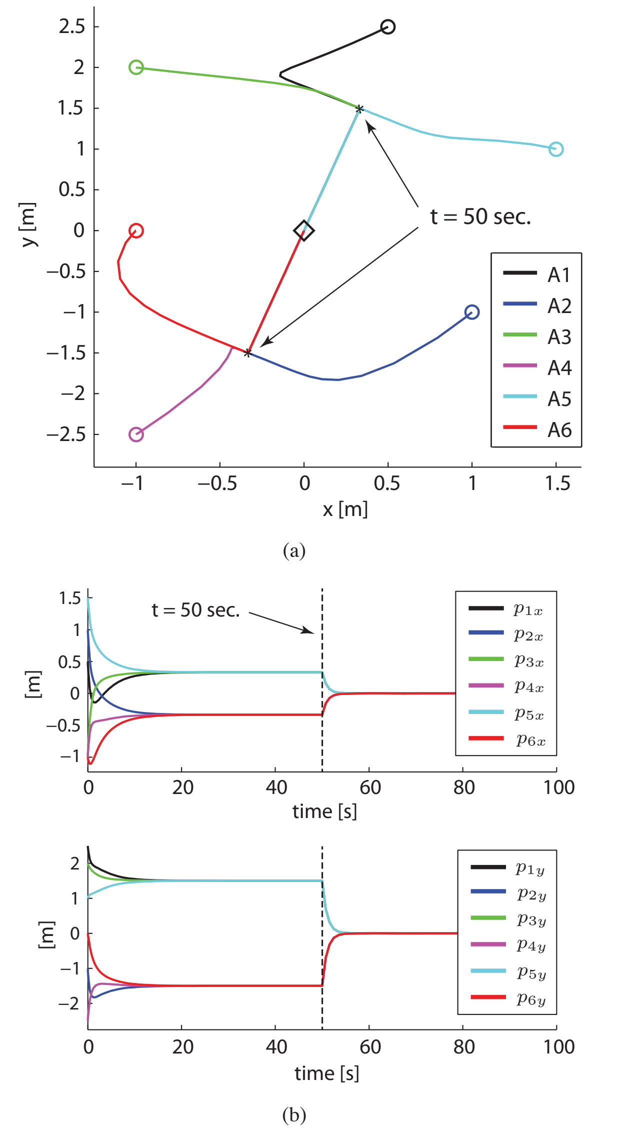


Fig. 2. *Simulation results:* (a) Trajectory of the 6 vehicles: the communication topology is the path graph P_6; (b) Time evolution of the x-, y-coordinates of the vehicles (top and bottom, respectively).

V. Conclusions and future work

In this paper we have presented a generalization of the classical consensus protocol, called *deformed consensus protocol*, and we have analyzed its stability properties for some special families of undirected graphs. Theoretical results for graphs of arbitrary topology are also provided. The theory has been illustrated via examples and numerical simulations.

In future works, we will delve into the peculiar grouping behaviors of Example 1, and we aim at studying the properties of the deformed consensus protocol when the (weighted) communication graph is directed and/or changes over time.

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