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ON THE PROPERTIES OF THE SECOND MOMENT OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL-FUNCTIONAL EQUATIONS WITH VARYING COEFFICIENTS

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ABSTRACT. Sufficient conditions for the mean square stability of solutions of linear stochastic differential-functional Itô–Skorokhod equations with unbounded aftereffect are obtained in the paper. The critical case is also studied.

1. Asymptotic mean square stability

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and

$$\{\mathcal{F}_t, t \geq 0\}, \qquad \mathcal{F}_t \subset \mathcal{F},$$

a current of minimal σ-algebras. Further let $\mathbb{R}$ be the real one-dimensional Euclidean space equipped with the norm $|\cdot|$, and D_0 the Skorokhod space of functions $\{\varphi(\theta)\} \subset \mathbb{R}$ defined on $(-\infty, 0]$. Every function $\varphi(\theta)$ of D_0 has no discontinuities of the second kind, has the left limit at every point of discontinuity, is right continuous, and has the limit as $\theta \to -\infty$ [4].

Let $\{x(t) \equiv x(t, \omega)\} \subset \mathbb{R}$ be a stochastic process defined for $t \geq 0$ by the stochastic differential-functional Itô–Skorokhod equation

$$dx(t) = a(t, x_t)\, dt + b(t, x_t)\, dw(t) + \int_{\mathbb{R}} g(t, x_t, u)\, \overline{v}(dt, du), \tag{1}$$

$$x(t) = \varphi(t) \quad \text{for all } t \in (-\infty, 0] \tag{2}$$

(see [1], [2], [5]) where $\varphi \in D_0$; in what follows the trajectory $\{x(t)\} \subset \mathbb{R}$ up to the moment $t \geq 0$ is denoted by $x_t \equiv \{x(t+\theta), -\infty < \theta \leq 0\}$; $\{w(t) \equiv w(t, \omega)\} \subset \mathbb{R}$ is a one-dimensional Wiener process; $\overline{v}(t, A) \equiv v(t, A) - t\Pi(A)$, $A \subset \mathbb{R}$, is a centered Poisson measure in $\mathbb{R}$ with parameter $t\Pi(A) \equiv \mathsf{E}\{\nu(t, A)\}$ where $\{w(t)\}$ and $\{\overline{v}(t, A)\}$ are independent and $\mathcal{F}_t$-measurable for $t \geq 0$.

The coefficients a, b, and g are linear functionals for any $t \geq 0$ defined on $\mathbb{R}_+ \times D_0$, $\mathbb{R}_+ \times D_0$, and $\mathbb{R}_+ \times D_0 \times \mathbb{R}$, respectively. We treat D_0 as a metric space with the Skorokhod metric ρ_D (see [4, Chapter VI, §5]).

To facilitate the discussion of the behavior of stochastic processes $\{x(t)\} \subset \mathbb{R}$ without discontinuities of the second kind, a simpler metric is often considered (see [3]). This metric is generated by the seminorm

$$\|\varphi\|_* \equiv \left\{ \int_{-\infty}^{0} |\varphi(\theta)|^2\, \mathbb{K}(d\theta) \right\}^{1/2} \tag{3}$$

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where $\mathbb{K}(\cdot)$ is some finite measure defined on the Borel sets of $(-\infty, 0]$, that is,

$$\mathbb{K}(-\infty, 0) = \mathbb{K} < \infty.$$

Definition 1. Let L_t be the σ-algebra of Borel sets on $(-\infty, t]$. A separable stochastic process $\{x(t)\} \subset \mathbb{R}$ defined by relation (2) for $t \in (-\infty, 0]$, measurable with respect to the σ-algebra $\mathcal{F}_t \times L_t$, and satisfying for all $t \geq 0$ the integral Itô–Skorokhod equation

$$x(t) = \varphi(0) + \int_0^t a(s, x_s)\, ds + \int_0^t b(s, x_s)\, dw(s) + \int_0^t \int_{\mathbb{R}} g(s, x_s, u)\, \overline{v}(ds, du) \tag{4}$$

with probability one, is called a solution of the stochastic differential-functional equation (1) with initial condition (2).

