J. Math. Soc. Japan
Vol. 13, No. 4, 1961

On the quadratic norm symbol in local number fields

By Katsumi SHIRATANI

(Received April 17, 1961)

1. Introduction.

The theory of norm residue symbol in algebraic number fields has been variously treated. An explicit formula of the local norm symbol has been given as the Šafarevič symbol by Šafarevič [8]. Hasse [5] and Kneser [7] improved the result of Šafarevič and supplied a link for the 2^n-th symbols.

As for a more special case than that of Šafarevič, Yamamoto [11] has proved the local reciprocity law of Kummer-Hilbert, on which the present author gave a note [9].

The structure of norm group of Kummer extension of prime degree was characterized to a certain extent by Hensel-Hasse [6]. The specially important formula of Hasse

$$\left(\frac{\nu}{\mu}\right)\left(\frac{\mu}{\nu}\right)=(-1)^{S\left(\frac{1-\nu}{2}\cdot\frac{1-\mu}{2}\right)}, \qquad (\mu\equiv\nu\equiv 1\ (2)),$$

is widely known. Here μ, ν mean two total-positive numbers in an algebraic number field which are mutually prime, and S denotes the trace from this field to the field of rational numbers.

Recently Siegel [10] proved the formula of Hasse from the viewpoint of the Gauss-Hecke sum in the theta function theory.

In this paper it is our purpose to give a local refinement of the formula of Hasse, from which we also show that the Šafarevič-Hasse-Kneser formula [5], [7] in the quadratic case can be readily derived.

Our method is to calculate explicitly the norm elements in the quadratic case by means of an idea of Yamamoto [11] and the present author [9]. In order to make this paper self-contained we shall prove several lemmas analogous to those given in [11], [9].

2. Several preliminary lemmas.

Let k be a local number field of finite degree over the field of rational 2-adic numbers R_2, e its ramification order, f its residue class degree and k_T the field of inertia, i.e., the maximal unramified field, contained in k. We denote by $\mathfrak{O}_{R_2}$, $\mathfrak{O}_k$, $\mathfrak{O}_{k_T}$ the rings of integers in R_2, k, k_T respectively and also by $\mathfrak{l}$, π

the prime ideal, a prime element in k.

We consider $K=k(\sqrt{\mu})$ with the ramification constant v, in which μ means a principal unit in k. If $\mathfrak{L}$ denotes the prime ideal in K, then of course $\mathfrak{l}=\mathfrak{L}^2$, the different of K/k is $\mathfrak{D}_{K/k}=\mathfrak{L}^{v+1}$.

LEMMA 1. *If K/k is ramified, we can select a suitable unit μ such that* $\mathrm{Ord}_{\mathfrak{l}}(1-\mu)=s$, $s\equiv 1\ (2)$, *and then we have $s+v=2e$. Therefore the number s is uniquely determined by K* [2].

PROOF. If we have $K=k(\sqrt{\mu})$, $s\equiv 0\ (2)$, then select a number $\mu'=\mu(1+b_{\frac{s}{2}}\pi^{\frac{s}{2}})^{-2}$ with $b_{\frac{s}{2}}=a_s{}^{2^{f-1}}$, $\mu=1+a_s\pi^s+\cdots$, $a_s\in\mathfrak{O}_{kT}$. Because $s<2e$ in the ramified case, we have $\mathrm{Ord}_{\mathfrak{l}}(2b_{\frac{s}{2}}\pi^{\frac{s}{2}})=e+\frac{s}{2}>s$, from which $k(\sqrt{\mu'})=k(\sqrt{\mu})$, $\mathrm{Ord}_{\mathfrak{l}}(1-\mu')>s$ follows. Here if $s'\equiv 0\ (2)$ again occurs, select a unit μ'' from μ' as above. By continuing the same process if necessary, we can find some unit $\mu^{(i)}$ such that $k(\sqrt{\mu})=k(\sqrt{\mu^{(i)}})$, $s^{(i)}\equiv 1\ (2)$, because we have $s<s'<\cdots<2e$. Now set $s=2d+1$ with the non-negative rational integer d and $M=\sqrt{\mu}$, then $\Pi=(1-M)\pi^{-d}$ is a prime element of K. From the definition of the ramification constant $v=2e-s$ [2] directly follows.

