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On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables

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Introduction

By a hypersurface we here mean a $(2n-1)$-dimensional real analytic submanifold of the space of n complex variables, i. e. the n-dimensional complex Cartesian space $\boldsymbol{C}^n (n \geqq 2)$. A homeomorphism f of one hypersurface S onto another hypersurface S' is called a pseudo-conformal homeomorphism, if it can be extended to a complex analytic homeomorphism of a neighborhood of S onto a neighborhood of S' (Definition 1). In case such f exists, we say that the two hypersurfaces S and S' are mutually pseudo-conformally equivalent.

The main purpose of this paper is to study conditions for the pseudo-conformal equivalence of two hypersurfaces. In case $n=2$, this problem was first considered by H. Poincaré and was studied by B. Segre and E. Cartan. In his paper [**1**], E. Cartan gives a complete solution of the problem by the application of his own "method of the equivalence" [**3**]. We want to generalize his results to case $n \geqq 2$.

We introduce the notion of a non-degenerate hypersurface (Definition 2) which is a slight generalization of the notion of a hypersurface satisfying the so-called condition of Levi-Krzoska. Moreover, we introduce the notion of a regular hypersurface (Definition 3). Roughly speaking, a non-degenerate hypersurface is regular when it locally admits a non-trivial infinitesimal pseudo-conformal transformation (Proposition 5). Now, the main theorem (Theorem 4) in this paper may be stated as follows: To every regular non-degenerate hypersurface S there is associated, in an intrinsic manner, a principal fiber bundle P over the base space S together with an infinitesimal structure B in P, in terms of which the pseudo-conformal equivalence (of two regular non-degenerate hypersurfaces) can be characterized. The infinitesimal structure B stated above is a Cartan connection which we shall call the normal pseudo-conformal connection associated to the hypersurface S, cf. [**2**]. One finds that the situation is just analogous to the case of the Riemannian geometry of hypersurfaces. As an application of Theorem 4, it is shown that if a hypersurface S has a non-degenerate part, then the group $G(S)$ of all the pseudo-conformal transformations of S becomes a Lie group of dimension

$\leqq n^2+2n$ with respect to the natural topology (Theorem 5). Another application of Theorem 4 is concerned with a quadric Q of $\boldsymbol{C}^n$: Let us identify $\boldsymbol{C}^n$ with an open set of the n-dimensional complex projective space $P^n(\boldsymbol{C})$. We prove that every pseudo-conformal homeomorphism f of a connected open set U of Q with an open set U' of Q can be necessarily extended to a (unique) projective transformation σ of $P^n(\boldsymbol{C})$ (Theorem 6).

In Chapter I, we study conditions for the pseudo-conformal equivalence from the point of view of the affine geometry of hypersurfaces. We establish a theorem (Theorem 1) indicating that our problem is a special case of E. Cartan's equivalence problem [3]. In Chapter II, we give the definition of a non-degenerate hypersurface and of a regular hypersurface. This last definition has been suggested by a theorem (Chapter I, 17, p. 30) in E. Cartan [1]. Chapter III is preliminary to the subsequent chapters and deals with a quadric of $\boldsymbol{C}^n$ or rather an equivalent quadric of $P^n(\boldsymbol{C})$. In Chapter IV, it is shown that every regular non-degenerate hypersurface yields a family of Kählerian metrics defined on open sets of the space of $n-1$ complex variables. The family lies at the base of our construction of the normal pseudo-conformal connection. Chapters V and VI are concerned with the construction of the normal pseudo-conformal connection, which will be carried out following the construction of the normal conformal connection given in N. Tanaka [5]. However, the former is much more complicated. Finally in Chapter VII, we give some results (Theorems 4, 5 and 6) concerning pseudo-conformal transformations.

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Preliminary remarks

Throughout this paper, we always assume the differentiability of class C^ω. Let M be a manifold. We shall denote by $T_p(M)$ the tangent space to M at a point p of M and by df the differential of a mapping f of M into a manifold M'. By a submanifold N of M we shall mean a manifold satisfying the following conditions: 1) N is a subset of M; 2) The injection of N into M is real analytic and regular; 3) For each $p \in N$, we can find a system of coordinates $x_1, \cdots, x_n$ of M at p with coordinate neighborhood U such that the intersection $N \cap U$ is defined by $x_{r+1} = \cdots = x_n = 0$ $(r = \dim N)$.

Let P be a principal fiber bundle over a manifod M with structure group G. This principal fiber blundle will be symbolically denoted by $P(M, G)$. (1) R_σ will denote the right translation of P corresponding to an element σ of G. (2) A^* will denote the so-called fundamental vector field[1] on P corresponding

1) K. Nomizu, Lie groups and differential geometry, The Math. Soc. of Japan, 1956.

to a left invariant vector field A on G. (3) Let f be a homomorphism of G into G' and let $P'(M, G')$ be a principal fiber bundle over the base space M with structure group G'. A mapping $\bar{f}$ of P into P' will be called a homomorphism of $P(M, G)$ into $P'(M, G')$ corresponding to the homomorphism f, if $\bar{f}(x\cdot\sigma)=\bar{f}(x)\cdot f(\sigma)$ for all $x\in P$ and $\sigma\in G$ and if $\bar{f}$ induces the identity transformation of M. (4) Let H be a subgroup of G. A principal fiber bundle $Q(M, H)$ will be called a subbundle of $P(M, G)$ if Q is a submanifold of P and if the injection of Q into P gives a homomorphism of $Q(M, H)$ into $P(M, G)$ corresponding to the injection of H into G. (5) Let U be an open set of M. We shall denote by $P|U$ the restriction to U of the principal fiber bundle $P(M, G)$.

Let G be a Lie transformation group on a connected manifold M. For each tangent vector X to M and for each element σ in G, we shall denote by $\sigma\cdot X$ the tangent vector $dl_\sigma\cdot X$, l_σ being the transformation of M induced by σ. Now take a fixed point o of M and denote by H the isotropy group of G at o and by π the mapping $\sigma\rightarrow\sigma\cdot o$ of G into M. Under the assumption that G satisfies the second countability axiom and that G acts transitively on M, G may be considered as a principal fiber bundle over the base space M with structure group H with projection π. This fiber bundle will be denoted by $G(M, H)$.

Finally, let α be a complex number. We shall denote by $\bar{\alpha}$ the conjugate of α and by $\mathcal{R}\alpha$(resp. $\mathcal{I}\alpha$) the real part (resp. the imaginary part) of α.

I. Fundamental theorem

1. Let $\boldsymbol{C}^n$ $(n\geqq 2)$ be the n-dimensional complex Cartesian space and let $z_1, \cdots, z_n$ be the natural coordinates of it. We denote by x_i (resp. y_i) the real (resp. imaginary) part of the i-th coordinate z_i. Then the $2n$ functions $x_1, \cdots, x_n, y_1, \cdots, y_n$ form a system of coordinates of $\boldsymbol{C}^n$ as a $2n$-dimensional real Cartesian space, i.e. $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ in this sense. By a hypersurface we shall always mean a $(2n-1)$-dimensional submanifold of $\boldsymbol{C}^n$.

DEFINITION 1. A homeomorphism f of a hypersurface S with a hypersurface S' is called a pseudo-conformal homeomorphism if it can be extended to a complex analytic homeomorphism f of a neighborhood of S with a neighborhood of S'.

REMARK. Let f be a homeomorphism of a hypersurface S with another hypersurface. If f admits two complex analytic extensions $\tilde{f}$ and $\tilde{f}'$, then we have $\tilde{f}=\tilde{f}'$ on a sufficiently small neighborhood of S, cf. Lemma 2. We know from this fact that the property for a homeomorphism of being pseudo-conformal is of local character.

2. For each tangent vector X to $\boldsymbol{C}^n$, we denote by $\sqrt{-1}\cdot X$ the tangent vector to C^n defined by $(\sqrt{-1}\cdot X)z_i=\sqrt{-1}\cdot Xz_i$, i.e. $(\sqrt{-1}\cdot X)x_i=-Xy_i$ and $(\sqrt{-1}\cdot X)y_i=Xx_i$ $(1\leqq i\leqq n)$. Every tangent space $T_p(C^n)$ becomes an n-dimensional complex vector space with respect to this operation. We denote by $\partial/\partial z_i$[2] the vector field on C^n induced by the one parameter group of transformations $(z_1,\cdots,z_n)\rightarrow(z_1,\cdots,z_{i-1},z_i+t,z_{i+1},\cdots,z_n)$ on C^n. The n vectors $(\partial/\partial z_1)_p,\cdots,(\partial/\partial z_n)_p$ form a complex base of $T_p(C^n)$ at each $p\in C^n$. We set $e_i=(\partial/\partial z_i)_0$ $(1\leqq i\leqq n)$, o being the origin of C^n.

An affine transformation of C^n is a transformation $(z_i)\rightarrow(z_i')$ on C^n such that $z_i'=\xi_i+\sum_{j=1}^{n}a_{ij}z_j$, where $\det(a_{ij})\neq 0$. The group $A(n,C)$ of all the affine transformations of C^n may be represented as a subgroup of the general linear group $GL(n+1,C)$ of degree $n+1$ consisting of all the matrices σ of the form

$$\begin{pmatrix} 1 & 0 \\ \xi_i & a_{ij} \end{pmatrix} \qquad (1\leqq i,\ j\leqq n). \tag{2.1}$$

The linear isotropy group of $A(n,C)$ at the origin o may be identified with the general linear group $GL(n,C)$ of degree n. We denote by ϖ the projection of the fiber bundle $A(n,C)$ $(C^n,GL(n,C))$, i.e. $\varpi(\sigma)=(\xi_1,\cdots,\xi_n)$ if σ is expressed as (2.1). Now let σ be an element of $A(n,C)$ expressed as (2.1) and let $X=\sum_{i=1}^{n}X_i\cdot(\partial/\partial z_i)_p$ be a tangent vector to C^n at a point p. Setting $q=\sigma\cdot p$, then we have

$$\sigma\cdot X=\sum_{i,j=1}^{n}a_{ji}X_i\cdot(\partial/\partial z_j)_q\,.$$

The Maurer-Cartan form α of $A(n,C)$ may be represented as a matrix of the form

$$\begin{pmatrix} 0 & 0 \\ \alpha_i & \alpha_{ij} \end{pmatrix} \qquad (1\leqq i,\ j\leqq n).$$

The components α_i and α_{ij} are complex forms on $A(n,C)$. The n forms $\alpha_1,\cdots,\alpha_n$ will be called the *basic forms* of $A(n,C)$. Let $y_i(x)$ and $y_{ij}(x)$ be the components of a matrix x in $A(n,C)$ and let $(z_{ij}(x))$ be the inverse matrix of the matrix $(y_{ij}(x))$. Considering y_i, y_{ij} and z_{ij} as functions on $A(n,C)$, we get an explicit expression of α_i and α_{ij}:

$$\begin{aligned} \alpha_i &= \sum_{j=1}^{n} z_{ij}dy_j\,; \\ \alpha_{ij} &= \sum_{k=1}^{n} z_{ik}dy_{kj} \qquad (1\leqq i,\ j\leqq n). \end{aligned} \tag{2.2}$$

3. We denote by $\mathfrak{m}$ the $(2n-1)$-dimensional subspace of $T_o(C^n)$ spanned

2) We have $\partial/\partial z_i=\partial/\partial x_i$ and $\sqrt{-1}\cdot\partial/\partial z_i=\partial/\partial y_i$ $(1\leqq i\leqq n)$.

(over the field $\boldsymbol{R}$ of real numbers) by the $2n-1$ vectors $e_1, \cdots, e_n, \sqrt{-1}\cdot e_1, \cdots, \sqrt{-1}\cdot e_{n-1}$. Let $\mathcal{G}$ be the Grassmann manifold consisting of all the $(2n-1)$-dimensional contact elements to $\boldsymbol{C}^n$. Then the group $A(n, \boldsymbol{C})$ acts transitively on $\mathcal{G}$ through the mapping $A(n, \boldsymbol{C}) \times \mathcal{G} \ni (x, E) \to x\cdot E \in \mathcal{G}$. We denote by H the isotropy group of $A(n, \boldsymbol{C})$ at the point $\mathfrak{m} \in \mathcal{G}$. The group H explicitly consists of all the matrices $\sigma \in GL(n, \boldsymbol{C})$ of the form

$$\begin{pmatrix} b_{ij} & c_i \\ 0 & a \end{pmatrix} \qquad (1 \leqq i,\ j \leqq n-1), \tag{3.1}$$

where a is real. Since H acts effectively on $\mathfrak{m}$, it may be identified with a subgroup of the general linear group $GL(\mathfrak{m})$ of the real vector space $\mathfrak{m}$.

Let S be a hypersurface. We denote by $P^*(S, H)$ the induced principal fiber bundle from the principal bundle $A(n, \boldsymbol{C})(\mathcal{G}, H)$ by the mapping $p \to T_p(S)$. The point set of P^* may be defined as the subset of $A(n, \boldsymbol{C})$ consisting of all the points x such that $\varpi(x) \in S$ and $x\cdot \mathfrak{m} = T_p(S)$, where $p = \varpi(x)$. In this case, we have the following statements: (1) to (4). (1) P^* is a submanifold of $A(n, \boldsymbol{C})$; (2) The action on P^* of H is defined by the group multiplication of $A(n, \boldsymbol{C})$; (3) The mapping $x \to \varpi(x)$ gives the projection of P^* onto S; (4) H being identified with a subgroup of $GL(\mathfrak{m})$, $P^*(S, H)$ may be regarded as a subbundle of the bundle of frames of S.

The restriction ω to P^* of the Maurer-Cartan form α of $A(n, \boldsymbol{C})$ will be called the *Maurer-Cartan form* of P^*. The components ω_i and ω_{ij} of ω, which are complex valued forms on P^*, satisfy the following conditions:

(3.2) 1) $R^*_\sigma(\omega_i) = \sigma^{-1}\cdot(\omega_i)$,
$R^*_\sigma(\omega_{ij}) = \sigma^{-1}\cdot(\omega_{ij})\cdot\sigma,\ \sigma \in H$;

2) $\omega_i(A^*) = 0$,
$\omega_{ij}(A^*) = A_{ij}\ (1 \leqq i,\ j \leqq n)$
for any element $A = (A_{ij})$ in the Lie algebra of H;

3) $d\omega_i + \sum_{j=1}^n \omega_{ij} \wedge \omega_j = 0$,
$d\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} = 0 \quad (1 \leqq i,\ j \leqq n)$;

4) ω_n is a real form and we have
$d\varpi \cdot X = x\cdot(\sum_{i=1}^n \omega_i(X) e_i)$
for all $x \in P^*$ and $X \in T_x(P^*)$.

4) is clear from the equality: $d\varpi\cdot X = x\cdot(\sum_{i=1}^n \alpha_i(X) e_i)$ for all $x \in A(n, \boldsymbol{C})$ and $X \in T_x(A(n, \boldsymbol{C}))$. From 4), it follows that the $2n-1$ forms $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_{n-1}$ are linearly independent at each point of P^* over the field $\boldsymbol{C}$ of complex numbers. The n forms $\omega_1, \cdots, \omega_n$ will be called the *basic forms* of P^*.

4. THEOREM 1. *Let S (resp. S') be a hypersurface and let $P^*(S, H)$ (resp.*

$'P^*(S', H)$) be the corresponding H-bundle (See 3). Let ω_i (resp. ω'_i) be the basic forms of P^* (resp. $'P^*$). If f is a pseudo-conformal homeomorphism of S with S', there corresponds to f a unique isomorphism φ of $P^*(S, H)$ with $'P^*(S', H)$ such that φ induces the given f and such that $\varphi^*\omega'_i = \omega_i$ $(1 \leqq i \leqq n)$. Conversely, every isomorphism φ of $P^*(S, H)$ with $'P^*(S', H)$ satisfying this last condition induces a pseudo-conformal homeomorphism f of S with S'.