We denote by H_0 the space of $\mathcal{F}_0$-measurable functions $\varphi\colon (-\infty, 0] \times \Omega \to \mathbb{R}$ equipped with the norm

$$\|\varphi\|_0 \equiv \left\{ \sup_{-\infty<\theta\leq 0} \mathsf{E}\left\{ |\varphi(\theta)|^2 \right\} \right\}^{1/2}. \tag{5}$$

Assume that $a, b\colon \mathbb{R}_+ \times D_0 \to \mathbb{R}$ and $g\colon \mathbb{R}_+ \times D_0 \times \mathbb{R} \to \mathbb{R}$ are functionals, measurable with respect to all their arguments and such that

$$|a(t,\varphi)|^2 + |b(t,\varphi)|^2 + \int_{\mathbb{R}} |g(t,\varphi,u)|^2\, \Pi(du) \leq L \int_{-\infty}^0 \left(1 + |\varphi(\theta)|^2\right) dK(\theta), \tag{6}$$

$$\begin{aligned} &|a(t\varphi) - a(t,\psi)|^2 + |b(t,\varphi) - b(t,\psi)|^2 + \int_{\mathbb{R}} |g(t,\varphi,u) - g(t,\psi,u)|^2\, \Pi(du) \\ &\quad \leq L \int_{-\infty}^0 \left(|\varphi(\theta) - \psi(\theta)|^2\right) dK(\theta) \end{aligned} \tag{7}$$

for all $t \geq 0$ where $L > 0$ is a constant.

Then according to [4, Chapter II, §1, Theorems 3 and 4] and [3, Theorem 6.2.1], in the space D_0 there exists a unique (up to stochastic equivalence) solution of the problem (1), (2) such that

$$\begin{aligned} &\mathsf{E}\left\{ \sup_{0\leq s\leq T} |x(s)|^2 \,/\, \mathcal{F}_t \right\} \leq A\left(1 + \|\varphi\|_0^2\right), \\ &\mathsf{E}\left\{ \sup_{t\leq s\leq t+h} |x(s) - x(t)|^2 \,/\, \mathcal{F}_t \right\} \leq B\left(1 + \|\varphi\|_0^2\right) h \end{aligned} \tag{8}$$

where $A > 0$ and $B > 0$ are constants depending on $T > 0$, $L > 0$, and $K > 0$.

Denote by $\{h(t,s)\} \subset \mathbb{R}$ the fundamental solution of the deterministic equation

$$dy(t) = a(t, y_t)\, dt \tag{9}$$

with the initial function η such that $\eta(t) = 0$ for $t < s$ and $\eta(t) = 1$ for $t = s$. Using $\{h(t,s)\}$, the solution of the problem (1), (2) can be rewritten in the integral form [8]:

$$x(t) = y(t) + \int_0^t h(t,s) b(s, x_s)\, dw(s) + \int_0^t \int_{\mathbb{R}} h(t,s) g(s, x_s, u)\, \overline{v}(ds, du) \tag{10}$$

where $\{y(t)\}$ is a solution of equation (9) with nonrandom initial function $\{\varphi(t)\}$ (see (2)).

Now we obtain sufficient conditions for the mean square asymptotic stability of the trivial solution of the problem (1), (2).

Theorem 1. *Assume that*

1) *the trivial solution of equation* (9) *is exponentially stable;*

2) *the fundamental solution of equation* (9) *is such that*

$$d = \overline{\lim_{t\to\infty}}\, d(t) = \overline{\lim_{t\to\infty}} \int_0^t \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u)\Pi(du) \right] ds < 1; \tag{11}$$

3) *the functionals* $a(t,\cdot)$, $b(t,\cdot)$, *and* $g(t,\cdot,u)$ *are uniformly bounded with respect to* t *and* $u \in \mathbb{R}$ *in the norm* (5) *of the space* D_0 *where* $\Pi(du) = du/|u|^2$.

Then the trivial solution of the problem (1), (2) *is mean square asymptotically stable.*

Proof. Using equation (10) at the moment $t+\theta$ and applying the linear operators $b(t,\cdot)$ and $g(t,\cdot,u)$ [6] we obtain

$$\begin{aligned} b(t, x_t) &= b(t, y_t) + \int_0^t b(t, h_t(\theta, s)) b(s, x_s)\, dw(s) \\ &\quad + \int_0^t \int b(t, h_t(\theta, s)) g(s, x_s, u)\, \overline{v}(ds, du), \\ g(t, x_t, u) &= g(t, y_t, u) + \int_0^t g(t, h_t(\theta, s), u) b(s, x_s)\, dw(s) \\ &\quad + \int_0^t \int g(t, h_t(\theta, s), u) g(s, x_s, u)\, \overline{v}(ds, du). \end{aligned}$$

Now we square both sides of the latter two equations, then take the mathematical expectation ($\mathsf{E}\{\cdot\}$), and use some properties of the stochastic integral [3]. As a result we get

$$\begin{aligned} \mu_b(t) &= b^2(t, y_t) + \int_0^t b^2(t, h_t, (\theta, s))\mu_b(s)\, ds + \int_0^t \int_{\mathbb{R}} b^2(t, h_t(\theta, s))\mu_g(s, u)\Pi(du)\, ds, \\ \int_{\mathbb{R}} \mu_g(t, u)\, \Pi(du) &= \int_{\mathbb{R}} g^2(t, y_t, u)\, \Pi(du) + \int_0^t \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u)\mu_b(s)\, \Pi(du)\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u)\mu_g(s, u_1)\, \Pi(du)\, \Pi(du_1)\, ds \end{aligned}$$

where

$$\mu_b(t) \equiv \mathsf{E}\left\{ b^2(t, x_t) \right\}, \qquad \mu_g(t, u) \equiv \mathsf{E}\left\{ g^2(t, x_t, u) \right\}.$$

After the summation of these equations we get

$$m(t) = f(t) + \int_0^t \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u)\, \Pi(du) \right] m(s)\, ds \tag{12}$$

where

$$m(t) \equiv \mu_b(t) + \int_{\mathbb{R}} \mu_g(t, u)\, \Pi(du), \qquad f(t) \equiv b^2(t, y_t) + \int_{\mathbb{R}} g^2(t, y_t, u)\, \Pi(du).$$

According to the definition of sup, for a given $\varepsilon > 0$ we find $T > 0$ such that $\sup_{s\geq T} d(s) \leq d + \varepsilon$ for $t \geq T$. Put

$$M(t, T) \equiv \max_{T\leq s\leq t} m(s).$$

Taking into account (11) we obtain

$$\begin{aligned} M(t, T) \leq \max_{T\leq\tau\leq t} f(\tau) &+ \max_{T\leq\tau\leq t} \int_0^T \left[b^2(\tau, h_\tau(\theta, s)) + \int_{\mathbb{R}} g^2(\tau, h_\tau(\theta, s), u)\, \Pi(du) \right] m(s)\, ds \\ &+ \max_{T\leq\tau\leq t} d(\tau) M(t, T). \end{aligned}$$

The contraction principle [6] implies that

$$\varlimsup_{t\to\infty} M(t,T) < \infty.$$

Relation (11) yields

$$\begin{aligned} m(t) \le f(t) + \int_0^{T_1} \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta,s),u)\,\Pi(du) \right] ds \\ + \sup_{\tau \ge T_1} d(\tau) M(t, T_1) \end{aligned} \tag{13}$$

for all t and T_1 such that $t \ge T_1$.

The assumptions of Theorem 1 allow one to apply the Lebesgue dominated convergence theorem [6], since $m(t) < \infty$. Thus

$$\varlimsup_{t\to\infty} \left\{ f(t) + \int_0^{T_1} \left[b^2\left(t, h_t(\theta,s)\right) + \int_{\mathbb{R}} g^2(t, h_t(\theta,s),u)\,\Pi(du) \right] \right\} ds = 0.$$

Relation (12) implies that

$$\varlimsup_{t\to\infty} m(t) \le (d+\varepsilon) \varlimsup_{t\to\infty} M\left(t, T_1\right) \le (d+\varepsilon) \varlimsup_{t\to\infty} m(t).$$

If $\varepsilon > 0$ is sufficiently small, then it follows from (11) that $\varlimsup_{t\to\infty} m(t) = 0$.

Now equation (10) yields

$$\mathsf{E}\left\{x^2(t)\right\} = y^2(t) + \int_0^t h^2(t,s) m(s)\,ds.$$

It is clear that for any $\varepsilon > 0$ there exists $T > 0$ such that $m(t) < \varepsilon$ for all $t \ge T$. Thus assumption 2) of Theorem 1 implies that

$$\mathsf{E}\left\{x^2(t)\right\} \le y^2(t) + \int_0^T h^2(t,s) m(s)\,ds + p\varepsilon.$$

Now, using assumption 1) of Theorem 1, we pass to the limit in the last inequality and obtain

$$\varlimsup_{t\to\infty} \mathsf{E}\left\{x^2(t)\right\} \le p\varepsilon,$$

which means that $\lim_{t\to\infty} \mathsf{E}\left\{x^2(t)\right\} = 0$.