Taking account of this theorem, we shall say that the principal fiber bundle $P^*(S, H)$, defined in 3, is the *pseudo-conformal H-bundle associated to the hypersurface S.*

5. In order to prove Theorem 1, we need auxiliary lemmas. Set $\mathcal{V} = \boldsymbol{C}^n \times \boldsymbol{C}$ and consider the system of coordinates $z_1, \cdots, z_n, w$ of $\mathcal{V}$ which is defined by the coordinates $z_1, \cdots, z_n$ of $\boldsymbol{C}^n$ and the coordinate w of $\boldsymbol{C}$. Let Σ be the exterior differential system[3] on $\mathcal{V}$ generated by the real part and imaginary part of the $(n+1)$-form $dz_1 \wedge \cdots \wedge dz_n \wedge dw$ on $\mathcal{V}$, and let Ω be the system of independent variables on $\mathcal{V}$ generated by the $2n$ Pfaffian forms $dx_1, \cdots, dx_n, dy_1, \cdots, dy_n$ on $\mathcal{V}$. Then the pair (Σ, Ω) forms an exterior differential system on $\mathcal{V}$ with $2n$ independent variables.

LEMMA 1. *Every $(2n-1)$-dimensional integral element of (Σ, Ω) is regular, and it is contained in a unique $2n$-dimensional integral element of (Σ, Ω).*

The proof of Lemma 1, which is omitted, makes use of E. Cartan's criterion[4] for an exterior differential system to be in involution and it does not require any difficulty.

Now let S be a hypersurface and let f be a function on S. If f can be extended to a holomorphic function on a neighborhood of S, then it satisfies clearly the condition: $df \wedge d(z_1 \circ \iota) \wedge \cdots \wedge d(z_n \circ \iota) = 0$, ι being the injection of S into $\boldsymbol{C}^n$. Conversely, suppose that f satisfies this condition. Then the totality V^{2n-1} of all the pairs $(p, f(p))$ with $p \in S$, which is a $(2n-1)$-dimensional submanifold of $\mathcal{V}$, turns out to be a $(2n-1)$-dimensional integral of (Σ, Ω). Therefore by Lemma 1 and E. Cartan's existence theorem[4], we can take a "unique" $2n$-dimensional integral V^{2n} of (Σ, Ω) such that $V^{2n-1} \subset V^{2n}$. It follows easily that f can be extended to a "unique" holomorphic function on a neighborhood of S.

We have thereby proved

LEMMA 2. *Let S be a hypersurface and let f be a function on S. Then, f can be extended to a holomorphic function defined on a neighborhood of S if and*

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3) As for an exterior differential system, we adopt the definitions and notations given in M. Kuranishi, On E. Cartan's prolongation theorem of exterior differential systems, Amer. J. Math., 1957, Vol. LXXIX, no. 1.

4) E. Cartan, Les systemes différentiels extérieurs et leurs applications géométriques, Paris, 1945.

only if it satisfies the condition: $df \wedge d(z_1 \circ \iota) \wedge \cdots \wedge d(z_n \circ \iota) = 0$, *$\iota$ being the injection of S into $\boldsymbol{C}^n$. Furthermore, if f admits two holomorphic extensions $\tilde{f}$ and $\tilde{f}'$, then we have $\tilde{f} = \tilde{f}'$ on a sufficiently small neighborhood of S.*

6. PROOF OF THEOREM 1. First, suppose that there is given a pseudo-conformal homeomorphism f of S with S'. Under this condition, we can find, for each $p \in S$, a unique element $u(p)$ of $A(n, \boldsymbol{C})$ such that $df \cdot X = u(p) \cdot X$ for all $X \in T_p(S)$. It is easily seen that the mapping $S \ni p \to A(n, \boldsymbol{C}) \ni u(p)$ is real analytic. Now take a point x of P^* and set $\varphi(x) = u(p) \cdot x$, where $p = \varpi(x)$. Then we have

$$\varphi(x) \cdot \mathfrak{m} = u(p) \cdot (x \cdot \mathfrak{m}) = u(p) \cdot T_p(S)$$
$$= df \cdot T_p(S) = T_{f(p)}(S'),$$

meaning that $\varphi(x)$ is in $'P^*$. The mapping $x \to \varphi(x)$ clearly gives an isomorphism of $P^*(S, H)$ with $'P^*(S', H)$, and it satisfies the condition: $\varpi \circ \varphi = f \circ \varpi$ on P^*. We must show that $\varphi^* \omega_i' = \omega_i$ $(1 \leqq i \leqq n)$. We have, for each $x \in P^*$ and $X \in T_x(P^*)$,

$$d(f \circ \varpi) \cdot X = df \cdot (d\varpi \cdot X) = df \cdot (x \cdot \sum_{i=1}^{n} \omega_i(X) e_i)$$
$$= \varphi(x) \cdot (\sum_{i=1}^{n} \omega_i(X) e_i)$$

and we have

$$d(f \circ \varpi) \cdot X = d(\varpi \circ \varphi) \cdot X = d\varpi \cdot (d\varphi \cdot X)$$
$$= \varphi(x) \cdot (\sum_{i=1}^{n} \varphi^* \omega_i'(X) e_i).$$

If follows immediately that $\varphi^* \omega_i' = \omega_i$ $(1 \leqq i \leqq n)$. Now let us prove the uniqueness of φ. Let φ be an arbitrary isomorphism of $P^*(S, H)$ with $'P^*(S', H)$ such that $f \circ \varpi = \varpi \circ \varphi$ on P^* and $\varphi^* \omega_i' = \omega_i$ $(1 \leqq i \leqq n)$. From the above consideration, we can deduce that $\varphi(x) \cdot \xi = df \cdot (x \cdot \xi)$ for all $x \in P^*$ and $\xi \in \mathfrak{m}$; Hence $\varphi(x)$ can be written in the form $u(p) \cdot x$ with the mapping u defined above, proving the uniqueness of φ. Thus we have completed the proof of the first half of Theorem 1.

Conversely, suppose that there is given an isomorphism φ of $P^*(S, H)$ with $'P^*(S', H)$ such that $\varphi^* \omega_i' = \omega_i$ $(1 \leqq i \leqq n)$ and denote by f the homeomorphism of S with S' induced by φ. Let p be a point of S and let g be a local cross-section of $P^*(S, H)$ defined over a neighborhood U of p in S. By using the notations in 2, then we have

$$g^* \omega_i = g^* \alpha_i = \sum_{j=1}^{n} (z_{ij} \circ g) \cdot d(z_j \circ \iota)$$

and

$$(\varphi \circ g)^* \omega_i' = (\varphi \circ g)^* \alpha_i = \sum_{j=1}^{n} (z_{ij} \circ \varphi \circ g) \cdot d(z_j \circ f),$$

where ι denotes the injection of U into $\boldsymbol{C}^n$. Since $\varphi^*\omega'_i=\omega_i$, it follows that

$$d(z_i\circ f)\equiv 0 \qquad \bmod d(z_1\circ\iota),\cdots,d(z_n\circ\iota).$$

p being arbitrary, this means that the n functions $z_1\circ f,\cdots,z_n\circ f$ on S satisfies the condition in Lemma 2. Therefore we see from Lemma 2 that each function $z_i\circ f$ can be extended to a holomorphic function defined on a neighborhood of S; Hence f can be extended to a holomorphic mapping $\tilde f$ of a neighborhood of S into $\boldsymbol{C}^n$. In the same way, we get a holomorphic extension $\tilde f^{-1}$ of the inverse f^{-1} of f. We have $\tilde f\circ\tilde f^{-1}(p)=\tilde f^{-1}\circ\tilde f(p)=p$, provided p is in S and hence provided p is in a sufficiently small neighborhood of S (Lemma 2). This clearly means that $\tilde f$ gives a homeomorphism of a neighborhood of S with a neighborhood of S'. We have thereby proved that f is a pseudo-conformal homeomorphism of S with S'.

7. PROPOSITION 1. *Every infinitesimal pseudo-conformal transformation X on a hypersurface S can be extended to a holomorphic infinitesimal transformation X defined on a neighborhood of S.*

PROOF. Let X be an infinitesimal pseudo-conformal transformation S and let Φ_t be the local one parameter group of local transformations on S which is generated by X. If we denote by X_i the function on S defined by $X_i(p)=X_pz_i$, then we have $X_p=\sum_{i=1}^{n}X_i(p)\cdot(\partial/\partial z_i)_p$ and $X_i(p)=\partial/\partial t_{t=0}z_i\circ\Phi_t(p)$. Since Φ_t is a local pseudo-conformal homeomorphism on S, we see that the n functions $z_1\circ\Phi_t,\cdots,z_n\circ\Phi_t$ satisfy the condition in Lemma 2 and hence the n functions $X_1,\cdots,X_n$ satisfy the same condition. Therefore by Lemma 2, each function X_i can be extended to a holomorphic function $\tilde X_i$ defined on a neighborhood of S; Hence X can be extended to a holomorphic infinitesimal transformation $\tilde X$ (i.e. $=\sum_{i=1}^{n}\tilde X_i\cdot\partial/\partial z_i$) defined on a neighborhood of S.

II. Non-degenerate hypersurfaces

8. LEMMA 3. *Let S be a hypersurface. Let $P^*(S,H)$ be the pseudo-conformal H-bundle associated to S and let ω_i and ω_{ij} be the components of the Maurer-Cartan form of P^*. Then, the n forms ω_{ni} $(1\leqq i\leqq n-1)$ and $\omega_{nn}-\bar\omega_{nn}$ are expressed as linear combinations of $\omega_1,\cdots,\omega_n,\bar\omega_1,\cdots,\bar\omega_{n-1}$:*

$$\omega_{ni}=\sum_{j=1}^{n-1}K_{ji}\omega_j+\sum_{j=1}^{n-1}L_{ji}\bar\omega_j+M_i\omega_n \qquad (1\leqq i\leqq n-1)$$

$$\omega_{nn}-\bar\omega_{nn}=\sum_{j=1}^{n-1}M_j\omega_j-\sum_{j=1}^{n-1}\bar M_j\bar\omega_j+N\omega_n.$$

The coefficients K_{ij}, L_{ij}, M_i and N, which are complex valued functions on P^*, satisfy the following relations: $K_{ji}=K_{ij}$, $L_{ji}=-\bar L_{ij}$ and $N=-\bar N$.

PROOF. We have $d\omega_n+\sum_{i=1}^{n}\omega_{ni}\wedge\omega_i=0$ (3) of (3.2)). Since ω_n is a real form, it follows that

$$\sum_{i=1}^{n-1}\omega_{ni}\wedge\omega_i-\sum_{i=1}^{n-1}\bar{\omega}_{ni}\wedge\bar{\omega}_i+(\omega_{nn}-\bar{\omega}_{nn})\wedge\omega_n=0\,.$$

Lemma 3 is then an immediate consequence of this equality and the fact that $\omega_1,\cdots,\omega_n,\ \bar{\omega}_1,\cdots,\bar{\omega}_{n-1}$ are linearly independent at each point of P^*.

LEMMA 4. *Consider the functions L_{ij} in Lemma 3. Let x be a point of P^* and let σ be an element of H expressed as* (3.1). *Setting $L(x)=(L_{ij}(x))$ and $b=(b_{ij})$, then we have*

$$L(x\cdot\sigma)=a^{-1}\cdot{}^t\bar{b}\cdot L(x)\cdot b\,.$$

This follows easily from 1) of (3.2).

LEMMA 5. *Let S (resp. S') be a hypersurface. We use the notations in Theorem* 1. *Let L_{ij} (resp. L'_{ij}) be the functions on P^* (resp. $'P^*$) corresponding to S (resp. S') (See Lemma* 3). *If φ is an isomorphism of $P^*(S, H)$ with $'P^*(S', H)$ such that $\varphi^*\omega'_i=\omega_i$ $(1\leqq i\leqq n)$, then we have $L'_{ij}\circ\varphi=L_{ij}$ $(1\leqq i,\ j\leqq n-1)$.*

PROOF. We have $d\omega_n+\sum_{i=1}^{n}\omega_{ni}\wedge\omega_i=0$ and $d\omega'_n+\sum_{i=1}^{n}\omega'_{ni}\wedge\omega'_i=0$. Since $\varphi^*\omega'_i=\omega_i$ $(1\leqq i\leqq n)$, it follows that $\sum_{i=1}^{n}(\varphi^*\omega'_{ni}-\omega_{ni})\wedge\omega_i=0$. This means that $\varphi^*\omega'_{ni}-\omega_{ni}$ are linear combinations of $\omega_1,\cdots,\omega_n$. On the other hand, we have

$$\varphi^*\omega'_{ni}-\omega_{ni}=\sum_{j=1}^{n-1}(K'_{ji}\circ\varphi-K_{ji})\omega_j+\sum_{j=1}^{n-1}(L'_{ji}\circ\varphi-L_{ji})\bar{\omega}_j$$
$$+(M'_i\circ\varphi-M_i)\omega_n\qquad(1\leqq i\leqq n-1)\,.$$

Consequently we must have $L'_{ji}\circ\varphi=L_{ji}$ $(1\leqq i,\ j\leqq n-1)$.

By utilizing the above lemmas, we now define two pseudo-conformal invariants μ and λ, which are integer valued functions on any hypersurface S. Let p be a point of S and let x be a point of P^* lying over the point p. From Lemma 3, the matrix $L(x)$ is skew-hermitian. Considering the hermitian matrix $\sqrt{-1}\cdot L(x)$, we define $\mu(p)$ to be the multiplicity of the eigen-value 0 and $\lambda(p)$ to be the minimum of the number of the positive eigen-values and the number of the negative ones. The integers $\mu(p)$ and $\lambda(p)$ are well defined by Lemma 3.

PROPOSITION 2. *The functions μ and λ defined above are pseudo-conformal invariants: More precisely, let f be a pseudo-conformal homeomorphism of a hypersurface S with a hypersurface S', and let μ and λ (resp. μ' and λ') be the corresponding functions on S (resp. S'). Then, we have $\mu'\circ f=\mu$ and $\lambda'\circ f=\lambda$.*

This is clear from Theorem 1 and Lemma 5.

Proposition 2 enables us to give the following

DEFINITION 2. Let S be a hypersurface and let p be a point of S. (1) S

is called non-degenerate at p if $\mu(p)=0$. (2) S is called of index r at p if $\lambda(p)=r$.

9. Let p_0 be a fixed point of a hypersurface S. By replacing, if necessary, the hypersurface S by the hypersurface $\sigma\cdot S$ with an affine transformation σ, we can assume without loss of generality that $p_0=o$ and $T_{p_0}(S)=\mathfrak{m}$. In this case, a sufficiently small neighborhood of o in S is defined by a local equation of the form: $y_n=f(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1};x_n)$, where we assume that the function $f(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1};t)$ is defined on a neighborhood D of the origin of $\boldsymbol{C}^{n-1}\times\boldsymbol{R}=\boldsymbol{R}^{2n-1}$. For the sake of simplicity, we furthermore assume that the hypersurface S is globally defined by the above equation; Hence, the mapping $(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1};t)\to(x_1,\cdots,x_{n-1},t;y_1,\cdots,y_{n-1},f(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1};t))$ maps D onto S homeomorphically. In this sense, $x_1,\cdots,x_{n-1},y_1,\cdots,y_{n-1},t$ may be considered as a system of coordinates of S.