Note that $\mathsf{E}\left\{x^2(t)\right\}$ continuously depends on the initial function $\{\varphi(\theta)\}$, since the operators defined by the integral equation for $\mathsf{E}\left\{x^2(t)\right\}$ are bounded (see [6]). This completes the proof of Theorem 1. □

2. The critical case

The critical case is considered in [1, 2, 8] for the solution of a stochastic differential-functional equation with aftereffect. Consider the problem of the mean square stability of a trivial solution of the stationary stochastic differential-functional equation with unbounded aftereffect:

$$dz(t) = a(z_t)\,dt + b(z_t)\,dw(t) + \int g(z_t, u)\,\overline{v}(dt, du) \tag{14}$$

for the critical case

$$\int_0^\infty \left[b^2(h_t) + \int_{\mathbb{R}} g^2(h_t, u)\,\Pi(du) \right] dt = 1. \tag{15}$$

It is known [10] that there exists an initial function $\{\varphi(\theta)\}$ for the stochastic differential-functional equation (14) such that

$$\lim_{t\to\infty} \mathsf{E}\left\{x^2(t)\right\} \neq 0 \quad (\neq \infty).$$

Let a stochastic process $x \in \mathbb{R}$ be defined by the equation

$$dx(t) = a(x_t)\,dt + (1+\beta(t))b(x_t)\,dw(t) + \int_{\mathbb{R}} (1+\gamma(t))g(x_t,u)\,\overline{v}(dt,du), \tag{16}$$

$$x(\theta) = \varphi(\theta) \quad \text{for all } \theta \in (-\infty, 0]\,, \tag{17}$$

where $\varphi \in D_0$; a, b, and g are functionals defined on D_0 and $D_0 \times \mathbb{R}$, respectively; $\{w(t)\}$ is a homogeneous Wiener process; and $\overline{v}\,(t,A)$ is a centered Poisson measure with parameter $t\Pi\,(A)$. We assume that $w(t)$ and $\overline{v}$ are independent.

We denote by $\mathcal{N}_1$ the set of scalar functions $\{\alpha(t)\}$ continuous on $[0,\infty)$ and such that $\int_0^\infty \left|2\alpha(t)+\alpha^2(t)\right| dt < \infty$, and by $\mathcal{N}_2$, the set of scalar functions $\Delta\,(t) \ge 0$ continuous on $[0,\infty)$ and such that $\int_0^\infty \left(2\Delta(t)+\Delta^2(t)\right) dt = \infty$.

Theorem 2. *Assume that*

1) *the trivial solution of equation* (9) *is exponentially stable;*
2) *condition* (15) *holds.*

Then the trivial solution of the stochastic differential-functional equation (15) *is*

I) *mean square stable if* $\alpha(t) \equiv \max\{\beta(t),\gamma(t)\} \in \mathcal{N}_1$ *is decreasing;*
II) *mean square unstable if* $\alpha(t) \equiv \min\{\beta(t),\gamma(t)\} \in \mathcal{N}_2$ *is increasing.*

The symbol $\max\{\beta(t),\gamma(t)\}$ *stands for the supremum of functions* $\{\alpha(t)\}$ *such that*

$$\alpha(t) \ge \beta(t) \quad \textit{and} \quad \alpha(t) \ge \gamma(t) \quad \textit{for all } t \ge 0,$$

while $\min\{\beta(t),\gamma(t)\}$ *stands for the infimum of functions* $\{\overline{\alpha}(t)\}$ *such that*

$$\overline{\alpha}(t) \le \beta(t) \quad \textit{and} \quad \overline{\alpha}(t) \le \gamma(t) \quad \textit{for all } t \ge 0.$$

Proof. Let $\beta \in \mathcal{N}_1 \vee \mathcal{N}_2$. If $\{h(t)\}$ is the fundamental solution of equation (9), then a solution of problem (16), (17) can be represented in the form

$$\begin{aligned} x(t) = y(t) &+ \int_0^t h(t-s)(1+\beta(s))b(x_s)\,dw(s) \\ &+ \int_0^t \int_{\mathbb{R}} h(t-s)(1+\gamma(s))g(x_s,u)\,\overline{v}(ds,du) \end{aligned} \tag{18}$$

(see [1]) where $\{y(t)\}$ is a solution of stationary equation (9) constructed for initial condition (17).