Take a point p of S and define an element $g(p)$ of $A(n,\boldsymbol{C})$ as[5)]

$$\begin{pmatrix} 1 & 0 & 0 \\ z_i & \left(1+\left(\frac{\partial f}{\partial t}\right)^2\right)\delta_{ij} & 0 \\ t+\sqrt{-1}f & 2\sqrt{-1}\left(1+\sqrt{-1}\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial z_j} & \left(1+\sqrt{-1}\frac{\partial f}{\partial t}\right) \end{pmatrix} \tag{9.1}$$

Then, it can be shown that the mapping $p\to g(p)$ gives a cross-section of $P^*(S,H)$ and that[6)]

$$\begin{aligned} \frac{1}{2\sqrt{-1}}L_{ji}(g(p)) &= \frac{\partial^2 f}{\partial t^2}\cdot\frac{\partial f}{\partial z_i}\cdot\frac{\partial f}{\partial \bar z_j}-\left(\frac{\partial f}{\partial t}+\sqrt{-1}\right)\frac{\partial^2 f}{\partial z_i\partial t}\cdot\frac{\partial f}{\partial \bar z_j} \\ &\quad -\left(\frac{\partial f}{\partial t}-\sqrt{-1}\right)\frac{\partial f}{\partial z_i}\cdot\frac{\partial^2 f}{\partial \bar z_j\partial t}+\left(1+\left(\frac{\partial f}{\partial t}\right)^2\right)\frac{\partial^2 f}{\partial z_i\partial \bar z_j}. \end{aligned} \tag{9.2}$$

The proof of (9.2) makes use of the expressions (2.2) of α_i and α_{ij}. Now, set $F(x_1,\cdots,x_n;y_1,\cdots,y_n)=f(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1};x_n)-y_n$, which is a function defined on a neighborhood of S. Let p be a point of S and let $\xi_1,\cdots,\xi_n$ be arbitrary n complex numbers subject to the condition: $\sum_{i=1}^{n}\frac{\partial F}{\partial z_i}(p)\xi_i=0$. Then we get, from (9.2),

$$\frac{1}{2\sqrt{-1}}\cdot\sum_{i,j=1}^{n-1}L_{ji}(g(p))\xi_i\bar\xi_j=c\cdot\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial z_i\partial\bar z_j}(p)\xi_i\bar\xi_j,$$

5) and 6) $\delta_{ij}=1$ $(i=j)$, $=0$ $(i\neq j)$.

$$\frac{\partial f}{\partial z_i}=\frac{1}{2}\left(\frac{\partial f}{\partial x_i}-\sqrt{-1}\frac{\partial f}{\partial y_i}\right) \quad\text{and}\quad \frac{\partial f}{\partial \bar z_i}=\frac{1}{2}\left(\frac{\partial f}{\partial x_i}+\sqrt{-1}\frac{\partial f}{\partial y_i}\right).$$

The functions of the right-hand side of (9.1) and of (9.2) should be evaluated at the point of D with the coordinates: $x_i=x_i(p)$, $y_i=y_i(p)$ $(1\leqq i\leqq n-1)$ and $t=x_n(p)$.

where c is a suitable real number ($\neq 0$) depending only on p.

Recalling the definition[7] of the condition of Levi-Krzoska, we have proved

PROPOSITION 3. *Let S be a hypersurface and let p be a point of S. Then S is non-degenerate and of index 0 at p if and only if it satisfies the condition of Levi-Krzoska at p.*

10. DEFINITION 3. A hypersurface S is called regular at a point p of S, if there exists an infinitesimal pseudo-conformal transfomation X defined on a neighborhood of p in S such that $\sqrt{-1} \cdot X_p \notin T_p(S)$, i.e. if X_p does not belong to the maximum complex subspace of $T_p(S)$.

PROPOSITION 4. *If a hypersurface S is regular at a point p of S, then we can find a system of complex coordinates $w_1, \cdots, w_n$ at p and a real valued function $f(x_1, \cdots, x_{n-1}; y_1, \cdots, y_{n-1})$ defined on an open set of $\boldsymbol{C}^{n-1} = \boldsymbol{R}^{2n-2}$ such that a sufficiently small neighborhood of p in S is defined by the equation: $v_n = f(u_1, \cdots, u_{n-1}; v_1, \cdots, v_{n-1})$, where u_i (resp. v_i) denotes the real (resp. imaginary) part of the i-th coordinate w_i.*

PROOF. S being regular at p, there exists an infinitesimal transformation X defined on a neighborhood of p in S such that $\sqrt{-1} \cdot X_p \notin T_p(S)$. By Proposition 1, X can be extended to a holomorphic infinitesimal transformation $\tilde{X}$ defined on a neighborhood of p in $\boldsymbol{C}^n$. Since $\tilde{X}_p = X_p \neq 0$, we can find a system of complex coordinates $w_1, \cdots, w_n$ of $\boldsymbol{C}^n$ at p such that $\tilde{X} = \partial/\partial w_n$ on a neighborhood of p in $\boldsymbol{C}^n$. u_i and v_i being as in Proposition 4, then we have $(\partial/\partial v_n)_p = \sqrt{-1} \cdot (\partial/\partial w_n) = \sqrt{-1} \cdot X_p \notin T_p(S)$. It follows that there exists a real valued function $f(x_1, \cdots, x_{n-1}; y_1, \cdots, y_{n-1}; t)$ defined on an open set of $\boldsymbol{C}^{n-1} \times \boldsymbol{R} = \boldsymbol{R}^{2n-1}$ such that a sufficiently small neighborhood of p in S is defined by the local equation $v_n = f(u_1, \cdots, u_{n-1}; v_1, \cdots, v_{n-1}; u_n)$. The infinitesimal transformation $\partial/\partial w_n = \partial/\partial u_n$ is tangent to S in a neighborhood of p, so that the function $f(x_1, \cdots, x_{n-1}; y_1, \cdots, y_{n-1}; t)$ does not depend on the variable t. We have thereby proved Proposition 4.

The set S^* of all the regular points of a hypersurface S is obviously an open set of S.

PROPOSITION 5. *If a non-degenerate hypersurface S admits a non-trivial infinitesimal pseudo-conformal transformation, then the subset S^* is dense in S.*

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7) We say that a hypersurface S satisfies the condition of Levi-Krzoska at a point p of S when we can find a regular local equation $F=0$ of S at p which satisfies the following condition: $\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}(p)\xi_i\bar{\xi}_j > 0$ for arbitrary n complex numbers $\xi_1, \cdots, \xi_n$ such that $(\xi_1, \cdots, \xi_n) \neq 0$ and $\sum_{i=1}^{n} \frac{\partial F}{\partial z_i}(p)\xi_i = 0$.

Proposition 5 is an immediate consequence of the following

LEMMA 6. *Let S be a non-degenerate hypersurface and let X be an infinitesimal pseudo-conformal transformation on S. If the vector $\sqrt{-1}\cdot X_p$ is tagent to S at each point p of S, then we have $X=0$.*

PROOF. We use the notations in Lemma 3. By Theorem 1, every infinitesimal pseudo-conformal transformation Z on S induces a unique infinitesimal transformation Z^* on P^* satisfying the following conditions: 1) $d\varpi\cdot Z_x^*=Z_p$ for each point x of P^*, where $p=\varpi(x)$; 2) Z^* is invariant under the right translations on P^*; 3) $\mathcal{L}_{Z^*}\omega_i=0$[8] $(1\leqq i\leqq n)$. By Proposition 1, the symbol $\sqrt{-1}\cdot X$ also defines an infinitesimal pseudo-conformal transformation on S. Now take a point p of S and a point x of P^* lying over the point p. Then we have $X_p=x\cdot(\sum_{i=1}^{n}\omega_i(X_x^*)e_i)$ and $\sqrt{-1}\cdot X_p=x\cdot(\sum_{i=1}^{n}\omega_i((\sqrt{-1}\,X)_x^*)e_i)$ (4) of (3.2)), whence $\omega_n((\sqrt{-1}\cdot X)_x^*)=\sqrt{-1}\cdot\omega_n(X_x^*)$. Since ω_n is a real form, we get $\omega_n(X^*)=0$. Now we assert that $\omega_i(X^*)=0$ $(1\leqq i\leqq n-1)$. We have $\mathcal{L}_{X^*}\omega_n=0$ and hence, for all vector field Y on P^*, $\mathcal{L}_{X^*}\omega_n(Y)=X^*\omega_n(Y)-\omega_n([X^*, Y])=d\omega_n(X^*, Y)+Y\omega_n(X^*)=0$. But, we have from Lemma 3,

$$d\omega_n+\sum_{i,j=1}^{n-1}L_{ji}\bar{\omega}_j\wedge\omega_i+\beta\wedge\omega_n=0\,,$$

where $\beta=\omega_{nn}-\sum_{i=1}^{n-1}M_i\cdot\omega_i$. It follows immediately that

$$\sum_{i,j=1}^{n-1}L_{ji}\bar{\omega}_j(X^*)\omega_i(Y)-\sum_{i,j=1}^{n-1}L_{ji}\omega_i(X^*)\bar{\omega}_j(Y)$$
$$+\beta(X^*)\omega_n(Y)-\omega_n(X^*)\beta(Y)-Y\omega_n(X^*)=0\,.$$

Since $\omega_n(X^*)=0$ and since $\omega_1,\cdots,\omega_n,\bar{\omega}_1,\cdots,\bar{\omega}_{n-1}$ are linearly independent, we have $\sum_{j=1}^{n-1}L_{ji}\omega_i(X^*)=0$ $(1\leqq i\leqq n-1)$. Therefore we get $\omega_i(X^*)=0$ $(1\leqq i\leqq n-1)$, because (L_{ij}) is non-degenerate, proving our assertion. We have thereby proved that $\omega_i(X^*)=0$ $(1\leqq i\leqq n)$ and hence $X=0$, proving Lemma 6.

III. Quadrics

11. Let $P^n(\boldsymbol{C})$ be the n-dimensional complex projective space and let $z_0,\cdots,z_n$ be the system of homogeneous coordinates of it. If we identify a point $(z_1,\cdots,z_n)$ of $\boldsymbol{C}^n$ with the point $(1, z_1,\cdots,z_n)$ of $P^n(\boldsymbol{C})$, we may identify $\boldsymbol{C}^n$ with an open submanifold of $P^n(\boldsymbol{C})$. A projective transformation of $P^n(\boldsymbol{C})$ is a transformation $(z_0,\cdots,z_n)\rightarrow(z_0',\cdots,z_n')$ on $P^n(\boldsymbol{C})$ such that $z_j'=\sum_{j=0}^{n}a_{ij}z_j$ $(0\leqq i\leqq n)$,

8) $\mathcal{L}_{Z^*}\omega_i$ means the Lie derivative of ω_i with respect to the infinitesimal transformation Z^*.

where $\det(a_{ij})\neq 0$. The group $P(n, \boldsymbol{C})$ of all the projective transformations of $P^n(\boldsymbol{C})$ may be represented as a factor group $GL(n+1, \boldsymbol{C})/\boldsymbol{C}^*$, where $\boldsymbol{C}^*$ is the center of $GL(n+1, \boldsymbol{C})$ identified with the multiplicative group of all the non-zero complex numbers. The affine transformation group $A(n, \boldsymbol{C})$ may be identified with the subgroup of $P(n, \boldsymbol{C})$ composed of all the projective transformations leaving invariant the subset $\boldsymbol{C}^n$.

Hereafter we shall always consider a fixed integer r with $0\leqq r\leqq\left[\frac{n-1}{2}\right]$.

Let S_r be the quadric of $\boldsymbol{C}^n$ defined by

$$-\sum_{i=1}^{r} z_i\bar{z}_i+\sum_{i=r+1}^{n} z_i\bar{z}_i=1. \tag{11.1}$$

The closure $\bar{S}_r$ of S_r in $P^n(\boldsymbol{C})$, being again a quadric of $P^n(\boldsymbol{C})$, is projectively equivalent to the quadric Q_r of $P^n(\boldsymbol{C})$ defined by

$$-\sqrt{-1}\, z_0\bar{z}_n+\sqrt{-1}\, z_n\bar{z}_0-\sum_{i=1}^{r} z_i\bar{z}_i+\sum_{i=r+1}^{n-1} z_i\bar{z}_i=0. \tag{11.2}$$

One remarks that "projectively equivalent" implies "pseudo-conformally equivalent".

We shall now explain the notations and identifications which are needed for our later considerations.

We denote by G the subgroup of $P(n, \boldsymbol{C})$ composed of all the projective transformations leaving invariant the quadric Q_r. Define hermitian matrices I and $\tilde{I}$ of degree $n-1$ and $n+1$ respectively by

$$I=\begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_{n-1} \end{pmatrix} \quad \text{and} \quad \tilde{I}=\begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & I & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}$$

where $\varepsilon_i=-1$ if $1\leqq i\leqq r$ and $=1$ otherwise, and denote by G_1 the subgroup of $GL(n+1, \boldsymbol{C})$ consisting of all the matrices σ such that ${}^t\bar{\sigma}\cdot\tilde{I}\cdot\sigma=\varepsilon\cdot\tilde{I}$ with an $\varepsilon=\pm 1$. Then the group G may be represented as a factor group $G_1/U(1)$. It follows in particular that the Lie algebra $\mathfrak{g}$ of G may be defined as follows: As a vector space, $\mathfrak{g}$ is identical with the vector space of all the matrices A of degree $n+1$ of the form

$$\begin{pmatrix} -u & -\sqrt{-1}\,\varepsilon_j\bar{w}_j & w_n \\ \xi_i & v_{ij} & w_i \\ \xi_n & \sqrt{-1}\,\varepsilon_j\bar{\xi}_j & u \end{pmatrix} \qquad (1\leqq i, j\leqq n-1), \tag{11.3}$$

where ${}^t\bar{v}\cdot I+I\cdot v=0$ $(v=(v_{ij}))$ and u, ξ_n, w_n are real; The bracket operation of $\mathfrak{g}$ is defined by the formula: $[A, B]=A\cdot B-B\cdot A-c\cdot I_{n+1}$ for all $A=(a_{ij})$, $B=(b_{ij})\in\mathfrak{g}$, where I_{n+1} is the unit matrix of degree $n+1$ and $c=\sqrt{-1}$

$\cdot \mathcal{J}(\sum_{k=1}^{n}(a_{ok}b_{ko}-b_{ok}a_{ko}))$.

As is easily seen, the group G operates effectively and transitively on Q_r; Hence it may be considered as a transitive transformation group on Q_r. Remarking that the origin o of $\boldsymbol{C}^n$ belongs to Q_r, we denote by G' the isotropy group of G at o. Since G is represented as $G_1/U(1)$, the group G' may be identified with a subgroup of G_1: More precisely, G' is a subgroup of G_1 consisting of all the matrices σ of the form:

$$\sigma=\tau\cdot\exp A, \tag{11.4}$$

where exp denotes the usual exponential mapping, and τ and A are respectively given by

$$\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b_{ij} & 0 \\ 0 & 0 & \varepsilon a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -\sqrt{-1}\,\varepsilon_j\bar{c}_j & c_n \\ 0 & 0 & c_i \\ 0 & 0 & 0 \end{pmatrix} \qquad (1 \leqq i,\ j \leqq n-1),$$

where ${}^t\bar{b}\cdot I\cdot b=\varepsilon\cdot I$, $\varepsilon^2=1$, $a>0$ and c_n is real. The notation being as in (11.3), the Lie algebra $\mathfrak{g}'$ of G' consists of all the matrices A such that $\xi_i=0$ $(1\leqq i\leqq n)$.

For each element $\xi=\sum_{i=1}^{n}\xi_i e_i$ of $\mathfrak{m}$, we define an element $\tilde{\xi}$ of $\mathfrak{g}$ as

$$\begin{pmatrix} 0 & 0 & 0 \\ \xi_i & 0 & 0 \\ \xi_n & \sqrt{-1}\,\varepsilon_j\bar{\xi}_j & 0 \end{pmatrix}. \tag{11.5}$$

The totality $\tilde{\mathfrak{m}}$ of all the elements $\tilde{\xi}$ with $\xi\in\mathfrak{m}$ forms a subalgebra of $\mathfrak{g}$. We have $\mathfrak{g}=\tilde{\mathfrak{m}}+\mathfrak{g}'$.

Let π be the projection of G onto the homogeneous space $G/G'=Q_r$. We have $d\pi\cdot\tilde{\xi}_e=\xi$ for all $\xi\in\mathfrak{m}$, where e is the identity element of G and where $\tilde{\xi}$ should be considered as a left invariant vector field on G. If follows that the tangent space $T_o(Q_r)$ coincides with the subspace $\mathfrak{m}$ of $T_o(\boldsymbol{C}^n)$.

Let σ be an element of G', i. e. $\sigma\in G$ and $\sigma\cdot o=o$. As a complex analytic transformation on $P^n(\boldsymbol{C})$, σ induces a complex automorphism $\xi\rightarrow\sigma\cdot\xi$ of $T_o(\boldsymbol{C}^n)$. Hence, there exists a unique element $l(\sigma)$ of $GL(n,\boldsymbol{C})$ such that $l(\sigma)\cdot\xi=\sigma\cdot\xi$ for all $\xi\in T_o(\boldsymbol{C}^n)$. Since $T_o(Q_r)=\mathfrak{m}$, $l(\sigma)$ is contained in the group H introduced in 3. A second characterization of $l(\sigma)$ is given by $\operatorname{Ad}\sigma\cdot\tilde{\xi}\equiv\widetilde{l(\sigma)\xi} \mod \mathfrak{g}'$ for all $\xi\in\mathfrak{m}$, from which we get an explicit expression of $l(\sigma)$: If σ is expressed as (11.4), then

$$l(\sigma)=\begin{pmatrix} ab_{ij} & 0 \\ 0 & \varepsilon a^2 \end{pmatrix}\begin{pmatrix} \delta_{ij} & c_i \\ 0 & 1 \end{pmatrix}. \tag{11.6}$$

It is clear that the mapping $\sigma\rightarrow l(\sigma)$ gives a homomorphism of G' into H. We denote by $\tilde{G}$ the image of G' by the homomorphism l, which is nothing but the linear isotropy group of G at o.

We denote by G_u the subgroup of $\tilde{G}$ composed of all the matrices σ of the form

$$\begin{pmatrix} b_{ij} & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{11.7}$$

where ${}^t\bar{b}\cdot I\cdot b=\varepsilon\cdot I$ and $\varepsilon^2=1$. Let us now define an injective homomorphism h of G_u into G' as follows: If σ is an element of G_u expressed as (11.7), then $h(\sigma)$ is defined as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{ij} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

We have clearly $l\circ h(\sigma)=\sigma$ for all $\sigma\in G_u$. Finally to each element σ of $\tilde{G}$ expressed as (11.6), we associate the element $s(\sigma)$ of G_u expressed as (11.7). Thus we get a homomorphism s of $\tilde{G}$ onto G_u.

12. We set $Q_r^*=Q_r\cap \boldsymbol{C}^n$, which turns out to be the hypersurface (hyperconic) defined by

$$y_n=\frac{1}{2}\sum_{i=1}^{n-1}\varepsilon_i(x_i^2+y_i^2).$$

From (9.2), we see that the hypersurface Q_r^* is non-degenerate[9] and of index r. Moreover, it is obviously regular. Denote by $P(Q_r^*, G')$ the restriction to Q_r^* of the principal fiber bundle $G(Q_r, G')$. Take an element z of P and set $p=\pi(z)$. As before, z induces a complex isomorphism $\xi\to z\cdot\xi$ of $T_o(\boldsymbol{C}^n)$ with $T_p(\boldsymbol{C}^n)$. Therefore we can find a unique element $\bar{l}(z)$ of $A(n,\boldsymbol{C})$ such that $\bar{l}(z)\cdot\xi=z\cdot\xi$ for all $\xi\in T_o(\boldsymbol{C}^n)$. Since $z\cdot\mathfrak{m}=T_p(Q_r^*)$, $\bar{l}(z)$ is in the subset P^*, the pseudo-conformal H-bundle associated to Q_r^*. Now let σ be an element of G'. Then we have easily $\bar{l}(z\cdot\sigma)=\bar{l}(z)\cdot l(\sigma)$, which implies that the mapping $z\to\bar{l}(z)$ defines a homomorphism of $P(Q_r^*, G')$ into $P^*(Q_r^*, H)$ corresponding to the homomorphism l of G' onto $\tilde{G}\subset H$.

The Maurer-Cartan form ω of G may be represented as a matrix of the form

$$\begin{pmatrix} -\alpha & * & * \\ \theta_i & * & * \\ \theta_n & \sqrt{-1}\,\varepsilon_j\bar{\theta}_j & \alpha \end{pmatrix}.$$

We shall denote by the same symbols α, θ_i, θ_n the restrictions to P of the components α, θ_i, θ_n respectively.

PROPOSITION 6. *We use the above notations. Let ω_i be the basic forms of*

9) A hypersurface S is called non-degenerate (resp. of index r, resp. regular) if it is non-degenerate (resp. of index r, resp. regular) everywhere.

P^* and let L_{ij} be the functions on P^* defined in Lemma 3. *Then, we have $\theta_i = \bar{l}^*\omega_i$ $(1 \leqq i \leqq n)$, and the image of P by the homomorphism $\bar{l}$ consists of all the points x of P^* such that $L_{ij}(x) = \sqrt{-1}\,\varepsilon_i\delta_{ij}$ $(1 \leqq i,\, j \leqq n-1)$.*

PROOF. We have $d\pi \cdot \tilde{\xi}_e = \xi = \sum_{i=1}^{n} \xi_i e_i = \sum_{i=1}^{n} \theta_i(\tilde{\xi}_e) e_i$ for all $\xi \in \mathfrak{m}$. It follows in general that, for all $z \in P$ and $X \in T_z(P)$, $d\pi \cdot X = z \cdot (\sum_{i=1}^{n} \theta_i(X) e_i) = \bar{l}(z) \cdot (\sum_{i=1}^{n} \theta_i(X) e_i)$. On the other hand, we have $d\pi \cdot X = d\varpi \circ \bar{l} \cdot X = d\varpi \cdot d\bar{l} \cdot X = \bar{l}(z) \cdot (\sum_{i=1}^{n} \omega_i(d\bar{l} \cdot X) e_i)$. Therefore we must have $\theta_i = \bar{l}^*\omega_i$ $(1 \leqq i \leqq n)$.

Now denote by P_u the totality of all the matrices σ of the form: $\exp \tilde{\xi} \cdot \tau$, where $\xi \in \mathfrak{m}$ and $\tau \in G_u$. P_u becomes a subgroup at the same time of $A(n, \boldsymbol{C})$ and G'. As for the group, we have the following statements: (1) to (3). (1) P_u acts transitively on Q_r^*. It follows that P_u is a principal fiber bundle over the base space Q_r^* with structure group G_u, which is a subbundle of $P^*(Q_r^*, H)$; (2) The injection $\bar{h}$ of P_u into P^* defines a homomorphism of $P_u(Q_r^*, G_u)$ into $P(Q_r^*, G')$ corresponding to the homomorphism h of G_u into G'. We have clearly $\bar{l} \circ \bar{h}(x) = x$ for all $x \in P_u$; (3) The Lie algebra of the group P_u consists of all the matrices A of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ \xi_i & v_{ij} & 0 \\ \xi_n & \sqrt{-1}\,\varepsilon_j \bar{\xi}_j & 0 \end{pmatrix},$$

where ${}^t\bar{v} \cdot I + I \cdot v = 0$ and ξ_n is real. Let us now complete the proof of Proposition 6. From (3), we see that $\omega_{ni} = \sqrt{-1}\,\varepsilon_i \cdot \bar{\omega}_i$ on P_u $(1 \leqq i \leqq n-1)$, implying that $L_{ij} = \sqrt{-1}\,\varepsilon_i\delta_{ij}$ on P_u $(1 \leqq i,\, j \leqq n-1)$. But by (2) and (3), we have $\bar{l}(P) = \bar{l}(P_u \cdot G') = P_u \cdot \tilde{G}$. Therefore we conclude from Lemma 4 and (11.6) that $\bar{l}(P)$ consists of all the points x of P^* such that $L_{ij}(x) = \sqrt{-1}\,\delta_{ij}$ $(1 \leqq i,\, j \leqq n-1)$.

PROPOSITION 7. *The notations being as above, we have:*

1) *If σ is an element of G' expressed as* (11.4), *then*

$$R_\sigma^*\alpha = \alpha - \varepsilon a^{-1}\mathcal{R}(\sqrt{-1}\sum_{i,j=1}^{n-1} \varepsilon_i \overline{b_{ij} c_j \theta_i}) + \varepsilon a^{-2} c_n \theta_n;$$

2) $$d\theta_n + \sqrt{-1}\sum_{i=1}^{n-1} \varepsilon_i \bar{\theta}_i \wedge \theta_i + 2\alpha \wedge \theta_n = 0.$$

PROOF. 1) follows from the equalities: $R_\sigma^*\omega = \operatorname{Ad}\sigma^{-1} \cdot \omega$ for all $\sigma \in G'$, and 2) from the equality: $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

IV. Pseudo-conformal $\tilde{G}$-bundles

13. In the following, by a non-degenerate hypersurface we always mean that of index r, for the rest, we use the notations in the previous chapters.

Let S be a non-degenerate hypersurface. We denote by $\tilde{P}$ the subset of P^* consisting of all the points x such that $L(x)=\sqrt{-1}\cdot I$. We see from Lemma 4 and Definition 2 that the mapping $x\rightarrow\varpi(x)$ maps $\tilde{P}$ onto S and from Lemma 4 and (11.6) that if $x\in\tilde{P}$ and $\sigma\in H$, then the condition $x\cdot\sigma\in\tilde{P}$ is equivalent to the condition $\sigma\in\tilde{G}$. Therefore $\tilde{P}$ becomes a principal fiber bundle over the base space S with $\tilde{G}$ as structure group, which is a subbundle of $P^*(S, H)$. The restrictions $\tilde{\theta}_i$ to $\tilde{P}$ of the basic forms ω_i of P^* will be called the *basic forms* of $\tilde{P}(S,\tilde{G})$.

PROPOSITION 8. *Let S (resp. S') be a non-degenerate hypersurface, and let $\tilde{P}(S,\tilde{G})$ (resp. $\tilde{P}'(S',\tilde{G})$) be the corresponding $\tilde{G}$-bundle. Let $\tilde{\theta}_i$ (resp. $\tilde{\theta}_i'$) be the basic forms of $\tilde{P}$ (resp. $\tilde{P}'$). If f is a pseudo-conformal homeomorphism of S with S', there corresponds to f a unique isomorphism φ of $\tilde{P}(S,\tilde{G})$ with $\tilde{P}'(S',\tilde{G})$ such that φ induces the given f and such that $\varphi^*\tilde{\theta}_i'=\tilde{\theta}_i$ $(1\leqq i\leqq n)$. Conversely, every isomorphism φ of $\tilde{P}(S,\tilde{G})$ with $\tilde{P}'(S',\tilde{G})$ satisfying this last condition induces a pseudo-conformal homeomorphism f of S with S'.*

This is clear from Theorem 1 and Lemma 5. Taking account of Proposition 8, we shall say that the principal fiber bundle $\tilde{P}(S,\tilde{G})$ defined above is the *pseudo-conformal $\tilde{G}$-bundle associated to the hypersurface S.*

14. Now let us identify the general linear group $GL(n-1,\boldsymbol{C})$ with a subgroup of H and set $G_u^*=G_u\cap GL(n-1,\boldsymbol{C})$. The Lie algebra of G_u^* coincides with the Lie algebra $\mathfrak{g}_u$ of G_u.

Let S be a non-degenerate hypersurface and let Z be an infinitesimal pseudo-conformal transformation on S such that $\sqrt{-1}\cdot Z_p\notin T_p(S)$ at each point p of S. Denote by $P_u(Z)$ the subset of $\tilde{P}$ consisting of all the points x such that $x\cdot e_n=\varepsilon\cdot Z_p$ with an $\varepsilon=\pm 1$. We assert that $P_u(Z)$ is a principal fiber bundle over the base space S with structure group G_u, which is a subbundle of $\tilde{P}(S,\tilde{G})$. Indeed let p be a point of S and let x be a point of $\tilde{P}$ lying over the point p. Since $\sqrt{-1}\cdot Z_p\notin T_p(S)$, $x^{-1}\cdot Z_p$ does not belong to the maximum complex subspace of $\mathfrak{m}$, i.e. the complex subspace of $T_o(\boldsymbol{C}^n)$ spanned by the $n-1$ vectors $e_1,\cdots,e_{n-1}$; Hence $x^{-1}\cdot Z_p$ can be written in the form: $\varepsilon(\sum_{i=1}^{n-1}a\cdot w_i\cdot e_i+a^2\cdot e_n)$, where $a>0$ and $\varepsilon^2=1$. Therefore we can find an element σ of $\tilde{G}$ such that $x\cdot\sigma\in P_u(Z)$, showing that the mapping $x\rightarrow\varpi(x)$ maps $P_u(Z)$ onto S. Moreover, it is easy to see that if $x\in P_u(Z)$ and $\sigma\in\tilde{G}$, then the condition $x\cdot\sigma\in P_u(Z)$ is equivalent to the condition $\sigma\in G_u$, proving our assertion.

THEOREM 2. *Let S be a non-degenerate hypersurface and let Z be an infinitesimal pseudo-conformal transformation on S such that $\sqrt{-1}\cdot Z_p\notin T_p(S)$ at each point p of S. Then, the subbundle $P_u(Z)(S,G_u)$ of $\tilde{P}(S,\tilde{G})$ satisfies the following conditions:*

1) $d\tilde{\theta}_n+\sqrt{-1}\cdot\sum_{i=1}^{n-1}\varepsilon_i\bar{\tilde{\theta}}_i\wedge\tilde{\theta}_i=0$ *on* $P_u(Z)$;

2) *There exists a connection* (χ_{ij}) *in* $P_u(Z)$ *satisfying the condition:*
$d\tilde{\theta}_i+\sum_{j=1}^{n-1}\chi_{ij}\wedge\tilde{\theta}_j=0$ $(1\leq i\leq n-1)$ *on* $P_u(Z)$.

PROOF. First we deal with the special case where $Z=\partial/\partial z_n$ on S and where S is globally defined by an equation of the form:

$$y_n=f(x_1,\cdots,x_{n-1};y_1,\cdots,y_{n-1}).$$

We assume that the function f is defined on an open set D' of C^{n-1}. Regarding f as a function on $D=D'\times R$, we now apply the argument in 9 to the hypersurface S. Consider the cross-section g of $P^*(S,H)$. In this case, (9.1) (i.e. the matrix $g(p)$) reduces to the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ z_i & \delta_{ij} & 0 \\ t+\sqrt{-1}f & 2\sqrt{-1}\dfrac{\partial f}{\partial z_j} & 1 \end{pmatrix}, \tag{14.1}$$

and (9.2) to the equalities:

$$L_{ji}(g(p))=2\sqrt{-1}\frac{\partial^2 f}{\partial z_i\partial\bar{z}_j}.$$

Denote by Φ the matrix of degree $n-1$ whose (i,j)-component is given by $\dfrac{\partial^2 f}{\partial z_j\partial\bar{z}_i}$. Since S is non-degenerate, we see that the matrix Φ is non-degenerate at each point of D' and hence it defines a (definite or indefinite) Kählerian metric on D'.

As auxiliary tools, we shall introduce two principal fiber bundles $'P_u^*(D',G_u^*)$ and $P_u^*(S,G_u^*)$.

Fiber bundle $'P_u^*(D',G_u^*)$. Let $'P_u^*$ be the subset of the affine transformation group $A(n-1,C)$ of degree $n-1$ consisting of all the matrices x of the form:

$$\begin{pmatrix} 1 & 0 \\ z_i & b_{ij} \end{pmatrix},$$

where $z'=(z_1,\cdots,z_{n-1})\in D'$ and $2\cdot{}^t\bar{b}\cdot\Phi(z')\cdot b=I$. As usual, $'P_u^*$ becomes a principal fiber bundle over the base space D' with structure group G_u^*, which is a subbundle of $A(n-1,C)\,|\,D'$. The principal fiber bundle $'P_u^*(D',G_u^*)$ is nothing but the hermitian bundle associated to the Kählerian metric Φ on D'. As is well known, there exists a connection (χ'_{ij}) in $'P_u^*$ whose torsion is zero, the so-called Kählerian connection associated to the Kählerian metric Φ. If we denote by ω'_i the restrictions to $'P_u^*$ of the basic forms of $A(n-1,C)$, then we have

$$d\omega_i' + \sum_{j=1}^{n-1} \chi_{ij}' \wedge \omega_j' = 0 \qquad (1 \leqq i \leqq n-1), \tag{14.2}$$

which means that the connection has no torsion.