We apply the operators $(1+\beta(t))\,b\,(\cdot)$ and $(1+\gamma(t))\,g\,(\cdot,u)$ to both sides of equation (16). Then we proceed in the way that led to equation (11) and obtain for all $t \ge 0$ that

$$m(t) = f(t) + \int_0^t \left[(1+\beta(t))^2 b^2(h_{t-s}) + (1+\gamma(t))^2 \int_{\mathbb{R}} g^2(h_{t-s},u)\,\Pi(du)\right] m(s)\,ds \tag{19}$$

where

$$m(t) \equiv (1+\beta(t))^2 \mu_b(t) + (1+\gamma(t))^2 \int_{\mathbb{R}} \mu_g(t,u)\,\Pi(du);$$

$$\mu_b(t) \equiv \mathsf{E}\left\{b^2(x_t)\right\}; \qquad \mu_g(t,u) \equiv \mathsf{E}\left\{g^2(x_t,u)\right\};$$

$$f(t) \equiv (1+\beta(t))^2 b^2(y_t) + (1+\gamma(t))^2 \int_{\mathbb{R}} g^2(y_t,u)\,\Pi(du).$$

I) Let $\beta \in \mathcal{N}_1$ and $\gamma \in \mathcal{N}_1$ be decreasing. We check that $\{m(t)\}$ is bounded. First we show that $\{\overline{m}(t)\}$ is bounded and is a solution of the equation

$$\overline{m}(t) = f(t) + \int_0^t (1+\alpha(s))^2 k(t-s)\overline{m}(s)\,ds \tag{20}$$

where $\alpha(t) \equiv \max\{\beta(t), \gamma(t)\}$ for all $t \geq 0$, and $k(t) \equiv b^2(h_t) + \int_{\mathbb{R}} g^2(h_t, u)\,\Pi(du)$. It is clear that solutions of equations (19) and (20) are such that $m(t) \leq \overline{m}(t)$. Further, in view of the first assumption of Theorem 2 the functions $\{f(t)\}$ and $\{k(t)\}$ are such that

$$f(t) + k(t) \leq N e^{-\varepsilon t} \quad \text{for all } t \geq 0 \tag{21}$$

where $N > 0$ and $\varepsilon > 0$ are some constants.

Applying the Laplace transform we rewrite equation (17) as follows:

$$\overline{m}(t) = f(t) + \int_0^t H(t-s)H(t-s)f(s)\,ds + \int_0^t H(t-s)\tau(s)\overline{m}(s)\,ds \tag{22}$$

(see [7]) where $\tau(s) = 2\alpha(s) + \alpha^2(s)$,

$$H(t) \equiv k(t) + \int_0^t k(t-s)k(s)\,ds + \int_0^t \int_0^s k(t-s)k(s-s_1)k(s_1)\,ds\,ds_1 + \cdots .$$

Indeed, applying the Laplace transform to equation (22) we obtain

$$M(\lambda) = F(\lambda) + \overline{K}(\lambda)M(\lambda) + \overline{K}(\lambda)L\left\{\tau(s)\overline{m}(s)\right\}$$

(see [7]) where $\overline{K}$ and $\overline{m}$ are the Laplace transforms of K and m, respectively. Thus

$$M(\lambda) = \frac{F(\lambda)}{1-\overline{K}(\lambda)} + \frac{\overline{K}\,(\lambda)}{1-\overline{K}(\lambda)} L\left\{\tau(s)\overline{m}(s)\right\}.$$

Expanding the fraction $1/(1-\overline{K}(\lambda))$ into the series for $\lambda \neq 0$ we obtain from assumption 2) of Theorem 2 that

$$M(\lambda) = F(\lambda) + F\,(\lambda)\,[\overline{K}(\lambda) + \overline{K}^2\,(\lambda) + \cdots] + L\left\{\tau(s)\overline{m}(s)\right\}[\overline{K}(\lambda) + \overline{K}(\lambda) + \cdots].$$

Applying the inverse Laplace transform we reduce $\overline{m}\,(t)$ to the right-hand side of (22).