Fiber bundle $P_u^*(S, G_u^*)$. Let P_u^* be the subset of P^* of all the elements x of the form: $g(p) \cdot b$, where $p \in S$, $b \in GL(n-1, \boldsymbol{C})$ and $2 \cdot {}^t\bar{b} \cdot \Phi(\rho(p)) \cdot b = I$ $(\rho(p) = (z_1(p), \cdots, z_{n-1}(p))$. Exactly as in the case of $'P_u^*(D', G_u^*)$, P_u^* becomes a principal fiber bundle over the base space S with G_u^* as structure group, which is a subbundle of $P^*(S, H)$. Now let us study the properties of the principal fiber bundle $P_u^*(S, G_u^*)$. (1) By Lemma 4, we have $L = \sqrt{-1} \cdot I$ on P_u^* and by (14.1), we have $x \cdot e_n = (\partial/\partial z_n)_p = Z_p$ for each point x of P_u^* $(p = \varpi(x))$. It follows that $P_u^*(S, G_u^*)$ is a subbundle of $P_u(Z)(S, G_u)$. (2) By using (14.1) and (2.2), we can easily verify that $\omega_{nn} = 0$ and $M_i = 0$ $(1 \leqq i \leqq n-1)$ on P_u^*. But we have $d\omega_n + \sqrt{-1} \cdot \sum_{i,j=1}^{n-1} L_{ji}\bar{\omega}_j \wedge \omega_i + (\omega_{nn} - \sum_{i=1}^{n-1} M_i\omega_i) \wedge \omega_n = 0$ and therefore we get

$$d\omega_n + \sqrt{-1} \cdot \sum_{i=1}^{n-1} \varepsilon_i \bar{\omega}_i \wedge \omega_i = 0 \quad \text{on} \quad P_u^*. \tag{14.3}$$

(3) To each element $x = g(p) \cdot b$ of P_u^* we associate a matrix $\tilde{\rho}(x)$ of the form

$$\begin{pmatrix} 1 & 0 \\ z_i(p) & b_{ij} \end{pmatrix}.$$

Thus we get a mapping $\tilde{\rho}$ of P_u^* onto $'P_u^*$. We have clearly $\tilde{\rho}(x \cdot \sigma) = \tilde{\rho}(x) \cdot \sigma$ for all $x \in P_u^*$ and $\sigma \in G_u^*$; Furthermore, $\tilde{\rho}$ induces the mapping $p \to \rho(p)$ of S onto D'. Setting $\chi_{ij} = \tilde{\rho}^* \chi_{ij}'$, we see that the matrix (χ_{ij}) defines a connection in $P_u^*(S, G_u^*)$. (4) Finally we have $\tilde{\rho}^*\omega_i' = \omega_i$ on P_u^* and hence we get, from (14.2),

$$d\omega_i + \sum_{j=1}^{n-1} \chi_{ij} \wedge \omega_j = 0 \qquad (1 \leqq i \leqq n-1) \quad \text{on} \quad P_u^*. \tag{14.4}$$

We are now in a position to prove Theorem 2 in our special case. As we have already seen, $P_u^*(S, G_u^*)$ is a subbundle of $P_u(Z)(S, G_u)$. First, we see from (14.3) and 1) and 2) of (3.2) that $P_u(Z)$ satisfies the condition 1) in Theorem 2. Next, the connection (χ_{ij}) gives rise to a connection, denoted by the same symbol (χ_{ij}), in $P_u(Z)$. That the connection (χ_{ij}) in $P_u(Z)$ satisfies the condition 2) in Theorem 2 follows easily from (14.4) and 1) and 2) of (3.2).

Now let us return to the general case. Let p be a point of S. There exists a system of complex coordinates $w_1, \cdots, w_n$ of $\boldsymbol{C}^n$ at p such that $Z = \partial/\partial w_n$ on a neighborhood of p in S. From the proof of Proposition 4, we see that a sufficiently small neighborhood S' of p in S is defined by an equation of the form $v_n = f(u_1, \cdots, u_{n-1}; v_1, \cdots, v_{n-1})$. Let S'' be the hypersurface defined by $y_n = f(x_1, \cdots, x_{n-1}; y_1, \cdots, y_{n-1})$, which is clearly a hypersurface of the type considered above. There is a pseudo-conformal homeomorphism Φ of S'' with S' such that $d\Phi \cdot (\partial/\partial z_n)_q = (\partial/\partial w_n)_{\Phi(q)} = Z_{\Phi(q)}$ at each $q \in S''$. Therefore we know from Proposition 8 and the above argument that the restriction $P_u(Z)\mid S'$ to S' of $P_u(Z)$ satisfies the conditions in Theorem 2. However as in the case of the Riemannian connection, a connection (χ_{ij}) in $P_u(Z)\mid S'$ which satisfies the condition " $d\omega_i+\sum_{j=1}^{n-1}\chi_{ij}\wedge\omega_j=0$ on $P_u(Z)\mid S'$ " is uniquely determined. It follows that the whole principal fiber bundle $P_u(Z)(S, G_u)$ satisfies the conditions in Theorem 2. We have thereby completed the proof of Theorem 2.

The following proposition is a converse of Theorem 2.

PROPOSITION 9. *Let S be a non-degenerate hypersurface and let $P_u^*(S, G_u^*)$ be a subbundle of $\tilde{P}(S, \tilde{G})$ satisfying the conditions in Theorem 2 (where $P_u(Z)$ should be replaced by P_u^*). Then there exists an infinitesimal pseudo-conformal transformation Z on S such that $\sqrt{-1}\cdot Z_p \notin T_p(S)$ at each point p of S and such that the given $P_u^*(S, G_u^*)$ is a subbundle of $P_u(Z)(S, G_u)$.*

PROOF. Define an infinitesimal transformation Z^* on P_u^* by $\tilde{\theta}_i(Z_x^*)=0$ $(1\leqq i\leqq n-1)$, $\tilde{\theta}_n(Z_x^*)=1$ and $\chi_{ij}(Z_x^*)=0$ $(1\leqq i, j\leqq n-1)$ for each point x of P_u^*. As is easily seen, the vector field Z^* is invariant under the right translations on P_u^*. Denoting by $\tilde{\theta}_i'$ the restriction to P_u^* of $\tilde{\theta}_i$, we assert that $\mathcal{L}_{Z^*}\tilde{\theta}_i'=0$ $(1\leqq i\leqq n)$. Indeed, let Y be any vector field on P_u^*. Then we have, from 1) in Theorem 2, $\mathcal{L}_{Z^*}\tilde{\theta}_n'(Y)=d\tilde{\theta}_n'(Z^*, Y)+Y\tilde{\theta}_n'(Z^*)=0$. Analogously we have, from 2) in Theorem 2, $\mathcal{L}_{Z^*}\tilde{\theta}_i'(Y)=0$, proving our assertion. Now let Z be the infinitesimal transformation on S induced by Z^*. From Theorem 1, we see that Z is a pseudo-conformal transformation on S. Finally take a point x of P_u^* and set $p=\varpi(x)$. We have $Z_p=d\varpi\cdot Z_x^*=x\cdot(\sum_{i=1}^{n}\tilde{\theta}_i(Z_x^*)e_i)=x\cdot e_n$, meaning that x is in $P_u(Z)$ and hence $P_u^*(S, G_u^*)$ is a subbundle of $P_u(Z)(S, G_u)$.

15. Apart from the hypersurfaces, we now give the following

DEFINITION 4. Let $\tilde{P}$ be a principal fiber bundle over a base space M with structure group $\tilde{G}$. We shall say that $\tilde{P}(M, \tilde{G})$ is a pseudo-conformal $\tilde{G}$-bundle, if $\dim M=2n-1$ and if it is a subbundle of the bundle of frames of M.

Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle and let $\tilde{\pi}$ be the projection of $\tilde{P}$ onto M. Since $\tilde{P}(M, \tilde{G})$ is a subbundle of the bundle of frames of M, every element x of $\tilde{P}$ yields an isomorphism, denoted by $\xi\rightarrow x\cdot\xi$, of $\mathfrak{m}$ with $T_p(M)$ where $p=\tilde{\pi}(x)$. By making use of these isomorphisms, we now define n complex valued forms $\tilde{\theta}_1, \cdots, \tilde{\theta}_n$ on $\tilde{P}$, called the *basic forms* of $\tilde{P}$, by the formula: $\sum_{i=1}^{n}\tilde{\theta}_i(X)e_i=x^{-1}\cdot d\tilde{\pi}\cdot X$ for all $x\in\tilde{P}$ and $X\in T_x(\tilde{P})$. It is clear that $\tilde{\theta}_n$ is a real form and the $2n-1$ forms $\tilde{\theta}_1, \cdots, \tilde{\theta}_n, \bar{\tilde{\theta}}_1, \cdots, \bar{\tilde{\theta}}_{n-1}$ are linearly independent at each point of $\tilde{P}$.

DEFINITION 5. Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle and let $\tilde{\theta}_i$ be the basic forms of $\tilde{P}$. We shall say that $\tilde{P}(M, \tilde{G})$ satisfies condition (C), if there exists a subbundle $P_u(M, G_u)$ of $\tilde{P}(M, \tilde{G})$ satisfying the conditions in Theorem 2 (where $P_u(Z)$ should be replaced by P_u). Furthermore, we shall say that $\tilde{P}(M, \tilde{G})$ satisfies condition (LC), if, for each point p of M, there exists a neighborhood U of p such that the restriction to U of $\tilde{P}(M, \tilde{G})$ satisfies condition (C).

Definitions 3 and 5 and Theorem 2 lead us to the following proposition.

PROPOSITION 10. *Let S be a regular non-degenerate hypersurface. Then, the pseudo-conformal* $\tilde{G}$*-bundle associated to S satisfies condition* (LC).

V. Pseudo-conformal G'-bundles

16. Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundles satisfying condition (C) and let $\tilde{\theta}_i$ be the basic forms of $\tilde{P}(M, \tilde{G})$. Let $P_u(M, G_u)$ be a subbundle of $\tilde{P}(M, \tilde{G})$ satisfying the conditions in Theorem 2.

By making use of $P_u(M, G_u)$, we now construct a principal fiber bundle and three bundle homomorphisms. First, we define a homomorphism $\bar{s}$ of $\tilde{P}(M, \tilde{G})$ onto $P_u(M, G_u)$ corresponding to the homomorphism s of $\tilde{G}$ onto G_u by the requirement that $\bar{s}(x) = x$ for all $x \in P_u$. Next, the principal fiber bundle $P_u(M, G_u)$ and the injective homomorphism h of G_u into G' give rise to a principal fiber bundle $P(M, G')$ and an injective homomorphism $\bar{h}$ of $P_u(M, G_u)$ into $P(M, G')$. Finally, we define a homomorphism $\bar{l}$ of $P(M, G')$ onto $\tilde{P}(M, \tilde{G})$ corresponding to the homomorphism l of G' onto $\tilde{G}$ in such a way that $\bar{l} \circ \bar{h}(x) = x$ for all $x \in P_u$.

Since, for each point z of P, z and $\bar{h} \circ \bar{s} \circ \bar{l}(z)$ lie in the same fiber of P, we can take a unique mapping k of P into G' satisfying the condition

$$z = \bar{h} \circ \bar{s} \circ \bar{l}(z) \cdot k(z) \tag{16.1}$$

for all $z \in P$. We have $s \circ l(k(z)) = e$; Hence $k(z)$ can be written in the form: $\tilde{v}(z) \cdot \exp \tilde{w}(z)$, where $\tilde{v}(z)$ and $\tilde{w}(z)$ are respectively given by

$$\begin{pmatrix} v(z)^{-1} & 0 & 0 \\ 0 & \delta_{ij} & 0 \\ 0 & 0 & v(z) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\sqrt{-1}\ \varepsilon_j \overline{w_j(z)} & w_n(z) \\ 0 & 0 & w_i(z) \\ 0 & 0 & 0 \end{pmatrix}.$$

LEMMA 7. *Set* $\theta_i = \bar{l}^* \tilde{\theta}_i$ $(1 \leqq i \leqq n)$. *Then, we have*

1) $R_\sigma^*(\theta_i) = l(\sigma)^{-1} \cdot (\theta_i)$, $\sigma \in G'$;

2) $\theta_i(A^*) = 0$ $(1 \leqq i \leqq n)$, $A \in \mathfrak{g}'$.

LEMMA 8. *Set* $\eta_i = (\bar{s} \circ \bar{l})^* \tilde{\theta}_i$ $(1 \leqq i \leqq n)$. *Then, we have*

$$\eta_i = v(\theta_i + w_i \theta_n) \quad (1 \leqq i \leqq n-1)$$

$$\eta_n = v^2 \theta_n$$

$$d\eta_n + \sqrt{-1} \cdot \sum_{i=1}^{n-1} \varepsilon_i \bar{\eta}_i \wedge \eta_i = 0.$$

Lemma 7 follows immediately from the analogous equalities for $\tilde{\theta}_i$: 1) $R_\sigma^*(\tilde{\theta}_i)=\sigma^{-1}\cdot(\tilde{\theta}_i)$ for all $\sigma\in\tilde{G}$; 2) $\tilde{\theta}_i(A^*)=0$ $(1\leqq i\leqq n)$ for any element A of the Lie algebra of $\tilde{G}$. Lemma 8 is a consequence of Lemma 7 and the condition 1) in Theorem 2.

LEMMA 9. *Let z be an element of P and let σ be an element of G' expressed as τ or* $\exp A$ *in* (11.4).

1) *If* $\sigma=\tau$,

$$v(z\cdot\sigma)=av(z)$$
$$(w_i(z\cdot\sigma))=\begin{pmatrix}\varepsilon ab^{-1} & 0\\ 0 & \varepsilon a^2\end{pmatrix}(w_i(z))$$

2) *If* $\sigma=\exp A$,

$$v(z\cdot\sigma)=v(z);$$
$$w_i(z\cdot\sigma)=w_i(z)+c_i \qquad (1\leqq i\leqq n-1);$$
$$w_n(z\cdot\sigma)=w_n(z)+c_n+\mathcal{R}(\sqrt{-1}\cdot\sum_{i=1}^{n-1}\varepsilon_i\bar{c}_iw_i(z)).$$

PROPOSITION 11. *Let $\tilde{P}(M,\tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (C) and let $\tilde{\theta}_i$ be the basic forms of $\tilde{P}$. Then, there exists a collection $(P,\bar{l},\alpha)$ satisfying the following conditions: P is a principal fiber bundle over the base space M with structure group G'; $\bar{l}$ is a homomorphism of $P(M,G')$ onto $\tilde{P}(M,\tilde{G})$ corresponding to the homomorphism l of G' onto $\tilde{G}$; α is a real valued 1-form on P having the following properties: We set $\theta_i=\bar{l}^*\tilde{\theta}_i$ $(1\leqq i\leqq n)$.*

1) *If σ is an element of G' expressed as* (11.4), *then*

$$R_\sigma^*\alpha=\alpha-\varepsilon a^{-1}\mathcal{R}(\sqrt{-1}\cdot\sum_{i,j=1}^{n-1}\varepsilon_i\overline{b_{ij}c_j}\theta_i)+\varepsilon a^{-2}c_n\theta_n;$$

2) *If A is an element of $\mathfrak{g}'$ expressed as* (11.3), *then* $\alpha(A^*)=u$;

3) $d\theta_n+\sqrt{-1}\cdot\sum_{i=1}^{n-1}\varepsilon_i\bar{\theta}_i\wedge\theta_i+2\alpha\wedge\theta_n=0.$

PROOF. Construct P, $\bar{l}$, $\bar{h}$, $\bar{s}$ and k as above. We define α by

$$\alpha=v^{-1}dv-\mathcal{R}(\sqrt{-1}\cdot\sum_{i=1}^{n-1}\varepsilon_i\bar{w}_i\theta_i)+w_n\theta_n.$$

That α satisfies the conditions in Proposition 11 can be easily verified by using Lemmas 7, 8 and 9.

PROPOSITION 12. *Let $\tilde{P}(M,\tilde{G})$ (resp. $\tilde{P}'(M',\tilde{G})$) be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (C) and let $\tilde{\theta}_i$ (resp. $\tilde{\theta}'_i$) be the basic forms of $\tilde{P}$ (resp. $\tilde{P}'$). Let $(P,\bar{l},\alpha)$ (resp. $(P',\bar{l}',\alpha')$) be a collection which satisfies the conditions in Proposition* 11 *for $\tilde{P}(M,\tilde{G})$ (resp. $\tilde{P}'(M;\tilde{G})$). If $\tilde{\varphi}$ is an isomorphism of $\tilde{P}(M,\tilde{G})$ with $\tilde{P}'(M',\tilde{G})$ such that $\tilde{\varphi}^*\tilde{\theta}'_i=\tilde{\theta}_i$ $(1\leqq i\leqq n)$, there corresponds to $\tilde{\varphi}$ a unique isomorphism φ of $P(M,G')$ with $P'(M',G')$ satisfying the following conditions: 1) $\bar{l}'\circ\varphi=\tilde{\varphi}\circ\bar{l}$, 2) $\varphi^*\theta'_i=\theta_i$ $(1\leqq i\leqq n)$ and 3) $\varphi^*\alpha'=\alpha$. Conversely, if φ is an isomorphism of $P(M,G')$ with $P'(M',G')$ satisfying the condition 2) above, then there exists a unique isomorphism of $\tilde{P}(M,\tilde{G})$ with $\tilde{P}'(M',\tilde{G})$ satisfying the con-*

dition 1) *above.*

PROOF. Let $\tilde{\varphi}$ be an isomorphism of $\tilde{P}(M, \tilde{G})$ with $\tilde{P}'(M', \tilde{G})$ such that $\tilde{\varphi}^*\tilde{\theta}'_i = \tilde{\theta}_i$ $(1 \leqq i \leqq n)$.

We first prove the uniqueness of φ. Suppose that there are given two isomorphisms φ_1 and φ_2 satisfying the conditions in Proposition 12. Since, for each $z \in P$, $\varphi_1(z)$ and $\varphi_2(z)$ lie in the same fiber of P, we can take a unique mapping k of P into G' such that $\varphi_2(z) = \varphi_1(z) \cdot k(z)$ for all $z \in P$. We have $l(k(z)) = e$; Hence $k(z)$ may be expressed as

$$\text{(16.3)} \qquad \begin{pmatrix} 1 & 0 & w(z) \\ 0 & \delta_{ij} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the equalities 1) and 2) for α', it follows that $\varphi_2^*\alpha' = \varphi_1^*\alpha' + w \cdot \varphi_1^*\theta'_n$. However, we have $\varphi_1^*\alpha' = \varphi_2^*\alpha' = \alpha$ and $\varphi_1^*\theta'_n = \theta_n$ and hence $w = 0$. This means that $\varphi_1 = \varphi_2$.

Now we shall prove the existence of φ. By the uniqueness of φ, we may assume without loss of generality that both $P(M, G')$ and $P'(M', G')$ are trivial. This being said, we can find an isomorphism φ_1 of $P(M, G')$ with $P'(M', G')$ such that $\bar{l}' \circ \varphi_1 = \tilde{\varphi} \circ \bar{l}$. We have $\varphi_1^*\theta'_i = \theta_i$ $(1 \leqq i \leqq n)$ and hence the equalities 3) for α and α' yield the equality: $(\varphi_1^*\alpha' - \alpha) \wedge \theta_n = 0$; Therefore there is a unique function w on P such that $\alpha = \varphi_1^*\alpha' + w \cdot \theta_n$. Using the function w, we now define a mapping k of P into G' by (16.3). As is easily seen, k satisfies the condition: $k(z) \cdot \sigma = \sigma \cdot k(z \cdot \sigma)$ for all $z \in P$ and $\sigma \in G'$. Setting $\varphi(z) = \varphi_1(z) \cdot k(z)$, it follows that the mapping $z \to \varphi(z)$ gives an isomorphism of $P(M, G')$ with $P'(M', G')$, which will be the desired isomorphism. In fact, we have $\bar{l}' \circ \varphi = \bar{l}' \circ \varphi_1 = \tilde{\varphi} \circ \bar{l}$ and hence

$$\varphi^*\theta'_i = \theta_i \qquad (1 \leqq i \leqq n).$$

Exactly as above, we get $\varphi^*\alpha' = \varphi_1^*\alpha' + w \cdot \varphi_1^*\theta'_n$, whence $\varphi^*\alpha' = \varphi_1^*\alpha' + w \cdot \theta_n = \alpha$. We have thereby proved the first assertion in Proposition 12. The converse is almost evident.

Considering, in Proposition 12, the case where $\tilde{P}(M, \tilde{G}) = \tilde{P}'(M', \tilde{G})$ and where $\tilde{\varphi}$ is the identity trasformation of $\tilde{P}$, we know the following fact: To each pseudo-conformal $\tilde{G}$-bundle $\tilde{P}(M, \tilde{G})$ satisfying condition (C), there corresponds a "unique" collection $(P, \bar{l}, \alpha)$ satisfying the conditions in Proposition 11. Furthermore, we see from the uniqueness that this last statement is the case, even if condition (C) is replaced by condition (LC). We mention that Proposition 12 remains true under the replaced condition.

DEFINITION 6. Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (LC) and let $(P, \bar{l}, \alpha)$ be the collection satisfying the conditions in

Proposition 11. We shall say that $P(M, G')$ together with $\bar{l}$ and α is the *pseudo-conformal G'-bundle associated to the pseudo-conformal $\tilde{G}$-bundle $\tilde{P}(M, \tilde{G})$.*

The n forms $\theta_1, \cdots, \theta_n$ in Proposition 11 will be called the *basic forms* of P.

VI. Normal pseudo-conformal connection

17. We denote by $\mathfrak{m}_*$ the maximum complex subspace of $\mathfrak{m}$, i.e. the complex subspace of $T_0(\boldsymbol{C}^n)$ spanned by the $n-1$ vectors $e_1, \cdots, e_{n-1}$. Let $\xi = \sum_{i=1}^{n} \xi_i e_i$ and $\xi' = \sum_{i=1}^{n} \xi'_i e_i$ be vectors in $\mathfrak{m}$. We denote by $\langle \xi, \xi' \rangle$ (resp. $\rho(\xi, \xi')$) the real part of $\sum_{i=1}^{n} \varepsilon_i \xi_i \bar{\xi}'_i$ (resp. $2\sqrt{-1} \cdot \sum_{i=1}^{n-1} \varepsilon_i \xi_i \bar{\xi}'_i$). The symbol $\langle\ ,\ \rangle$ defines a (definite or indefinite) inner product of $\mathfrak{m}$; We have $\langle \sqrt{-1} \cdot \xi, \sqrt{-1} \cdot \xi' \rangle = \langle \xi, \xi' \rangle$ and $\langle b\xi, b\xi' \rangle = \varepsilon \cdot \langle \xi, \xi' \rangle$ for all $\xi, \xi' \in \mathfrak{m}_*$ and $\sigma \in G_u$, where σ is assumed to be expressed as (11.7). We have $[\tilde{\xi}, \tilde{\xi}'] = \rho(\xi, \xi')\tilde{e}_n$ for all $\xi, \xi' \in \mathfrak{m}$. For all $\sigma \in G'$ and $\xi \in \mathfrak{m}$, we denote by $D(\sigma, \xi)$ the $\mathfrak{g}'$-component of $\mathrm{Ad}\, \sigma \cdot \tilde{\xi}$ in the decomposition: $\mathfrak{g} = \tilde{\mathfrak{m}} + \mathfrak{g}'$. We have $D(\sigma, \xi) = \mathrm{Ad}\, \sigma \cdot \tilde{\xi} - \widetilde{l(\sigma) \cdot \xi}$ (See 11).

Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (*C*) and let $\tilde{\theta}_i$ be the basic forms of $\tilde{P}$. There is a subbundle $P_u(M, G_u)$ of $\tilde{P}(M, \tilde{G})$ satisfying the conditions in Theorem 2. Consider the connection (χ_{ij}) in P_u. For all $\xi = \sum_{i=1}^{n} \xi_i e_i \in \mathfrak{m}$, we define a vector field $B_u(\xi)$ on P_u by $\tilde{\theta}_i(B_u(\xi)) = \xi_i$ and $\chi_{ij}(B_u(\xi)) = 0$ $(1 \leqq i,\ j \leqq n-1)$. As for the vector fields $B_u(\xi)$, we know the following facts;

1) *The mapping $\xi \to B_u(\xi)$ is linear;*
2) $dR_\sigma \cdot B_u(\xi) = B_u(\sigma^{-1} \cdot \xi)$ *for all $\xi \in \mathfrak{m}$ and $\sigma \in G_u$;*
3) *Every tangent vector X to P_u can be uniquely written in the form: $B_u(\xi)_x + A^*_x$, where $\xi \in \mathfrak{m}$ and $A \in \mathfrak{g}_u$ and where x is the origin of X.*

By the conditions 1) and 2) in Theorem 2 and by 3) just above, we can find, for all $\xi, \xi' \in \mathfrak{m}$ and $x \in P_u$, a unique element $R_x(\xi, \xi')$ of $\mathfrak{g}_u$ such that

$$-[B_u(\xi), B_u(\xi')]_x = -\rho(\xi, \xi')B_u(e_n)_x + R_x(\xi, \xi')^*_x\,.$$

Exactly as in the Riemannian connection, then we have $\langle R_x(\xi, \xi')\eta, \eta' \rangle = \langle R_x(\eta, \eta')\xi, \xi' \rangle$ for all $\xi, \xi', \eta, \eta' \in \mathfrak{m}$ and $x \in P_u$. In particular, it follows that $R_x(\xi, e_n) = 0$ and $R_x(\sqrt{-1} \cdot \xi, \sqrt{-1} \cdot \xi') = R_x(\xi, \xi')$ for all $\xi, \xi' \in \mathfrak{m}_*$. Now let us define, for all $x \in P_u$, an endomorphism $\tilde{R}_x$ of $\mathfrak{m}_*$ by $\langle \tilde{R}_x(\xi), \xi' \rangle =$ the trace[10] of the endomorphism $\eta \to R_x(\xi, \eta)\xi'$ of $\mathfrak{m}_*$, where $\xi, \xi' \in \mathfrak{m}_*$. We have $\tilde{R}_x(\sqrt{-1} \cdot \xi) = \sqrt{-1} \cdot \tilde{R}_x(\xi)$ for all $\xi \in \mathfrak{m}_*$. We denote by R^*_x the trace of the endomorphism $\tilde{R}_x$ of $\mathfrak{m}_*$. Finally let f be a mapping of P_u into a finite dimensional vector

10) By the trace we always mean the real one.

space V. We denote by $\nabla_\xi f(x)$ the vector in V which is defined by $\nabla_\xi f(x) = \sum_{i=1}^{r} B_u(\xi)_x f_i \cdot v_i$, if $f = \sum_{i=1}^{r} f_i \cdot v_i$, where $\{v_1, \cdots, v_r\}$ is a base of V.

18. DEFINITION 7. Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (LC), and let $P(M, G')$ together $\bar{l}$ and α be the corresponding pseudo-conformal G'-bundle. A linear mapping B of $\mathfrak{m}$ into the vector space $\mathfrak{X}(P)$ of all the vector fields on P is called a pseudo-conformal connection in P, if it satisfies the following conditions;

1) $dR_\sigma \cdot B(\xi) = B(l(\sigma)^{-1} \cdot \xi) + D(\sigma^{-1}, \xi)^*$, $\xi \in \mathfrak{m}$, $\sigma \in G'$;
2) *Let θ_i be the basic forms of P. Then, we have $\theta_i(B(\xi)) = \xi_i$ $(1 \leqq i \leqq n)$ for all $\xi = \sum_{i=1}^{n} \xi_i e_i \in \mathfrak{m}$;*
3) $\alpha(B(\xi)) = 0$, $\xi \in \mathfrak{m}$.

The notation being as in Definition 7, suppose that there is given a pseudo-conformal connection. We shall investigate the fundamental properties of this connection.

(I) By the condition 2) in Definition 7, every tangent vector X to P can be uniquely written in the form: $B(\xi)_z + A_z^*$, where $\xi \in \mathfrak{m}$ and $A \in \mathfrak{g}'$ and where z is the origin of X. It follows that, for all $\xi, \xi' \in \mathfrak{m}$ and $z \in P$, we can find a unique element $T_z(\xi, \xi')$ of $\mathfrak{m}$ and a unique element $A_z(\xi, \xi')$ of $\mathfrak{g}'$ such that

$$-[B(\xi), B(\xi')]_z = B(T_z(\xi, \xi'))_z + A_z(\xi, \xi')_z^* . \tag{18.1}$$

We notice that by 3) in Proposition 11, we must necessarily have $T_z(\xi, \xi') \equiv -\rho(\xi, \xi')e_n \mod \mathfrak{m}_*$ for all $\xi, \xi' \in \mathfrak{m}$ and $z \in P$.

LEMMA 10. *Let $z \in P$, $\sigma \in G'$ and $\xi, \xi' \in \mathfrak{m}$ be arbitrary. Then, we have*

$$\begin{aligned} \operatorname{Ad}\sigma^{-1} \cdot \widetilde{T_z(\xi, \xi')} + \operatorname{Ad}\sigma^{-1} \cdot A_z(\xi, \xi') &= \widetilde{T_{z\cdot\sigma}(l(\sigma)^{-1}\xi, l(\sigma)^{-1}\xi')} \\ &+ A_{z\cdot\sigma}(l(\sigma)^{-1}\xi, l(\sigma)^{-1}\xi') - \operatorname{Ad}\sigma^{-1} \cdot [\tilde{\xi}, \tilde{\xi}'] + [\widetilde{l(\sigma)^{-1}\xi}, \widetilde{l(\sigma)^{-1}\xi'}] . \end{aligned}$$

PROOF. Let $A = \tilde{\xi} + U$ be any element of $\mathfrak{g}$, where $\xi \in \mathfrak{m}$ and $U \in \mathfrak{g}'$. We denote by A^* the vector field $B(\xi) + U^*$ on P. Then we have $dR_\sigma \cdot A^* = (\operatorname{Ad}\sigma^{-1}A)^*$ and $[U^*, A^*] = [U, A]^*$ for all $A \in \mathfrak{g}$, $U \in \mathfrak{g}'$ and $\sigma \in G'$. Moreover, we have clearly

$$-[\tilde{\xi}^*, \tilde{\xi}'^*]_z = (\widetilde{T_z(\xi, \xi')} + A_z(\xi, \xi'))_z^*$$

for all $\xi, \xi' \in \mathfrak{m}$ and $z \in P$, from which it follows that

$$\begin{aligned} &-[(\operatorname{Ad}\sigma^{-1} \cdot \tilde{\xi})^*, (\operatorname{Ad}\sigma^{-1} \cdot \tilde{\xi}')^*]_{z\cdot\sigma} \\ &= (\operatorname{Ad}\sigma^{-1} \cdot \widetilde{T_z(\xi, \xi')} + \operatorname{Ad}\sigma^{-1} \cdot A_z(\xi, \xi'))_{z\cdot\sigma}^* \end{aligned} \tag{18.2}$$

for all $\sigma \in G'$. But, the left-hand side of this equality is computed as follows:

$$\begin{aligned}(18.3)\qquad & -[\widetilde{l(\sigma)^{-1}\xi^*+D(\sigma^{-1},\xi)^*}, \widetilde{l(\sigma)^{-1}\xi'^*+D(\sigma^{-1},\xi')^*}]_{z\cdot\sigma} \\ &= -[\widetilde{l(\sigma)^{-1}\xi^*}, \widetilde{l(\sigma)^{-1}\xi'^*}]_{z\cdot\sigma} \\ &\quad -([\widetilde{l(\sigma)^{-1}\xi}, D(\sigma^{-1},\xi')]_{z\cdot\sigma}+[D(\sigma^{-1},\xi), \widetilde{l(\sigma)^{-1}\xi'}]_{z\cdot\sigma} \\ &\quad +[D(\sigma^{-1},\xi), D(\sigma^{-1},\xi')])^*_{z\cdot\sigma} \\ &= (\widetilde{T_{z\cdot\sigma}(l(\sigma)^{-1}\xi, l(\sigma)^{-1}\xi')}+A_{z\cdot\sigma}(l(\sigma)^{-1}\xi, l(\sigma)^{-1}\xi') \\ &\quad -\mathrm{Ad}\,\sigma^{-1}\cdot[\tilde{\xi},\tilde{\xi}']+[\widetilde{l(\sigma)^{-1}\xi}, \widetilde{l(\sigma)^{-1}\xi'}])^*_{z\cdot\sigma}.\end{aligned}$$

Lemma 10 now follows from (18.2) and (18.3).