Since $\int_0^\infty k(t)\,dt = 1$, we apply the Laplace transform and show that $H(t) = C + \delta(t)$ where $C = \text{const} > 0$ and $\{\delta(t)\}$ is a continuous function on $[0,\infty)$ such that

$$\lim_{t\to\infty} \delta(t) = 0. \tag{23}$$

Then

$$\overline{m}(t) \leq B + A\int_0^t |\tau(s)|\overline{m}(s)\,ds \quad \text{for all } t \geq 0$$

where $B \equiv \sup_{t\geq 0} \int_0^t H(t-s)f(s)\,ds \in (0,\infty)$ and $A \equiv \sup_{t\geq 0} H(t) \in (0,\infty)$. It follows from the Gronwall–Bellman lemma [4] that

$$\overline{m}(t) \leq B\exp\left\{A\int_0^t |\tau(s)|\,ds\right\} \quad \text{for all } t \geq 0.$$

Let $\alpha \in \mathcal{N}_1$. Then

$$\int_0^\infty |\tau(s)|\,ds < \infty,$$

whence $\sup_{t\geq 0} \overline{m}(t) < +\infty$ and $\sup_{t\geq 0} m(t) < +\infty$. Squaring both sides of equation (16) and applying the operator of the mathematical expectation we get

$$\mu(t) = y^2(t) + \int_0^t h^2(t-s)m(s)\,ds \quad \text{for all } t \geq 0 \tag{24}$$

where $\mu(t) \equiv \mathsf{E}\left\{x^2(t)\right\}$ and $\{m(t)\}$ is the function defined above. Note that

$$y^2(t) + h^2(t) \leq \mathsf{E}\exp(-\varepsilon_1 t) \quad \text{for all } t \geq 0 \tag{25}$$

for some positive constants $M > 0$ and $\varepsilon_1 > 0$.

According to (20), $\sup_{t\geq 0} m(t) < \infty$ and $\int_0^\infty |\tau(s)|\, ds < \infty$. Relation (24) implies that $\sup_{t\geq 0}\mu(t) < \infty$. This completes the proof of the first part of the theorem, since the function $\varphi \in D_0$ is arbitrary and the mean square stability of solutions of linear systems is equivalent to the mean square boundedness of every solution (see [2]).

II) Consider the increasing function $\overline{\alpha}(t) \equiv \min\{\beta(t), \gamma(t)\} \in \mathcal{N}_2$. We show that the function $\{m(t)\}$ defined by equation (19) is unbounded. First we check that the function $\overline{\overline{m}}(t) \leq m(t)$ is unbounded and is a solution of the equation

$$\overline{\overline{m}}(t) = f(t) + \int_0^t (1 + \overline{\alpha}(s))^2\, k(t-s)\overline{\overline{m}}(s)\, ds$$

or, equivalently, of the equation

$$\overline{\overline{m}}(t) = f(t) + \int_0^t H(t-s)f(s)\, ds + \int_0^t H(t-s)\tau(s)\overline{\overline{m}}(s)\, ds \tag{26}$$

where $\{f(t)\}$, $\{\tau(t)\}$, $\{k(t)\}$, and $\{H(t)\}$ are defined above.

Using assumption 2) we first consider the initial deterministic function $\varphi \in D_0$ for the solution $\{y(t)\}$ of equation (9) such that

$$\int_0^\infty f(t)\, dt > 0. \tag{27}$$

Consider equation (26). Using (18), (19), and (27) we get

$$\lim_{t\to\infty} \int_0^t H(t-s)f(s)\, ds > 0,$$

whence

$$\lim_{t\to\infty} \int_0^t H(t-s)\tau(s)\left[\int_0^s H(s-s_1)f(s_1)\, ds_1\right] ds = \infty, \tag{28}$$

since $\int_0^\infty \tau(s)\, ds = \infty$ (the latter integral is infinite in view of $\overline{\alpha} \in \mathcal{N}_2$).

Taking (27) and integral equality (26) into account we obtain

$$\overline{\overline{m}}(t) \geq \int_0^t H(t-s)f(s)\, ds + \int_0^t H(t-s)\tau(s)\left[\int_0^s H(s-s_1)f(s_1)\, ds_1\right] ds$$

for all $t \geq 0$.

Then it follows from (28) that $\lim_{t\to\infty} \overline{\overline{m}}(t) = \infty$. Hence $\lim_{t\to\infty} m(t) = \infty$, since $\overline{\overline{m}} \leq m(t)$. Relation (24) leads to $\lim_{t\to\infty}\mu(t) = \infty$, which completes the proof of part 2) of Theorem 2. □

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