(II) As an element of $\mathfrak{g}'$, $A_z(\xi,\xi')$ may be represented as a matrix of the form:

$$(18.4)\qquad \begin{pmatrix} -V_z(\xi,\xi') & -\sqrt{-1}\cdot {}^t\overline{K_z(\xi,\xi')}\cdot I & * \\ 0 & W_z(\xi,\xi') & K_z(\xi,\xi') \\ 0 & 0 & V_z(\xi,\xi') \end{pmatrix}.$$

By Lemma 10, we have easily

LEMMA 11. *Let $\xi,\xi'\in\mathfrak{m}$ and $z\in P$ be arbitrary and let σ be any element of G' expressed as* (11.4).

1) $T_{z\cdot\sigma}(\xi,\xi')=l(\sigma)^{-1}\cdot T_z(l(\sigma)\xi, l(\sigma)\xi')+\rho(l(\sigma)\xi, l(\sigma)\xi')l(\sigma)^{-1}\cdot e_n-\rho(\xi,\xi')e_n$;

2) *Assume that* $T=-\rho\cdot e_n$.

$$W_{z\cdot\sigma}(\xi,\xi')=b^{-1}\cdot W_z(l(\sigma)\xi, l(\sigma)\xi')\cdot b$$
$$V_{z\cdot\sigma}(\xi,\xi')=V_z(l(\sigma)\xi, l(\sigma)\xi')$$
$$K_{z\cdot\sigma}(\xi,\xi')=\varepsilon\cdot a\cdot b^{-1}\cdot K_z(l(\sigma)\xi, l(\sigma)\xi') +b^{-1}\cdot W_z(l(\sigma)\xi, l(\sigma)\xi')\cdot b\cdot c-V_z(l(\sigma)\xi, l(\sigma)\xi')\cdot c.$$

(III) For all $z\in P$, we define linear mappings $\widetilde{W}_z$ and $\widetilde{K}_z$ of $\mathfrak{m}$ into $\mathfrak{m}_*$ and $\boldsymbol{R}$ respectively as follows: Let $\xi\in\mathfrak{m}$ and $\xi'\in\mathfrak{m}_*$ be arbitrary. $\langle\widetilde{W}_z(\xi),\xi'\rangle$ = the trace of the endomorphism $\eta\to W_z(\xi,\eta)\xi'$ of $\mathfrak{m}_*$; $\widetilde{K}_z(\xi)$ = the trace of the endomorphism $\eta\to K_z(\xi,\eta)$ of $\mathfrak{m}_*$. For later uses, we further define W^*_z to be the trace of the endomorphism $\eta\to\widetilde{W}_z(\eta)$ of $\mathfrak{m}_*$.

LEMMA 12. *Let $z\in P$ and $\sigma\in G'$ be arbitrary.*

1) *If $T_z=-\rho\cdot e_n$, then $T_{z\cdot\sigma}=-\rho\cdot e_n$*;

2) *Assume that $T=-\rho\cdot e_n$. If $V_z=0$, then $V_{z\cdot\sigma}=0$*;
If $\widetilde{W}_z=0$, then $\widetilde{W}_{z\cdot\sigma}=0$;

3) *Assume that $T=-\rho\cdot e_n$, $V=0$ and $\widetilde{W}=0$. If $\widetilde{K}_z=0$, then $\widetilde{K}_{z\cdot\sigma}=0$.*

This is easy from Lemma 11.

(IV) Now, suppose that $\tilde{P}(M,\tilde{G})$ satisfies condition (C) and use the notations in 17. As is seen from the proof of Proposition 11, there is an injective homomorphism $\bar{h}$ of $P_u(M,G_u)$ into $P(M,G')$ such that $\bar{l}\circ\bar{h}(x)=x$ for all $x\in P_u$ and $\bar{h}^*\alpha=0$. Denote by k the mapping of P into G' defined by (16.1), i.e.

$z=\bar{h}\circ\bar{s}\circ\bar{l}(z)\cdot k(z)$ for all $z\in P$.

We have $\bar{h}^*\theta_i=\tilde{\theta}_i$ on P_u. Hence we can find, for all $x\in P_u$ and $\xi\in\mathfrak{m}$, a unique element $E_x(\xi)$ of $\mathfrak{g}'$ satisfying the condition:

$$d\bar{h}\cdot B_u(\xi)_x=B(\xi)_{\bar{h}(x)}+E_x(\xi)^*_{\bar{h}(x)}\,. \tag{18.5}$$

We have easily

LEMMA 13. *Let $z\in P$ and $\xi\in\mathfrak{m}$ be arbitrary.*

$$B(\xi)_z=dR_\sigma\cdot(d\bar{h}\cdot B_u(l(\sigma)\xi)_x-E_x(l(\sigma)\xi)^*_{\bar{h}(x)})-D(\sigma^{-1},l(\sigma)\xi)^*_z\,, \tag{18.6}$$

where $\sigma=k(z)$ and $x=\bar{s}\circ\bar{l}(z)$.

LEMMA 14. *Let $x\in P_u$ and $\xi,\xi'\in\mathfrak{m}$ be arbitrary.*

$$\begin{aligned}&\rho(\xi,\xi')\tilde{e}_n+\rho(\xi,\xi')E_x(e_n)-dh\cdot R_x(\xi,\xi')\\&=-\widetilde{T_{\bar{h}(x)}(\xi,\xi')}-A_{\bar{h}(x)}(\xi,\xi')+[E_x(\xi),\tilde{\xi}']+[\tilde{\xi},E_x(\xi')]\\&\quad+[E_x(\xi),E_x(\xi')]+\nabla_\xi E_x(\xi')-\nabla_{\xi'}E_x(\xi)\,.\end{aligned}$$

PROOF. For all $\xi\in\mathfrak{m}$, we define a vector field $C(\xi)$ on P by $C(\xi)_z=B(\xi)_z+E_{\bar{s}\circ\bar{l}(z)}(\xi)^*_z$ for all $z\in P$. It is clear that $C(\xi)$ is $\bar{h}$-related to $B_u(\xi)$, i.e. $C(\xi)_{\bar{h}(x)}=d\bar{h}\cdot B_u(\xi)_x$ at each $x\in P_u$; Hence $[C(\xi),C(\xi')]$ is also $\bar{h}$-related to $[B_u(\xi),B_u(\xi')]$ for all $\xi,\xi'\in\mathfrak{m}$. Take a point x of P_u and set $z=\bar{h}(x)$. Then we have

$$\begin{aligned}[C(\xi),C(\xi')]_z&=[B(\xi),B(\xi')]_z+[E_x(\xi)^*,B(\xi')]_z\\&\quad+[B(\xi),E_x(\xi')^*]_z+[E_x(\xi)^*,E_x(\xi')^*]_z\\&\quad+(\nabla_\xi E_x(\xi')-\nabla_{\xi'}E_x(\xi))^*_z\,.\end{aligned} \tag{18.7}$$

On the other hand, we have

$$\begin{aligned}d\bar{h}\cdot[B_u(\xi),B_u(\xi')]_x&=\rho(\xi,\xi')\cdot d\bar{h}\cdot B_u(e_n)_x\\&\quad-d\bar{h}\cdot R_x(\xi,\xi')^*_x\\&=\rho(\xi,\xi')\cdot B(e_n)_z+\rho(\xi,\xi')\cdot E_x(e_n)^*_z\\&\quad-(d\bar{h}\cdot R_x(\xi,\xi'))^*_z\,.\end{aligned} \tag{18.8}$$

Lemma 14 is then an immediate consequence of (18.7) and (18.8).

(V) Since $\bar{h}^*\alpha=0$, $E_x(\xi)$ may be represented as a matrix of the form

$$\begin{pmatrix}0&-\sqrt{-1}\cdot{}^t\overline{J_x(\xi)}\cdot I&J_x^n(\xi)\\0&U_x(\xi)&J_x(\xi)\\0&0&0\end{pmatrix} \tag{18.9}$$

for all $x\in P_u$ and $\xi\in\mathfrak{m}$.

From Lemma 14, we get

LEMMA 15. *Let $\xi,\xi'\in\mathfrak{m}_*$ be arbitrary.*

$T_{\bar{h}}(\xi,\xi')-U(\xi)\xi'+U(\xi')\xi+\rho(\xi,\xi')e_n=0$;

$T_{\bar{h}}(\xi,e_n)-J(\xi)+U(e_n)\cdot\xi=0$.

LEMMA 16. *$T=-\rho\cdot e_n$, if and only if $U(\xi)=0$ and $U(e_n)\cdot\xi=J(\xi)$ for all $\xi\in\mathfrak{m}_*$.*

This is easy from Lemma 15 and 1) of Lemma 12.

We denote by I_{n-1} the unit matrix of degree $n-1$. By Lemmas 14, 16, we have easily

LEMMA 17. *Let $\xi,\xi'\in\mathfrak{m}_*$ be arbitrary. Under the assumption that $T=-\rho\cdot e_n$, we have the followings:*

$$\sqrt{-1}\cdot W_{\bar h}(\xi,\xi')-\sqrt{-1}\cdot R(\xi,\xi')+J(\xi)\cdot{}^t\bar{\xi}'\cdot I-J(\xi')\cdot{}^t\bar{\xi}\cdot I$$
$$+\xi'\cdot{}^t\overline{J(\xi)}\cdot I-\xi\cdot{}^t\overline{J(\xi')}\cdot I+2\langle\xi',J(\xi)\rangle\cdot I_{n-1}$$
$$+2\sqrt{-1}\langle\xi,\sqrt{-1}\,\xi'\rangle U(e_n)=0;$$
$$\sqrt{-1}\cdot W_{\bar h}(\xi,e_n)-J(e_n)\cdot{}^t\bar{\xi}\cdot I-\xi\cdot{}^t\overline{J(e_n)}\cdot I-\langle\xi,J(e_n)\rangle I_{n-1}$$
$$-\sqrt{-1}\,\nabla_\xi U(e_n)=0;$$
$$V_{\bar h}(\xi,\xi')=0;$$
$$V_{\bar h}(\xi,e_n)+J^n(\xi)-\langle\sqrt{-1}\cdot J(e_n),\xi\rangle=0;$$
$$K_{\bar h}(\xi,\xi')+J^n(\xi)\cdot\xi'-J^n(\xi')\cdot\xi-\nabla_\xi J(\xi')+\nabla_{\xi'}J(\xi)$$
$$+2\langle\xi,\sqrt{-1}\,\xi'\rangle J(e_n)=0;$$
$$K_{\bar h}(\xi,e_n)-J^n(e_n)\cdot\xi+J(J(\xi))-\nabla_\xi J(e_n)+\nabla_{e_n}J(\xi)=0.$$

(VI) LEMMA 18. *Let $\xi,\xi'\in\mathfrak{m}_*$ be arbitrary.*

1) *Assume that $T=-\rho\cdot e_n$.*

$$\widetilde{W}_{\bar h}(\xi)-\frac{1}{4n}W^*_{\bar h}\cdot\xi+2(n+1)\sqrt{-1}\cdot J(\xi)-\tilde{R}(\xi)+\frac{1}{4n}R^*\cdot\xi=0;$$
$$\widetilde{W}_{\bar h}(e_n)+(2n-1)\cdot\sqrt{-1}\cdot J(e_n)$$
$$-\sum_{i=1}^{n-1}(\nabla_{e_i}J(Ie_i)+\nabla_{\sqrt{-1}\,e_i}J(\sqrt{-1}\,Ie_i))=0;$$
$$V_{\bar h}(\xi,\xi')=0;$$
$$V_{\bar h}(\xi,e_n)+J^n(\xi)-\langle\sqrt{-1}\cdot J(e_n),\xi\rangle=0.$$

2) *Assume that $T=-\rho\cdot e_n$, $\widetilde{W}=0$ and $V=0$.*

$$\tilde{K}_{\bar h}(\xi)=0;$$
$$\tilde{K}_{\bar h}(e_n)+2(n-1)\cdot J^n(e_n)-2\sum_{i=1}^{n-1}\langle J(J(e_i)),Ie_i\rangle$$
$$+\sum_{i=1}^{n-1}(\langle\nabla_{e_i}J(e_n),Ie_i\rangle+\langle\nabla_{\sqrt{-1}\,e_i}J(e_n),\sqrt{-1}\,Ie_i\rangle)=0.$$

We shall prove the first equality of 1) in Lemma 18. The others can be similarly dealt with. Before proceeding to the proof, we remark the following points: (1) By Lemma 16, we have $J(\xi)=U(e_n)\cdot\xi$ for all $\xi\in\mathfrak{m}_*$ and hence $\langle J(\xi),\xi'\rangle=-\langle\xi,J(\xi')\rangle$ and $\sqrt{-1}\cdot J(\xi)=J(\sqrt{-1}\cdot\xi)$ for all $\xi,\xi'\in\mathfrak{m}_*$. (2) We have $\langle e_i,Ie_j\rangle=\langle\sqrt{-1}\,e_i,\sqrt{-1}\,Ie_j\rangle=\delta_{ij}$ and $\langle\sqrt{-1}\,e_i,Ie_j\rangle=-\langle e_i,\sqrt{-1}\,Ie_j\rangle=0$ $(1\leqq i,j\leqq n-1)$. Let X be a vector in $\mathfrak{m}_*$ and let A be a (real) endomorphism of $\mathfrak{m}_*$. Then it follows that $X=\sum_{i=1}^{n-1}(\langle X,Ie_i\rangle e_i+\langle X,\sqrt{-1}\,Ie_i\rangle\sqrt{-1}\,e_i)$ and $T_r(A)$

$=\sum_{i=1}^{n-1}(\langle Ae_i, Ie_i\rangle+\langle A\cdot\sqrt{-1}\,e_i, \sqrt{-1}\,Ie_i\rangle)$.

From the first equality of Lemma 14, we get the following two equalities: Let $\xi, \eta\in\mathfrak{m}_*$ be arbitrary.

$$\begin{aligned}&\sum\langle W_{\bar h}(\xi, e_i)\eta, Ie_i\rangle-\sum\langle R(\xi, e_i)\eta, Ie_i\rangle+\sum\langle (I\eta)_i J(\xi), \sqrt{-1}\,Ie_i\rangle\\&+\langle\xi,\eta\rangle\sum\langle\sqrt{-1}\cdot J(e_i), Ie_i\rangle+(n-1)\langle\sqrt{-1}\,J(\xi),\eta\rangle\\&+2\sum\langle\sqrt{-1}\cdot J(\xi),\sqrt{-1}\,e_i\rangle\langle\eta,\sqrt{-1}\,Ie_i\rangle-\sum\langle J(\eta), e_i\rangle\langle\sqrt{-1}\,\xi, Ie_i\rangle\\&-\sum\langle\sqrt{-1}\cdot J(\eta), e_i\rangle\langle\xi, Ie_i\rangle-2\sum\langle\sqrt{-1}\,\xi, e_i\rangle\langle J(\eta), Ie_i\rangle=0\,;\\&\sum\langle W_{\bar h}(\xi,\sqrt{-1}\,e_i)\eta,\sqrt{-1}\,Ie_i\rangle-\sum\langle R(\xi,\sqrt{-1}\,e_i)\eta,\sqrt{-1}\,Ie_i\rangle\\&-\sum\langle (I\eta)_i J(\xi),\sqrt{-1}\,Ie_i\rangle+\langle\xi,\eta\rangle\sum\langle\sqrt{-1}\,J(e_i), Ie_i\rangle\\&+(n-1)\langle\sqrt{-1}\cdot J(\xi),\eta\rangle+2\cdot\sum\langle\sqrt{-1}\cdot J(\xi), e_i\rangle\langle\eta, Ie_i\rangle\\&+\sum\langle J(\eta), e_i\rangle\langle\sqrt{-1}\cdot\xi, Ie_i\rangle+\sum\langle\sqrt{-1}\cdot J(\eta), e_i\rangle\langle\xi, Ie_i\rangle\\&-2\sum\langle\sqrt{-1}\,\xi,\sqrt{-1}\,e_i\rangle\langle J(\eta),\sqrt{-1}\,Ie_i\rangle=0.\end{aligned}$$

Adding the above two equalities term by term and setting $c=\sum\langle\sqrt{-1}\cdot J(e_i), Ie_i\rangle$, we find

$$\langle\widetilde{W}_{\bar h}(\xi),\eta\rangle-\langle\widetilde{R}(\xi),\eta\rangle+2\cdot c\cdot\langle\xi,\eta\rangle+2(n+1)\langle\sqrt{-1}\cdot J(\xi),\eta\rangle=0\,,$$

namely

$$\widetilde{W}_{\bar h}(\xi)-\widetilde{R}(\xi)+2\cdot c\cdot\xi+2(n+1)\cdot\sqrt{-1}\cdot J(\xi)=0\,.$$

We have clearly

$$c=-\frac{1}{8n}\cdot W_{\bar h}^*+\frac{1}{8n}\cdot R^*$$

and hence we get the first equality of 1) in Lemma 18.

LEMMA 19.

1) *Assume that* $T=-\rho\cdot e_n$.

i) $\widetilde{W}=0$, *if and only if*

$$J(\xi)=\frac{\sqrt{-1}}{2(n+1)}\left(\frac{1}{4n}\cdot R^*\cdot\xi-\widetilde{R}(\xi)\right),\qquad \xi\in\mathfrak{m}_*,\tag{18.10}$$

$$J(e_n)=\frac{-\sqrt{-1}}{2n-1}\cdot\sum_{i=1}^{n-1}(\nabla_{e_i}J(Ie_i)+\nabla_{\sqrt{-1}\,e_i}J(\sqrt{-1}\,Ie_i))\,.\tag{18.11}$$

ii) $V=0$, *if and only if*

$$J^n(\xi)=\langle\sqrt{-1}\cdot J(e_n),\xi\rangle\,,\qquad \xi\in\mathfrak{m}_*\,.\tag{18.12}$$

2) *Assume that* $T=-\rho\cdot e_n$, $\widetilde{W}=0$ *and* $V=0$.

$\widetilde{K}=0$, *if and only if*

$$\begin{aligned}J^n(e_n)=\frac{1}{2(n-1)}\Big(2\cdot\sum_{i=1}^{n-1}\langle J(J(e_i)), Ie_i\rangle\\-\sum_{i=1}^{n-1}(\langle\nabla_{e_i}J(e_n), Ie_i\rangle+\langle\nabla_{\sqrt{-1}\,e_i}J(e_n),\sqrt{-1}\,Ie_i\rangle)\Big)\,.\end{aligned}\tag{18.13}$$

This is clear from Lemmas 12 and 18.

19. THEOREM 3. *Let $\tilde{P}(M, \tilde{G})$ be a pseudo-conformal $\tilde{G}$-bundle satisfying the condition (LC) and let $P(M, G')$ together with $\bar{l}$ and α be the corresponding pseudo-conformal G'-bundle. The notation being as in* 18, *there exists a unique pseudo-conformal connection B in P satisfying the following conditions: $T = -\rho \cdot e_n$, $V=0$, $\tilde{W}=0$ and $\tilde{K}=0$.*

PROOF. It is sufficient to make the proof in the case where $\tilde{P}(M, \tilde{G})$ satisfies condition (C). Take a fixed subbundle $P_u(M, G_u)$ of $\tilde{P}(M, \tilde{G})$ satisfying the conditions in Theorem 2 and use the notations in (IV) of 18.

Uniqueness. Suppose that there is given a pseudo-conformal connection B in P satisfying the conditions in Theorem 3. We see from Lemmas 16 and 19 that the linear mappings J_x, J_x^n and U_x are uniquely determined and therefore so is also the linear mapping E_x. Hence B is uniquely determined by Lemma 13.

Existence. Let x be any point of P_u. (1) We define a linear mapping J_x of $\mathfrak{m}$ into $\mathfrak{m}_*$ by (18.10) and (18.11). (2) Using J_x, we define a linear mapping J_x^n of $\mathfrak{m}$ into $\boldsymbol{R}$ by (18.12) and (18.13). (3) We have $J_x(\sqrt{-1} \cdot \xi) = \sqrt{-1} J_x(\xi)$ and $\langle J_x(\xi), \xi' \rangle + \langle \xi, J_x(\xi') \rangle = 0$ for all $\xi, \xi' \in \mathfrak{m}_*$. This being said, we define a linear mapping U_x of $\mathfrak{m}$ into $\mathfrak{g}_u$ by $U_x(\xi) = 0$ if $\xi \in \mathfrak{m}_*$ and $U_x(e_n) \cdot \eta = J_x(\eta)$ for all $\eta \in \mathfrak{m}_*$. (4) Using J_x, J_x^n and U_x, we define a linear mapping E_x of $\mathfrak{m}$ into $\mathfrak{g}'$ by (18.9). (5) Using E_x, we finally define a linear mapping B of $\mathfrak{m}$ into $\mathfrak{X}(P)$ by (18.6). Then the linear mapping B will be the desired connection. First of all, we have $J_{x\cdot\sigma}(\xi) = \varepsilon \cdot b^{-1} \cdot J_x(\sigma \cdot \xi)$, $J_{x\cdot\sigma}^n(\xi) = \varepsilon \cdot J_x^n(\sigma \cdot \xi)$ and $U_{x\cdot\sigma}(\xi) = b^{-1} \cdot U_x(\sigma \cdot \xi) \cdot b$ and hence

$$E_{x\cdot\sigma}(\xi) = \mathrm{Ad}\, h(\sigma)^{-1} \cdot E_x(\sigma \cdot \xi)$$

for all $x \in P_u$, $\xi \in \mathfrak{m}$ and $\sigma \in G_u$, where σ is assumed to be expressed as (11.7). That B satisfies the condition 1) in Definition 7 follows easily from this last equality. Next as for the conditions 2) and 3) in Definition 7, we use Lemma 7 and the equalities 1) and 2) in Proposition 11. We have thereby seen that B defines a pseudo-conformal connection in P. Finally we have $B(\xi)_{\bar{h}(x)} = d\bar{h} \cdot B_u(\xi)_x - E_x(\xi)^*_{\bar{h}(x)}$ for all $x \in P_u$ and $\xi \in \mathfrak{m}$. Therefore we see from Lemmas 16 and 19 that B satisfies the conditions in Theorem 3.

DEFINITION 8. The notation being as in Theorem 3, we shall say that the pseudo-conformal connection B in P whose existence and uniqueness are assured by Theorem 3 is the normal pseudo-conformal connection associated to the pseudo-conformal $\tilde{G}$-bundle $\tilde{P}(M, \tilde{G})$.

PROPOSITION 13. *Let $\tilde{P}(M, \tilde{G})$ (resp. $\tilde{P}'(M', \tilde{G})$) be a pseudo-conformal $\tilde{G}$-bundle satisfying condition (LC). Let $P(M, G')$ (resp. $P'(M', G')$) together with $\bar{l}$ and α (resp. $\bar{l}'$ and α') be the corresponding pseudo-conformal G'-bundle and let B (resp. B') be the corresponding normal pseudo-conformal connection in P (resp.*

P'). If φ is an isomorphism of $P(M, G')$ with $P'(M', G')$, a necessary and sufficient condition that $\varphi^*\theta'_i = \theta_i$ $(1 \leqq i \leqq n)$ and $\varphi^*\alpha' = \alpha$ is that $d\varphi \cdot B(\xi) = B'(\xi)$ for all $\xi \in \mathfrak{m}$, where θ_i (resp. θ'_i) are the basic forms of P (resp. P').

PROOF. We first prove the sufficiency. Take a tangent vector X to P and express it as $B(\xi)_z + A^*_z$, where $z \in P$, $\xi \in \mathfrak{m}$ and $A \in \mathfrak{g}'$. Then we have $\varphi^*\theta'_i(X) = \varphi^*\theta'_i(B(\xi)_z) + \varphi^*\theta'_i(A^*_z) = \theta'_i(B'(\xi)_{\varphi(z)}) + \theta'_i(A^*_{\varphi(z)}) = \xi_i = \theta_i(B(\xi)_z) + \theta_i(A^*_z)$ $= \theta_i(X)$, whence $\varphi^*\theta'_i = \theta_i$. Analogously we have $\varphi^*\alpha' = \alpha$. Let us prove the necessity. Set $B^*(\xi) = d\varphi^{-1} \cdot B'(\xi)$ for all $\xi \in \mathfrak{m}$. Since $\varphi^*\theta'_i = \theta_i$ and $\varphi^*\alpha' = \alpha$, we see easily that the assignment $\xi \to B^*(\xi)$ defines a pseudo-conformal connection in P. Since B' satisfies the condition in Theorem 3, so does the connection B^*. Consequently we have $B^* = B$ by Theorem 3, i.e. $d\varphi \cdot B(\xi) = B'(\xi)$ for all $\xi = \mathfrak{m}$.

VII. Pseudo-conformal transformations

20. Let S be a regular non-degenerate hypersurface. As we have already observed in Proposition 10, the corresponding pseudo-conformal $\tilde{G}$-bundle $\tilde{P}(S, \tilde{G})$ satisfies condition (LC). Therefore to such a hypersurface there are associated the pseudo-conformal G'-bundle $P(S, G')$ (together with $\tilde{l}$ and α) and the normal pseudo-conformal connection B in P. We have $\dim P = \dim G$ $= n^2 + 2n$; If we take a base $A_1, \cdots, A_s$ of $\mathfrak{g}'$, we see that the $n^2 + 2n$ vector fields $B(e_1), \cdots, B(e_n)$, $B(\sqrt{-1}\, e_1), \cdots, B(\sqrt{-1}\, e_{n-1})$, $A^*_1, \cdots, A^*_s$ are linearly independent at each point of P.

In virtue of Propositions 8, 12 and 13, we now arrive at the main theorem in this paper.

THEOREM 4. *Let S (resp. S') be a regular non-degenerate hypersurface. Let $P(S, G')$ (resp. $P'(S', G')$) be the corresponding pseudo-conformal G'-bundle and let B (resp. B') be the corresponding normal pseudo-conformal connection in P (resp. P'). If f is a pseudo-conformal homeomorphism of S with S', there corresponds to f a unique isomorphism φ of $P(S, G')$ with $P'(S', G')$ such that φ induces the given f and such that $d\varphi \cdot B(\xi) = B'(\xi)$ for all $\xi \in \mathfrak{m}$. Conversely, every isomorphism φ of $P(S, G')$ with $P'(S', G')$ satisfying this last condition induces a pseudo-conformal homeomorphism of S with S'.*

The pseudo-group $\Gamma(S)$ of all the local pseudo-conformal homeomorphisms of a hypersurface S is in general a continuous infinite pseudo-group of transformations. This fact can be easily verified by using Theorem 1. Theorem 4 and E. Cartan [**3**] indicate that $\Gamma(S)$ is of finite type and of dimension $\leqq n^2 + 2n$ in the case where S is regular and non-degenerate.

21. We denote by $G(S)$ the group of all the pseudo-conformal transformations of a hypersurface S and by $\mathfrak{g}(S)$ the Lie algebra of all the infinitesimal pseudo-conformal transformations on S.

PROPOSITION 14. *If a connected hypersurface S is non-degenerate at a point of S, then the Lie algebra* $\mathfrak{g}(S)$ *is of finite dimensional and* $\dim \mathfrak{g}(S) \leqq n^2+2n$.

PROOF. In the case where S is regular and non-degenerate, Proposition 14 is an immediate consequence[11] of Theorem 4. In the general case, assume that $\mathfrak{g}(S) \neq 0$. Let S^* be the set of all the points p of S at which S is non-degenerate. We see easily that S^* is open and dense in S and that the function λ introduced in 8 is constant on each connected component of S^*. Take a connected component $'S^*$ of S^* and assume that it is of index r. Now let $'S^{**}$ be the set of all the points p of $'S^*$ at which S is regular. Since $\mathfrak{g}(S) \neq 0$, we know from Proposition 5 that $'S^{**}$ is open and dense in $'S^*$. Let us again take a connected component $''S^{**}$ of $'S^{**}$. By considering the restrictions to $''S^{**}$ of the vector fields in $\mathfrak{g}(S)$, we may identify $\mathfrak{g}(S)$ with a subspace of $\mathfrak{g}(''S^{**})$. But, $\mathfrak{g}(''S^{**})$ is of finite dimensional and $\dim \mathfrak{g}(''S^{**}) \leqq n^2+2n$, because $''S^{**}$ is regular and non-degenerate. Therefore the same holds for $\mathfrak{g}(S)$, proving Proposition 14.

R. S. Palais [4] and Proposition 14 lead us to the following

THEOREM 5. *If a connected hypersurface S is non-degenerate at a point of S, then the group G(S) is a Lie group of dimension* $\leqq n^2+2n$ *with respect to the natural topology.*

As an immediate corollary to Theorem 5, we get

COROLLARY. *If S is a compact connected hypersurface, then the group G(S) is a Lie group of dimension* $\leqq n^2+2n$.

PROOF. Under the condition, there exists a point of S at which S satisfies the condition of Levi-Krzoska; Hence S is non-degenerate at this point by Proposition 3.

22. Now let us take up the hypersurface Q_r^* which has been already observed in 12. There we have seen that Q_r^* is non-degenerate, of index r and regular. Let $P(Q_r^*, G')$, $\bar{l}$, θ_i and α be as in 12. Then Propositions 6 and 7 mean that $P(Q_r^*, G')$ together with $\bar{l}$ and α is the pseudo-conformal G'-bundle associated to the hypersurface Q_r^* and that θ_i are the basic forms of P. For all $\xi \in \mathfrak{m}$, we denote by $B(\xi)$ the restriction to P of the left invariant vector field $\tilde{\xi}$ on G. Then the assignment $\xi \to B(\xi)$ clearly gives the normal pseudo-conformal connection in P.

We continue to identify $\boldsymbol{C}^n$ with an open submanifold of $P^n(\boldsymbol{C})$.

THEOREM 6. *Let* S_r *be the quadric of* $\boldsymbol{C}^n$ *defined by* (11.1). *If f is a pseudo-conformal homeomorphism of a connected open set of* S_r *with an open set of* S_r, *then f can be extended to a projective transformation of* $P^n(\boldsymbol{C})$.

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11) For a proof of this fact, see S. Kobayashi, Theory of connections, Annali. di, Math., **43** (1957), 119-194.

PROOF. In 11, we have remarked that the closure $\bar{S}_r$ of S_r in $P^n(\boldsymbol{C})$ is projectively equivalent to the quadric Q_r of $P^n(\boldsymbol{C})$ (which is also the closure of Q_r^* in $P^n(\boldsymbol{C})$). Therefore to prove Theorem 6, it is sufficient to prove the statement (of Theorem 6) in which S_r is everywhere replaced by Q_r^*. Let f be a pseudo-conformal homeomorphism of an open set U of Q_r^* with an open set U' of Q_r^*. By Theorem 4, f yields an isomorphism φ of $P|U$ with $P|U'$ such that φ induces the given f and such that $d\varphi \cdot B(\xi)_z = B(\xi)_{\varphi(z)}$, i.e. $d\varphi \cdot \tilde{\xi}_z = \tilde{\xi}_{\varphi(z)}$ for all $\xi \in \mathfrak{m}$ and $z \in P|U$. Moreover φ being a bundle isomorphism, we have $d\varphi \cdot A_z^* = A_{\varphi(z)}^*$, i.e. $d\varphi \cdot A_z = A_{\varphi(z)}$ for all $A \in \mathfrak{g}'$ and $z \in P|U$. It follows that $d\varphi \cdot X_z = X_{\varphi(z)}$ for all $X \in \mathfrak{g}$ and $z \in P|U$. Therefore we can find a unique element σ of G such that $\varphi(z) = \sigma \cdot z$ for all $z \in P|U$ and hence we get $f(p) = \sigma \cdot p$ for all $p \in U$. We have thereby completed the proof of Theorem 6.

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