LEMMA 2. *If $\binom{n}{m}$ denotes a binomial coefficient, then*

$$\mathrm{Ord}_2\Big\{\sum_t\binom{j}{2t}\binom{t}{n}\Big\}\geqq j-2n-1 \qquad \textit{for}\ \ j>2n.$$

PROOF. First we have

$$\sum_t\binom{j}{2t}\binom{2t}{k}=\binom{j}{k}\sum_t\binom{j-k}{2t-k}=\binom{j}{k}\sum_{l\equiv j\ (2)}\binom{j-k}{l}=2^{j-k-1}\binom{j}{k}\quad \text{if}\ \ j\geqq k+1.$$

Now, the Faá di Bruno formula on successive derivatives gives us immediately for $f(\xi)=\xi^{2t}$, $\xi^2=x$,

$$(t)_n=\sum_\alpha\frac{n!}{(1!)^{a_1}(2!)^{a_2}\cdots a_1!\,a_2!\cdots}\Big(\Big(\frac{1}{2}\Big)_1\Big)^{a_1}\Big(\Big(\frac{1}{2}\Big)_2\Big)^{a_2}\cdots\cdots(2t)_a,$$

in which the summation runs over all partition $\alpha: n=a_1+2a_2+\cdots$, $a=a_1+a_2+\cdots$.

Therefore we have

$$\sum_t 2^n\binom{j}{2t}\binom{t}{n}=\sum_t\sum_\alpha\frac{1}{(1!)^{a_1}(2!)^{a_2}\cdots}\ \frac{a!}{a_1!\,a_2!\cdots}(1-2)^{a_2}((1-2)(1-4))^{a_3}\cdots\binom{2t}{a}\binom{j}{2t},$$

$$=\sum_\alpha T_\alpha.$$

$$\mathrm{Ord}_2T_\alpha\geqq\mathrm{Ord}_2\binom{j}{a}+(j-a-1)-\sum a_i\mathrm{Ord}_2(i!),$$

$$\geqq \mathrm{Ord}_2\binom{j}{a}+(j-a-1)-\sum_i a_i(i-1)=\mathrm{Ord}_2\binom{j}{a}+j-n-1,$$
$$\geqq j-n-1.$$

Finally we obtain $\mathrm{Ord}_2\{\sum_t \binom{j}{2t}\binom{t}{n}\}\geqq j-2n-1$.

We denote by $N_{K/k}$ the norm from K to k and by $S_{K/k}$ the trace. Then we put $\gamma=N_{K/k}\Pi$, $\sigma_j=S_{K/k}(\Pi^j)$ $(j\geqq 1)$.

LEMMA 3.

$$\sigma_j\equiv 2\pi^{-jd}\beta^{[\frac{j}{2}]} \qquad (\mathfrak{l}^{v+1}).$$

Here $[x]$ *denotes the Gauss' symbol indicating the greatest integer* $\leqq x$ *and* $\beta=1-\mu$.

PROOF. $\sigma_j=\pi^{-dj}(1-M)^j-\pi^{-dj}(1+M)^j=\pi^{-dj}2\sum_t\binom{j}{2t}M^{2t}$
$=\pi^{-dj}2\sum_k\sum_t(-1)^k\binom{j}{2t}\binom{t}{k}\beta^k$. Now by Lemma 2 we obtain
$\mathrm{Ord}_{\mathfrak{l}}(2\pi^{-dj}(-1)^k\sum_t\binom{j}{2t}\binom{t}{k}\beta^k)\geqq e-jd+e(j-2k-1)+ks=-jd+ks+e(j-2k)$,
from which $-jd+ks+e(j-2k)\leqq v$ is valid if and only if $k=\left[\frac{j}{2}\right]$.

Therefore $\sigma_j\equiv 2\pi^{-jd}(-1)^{[\frac{j}{2}]}\binom{j}{2[\frac{j}{2}]}\beta^{[\frac{j}{2}]}$ $(\mathfrak{l}^{v+1})$. Observing that $\mathrm{Ord}_{\mathfrak{l}}(2\pi^{-jd}\beta^{[\frac{j}{2}]})\geqq\frac{v+j}{2}$, we obtain our proposition.

LEMMA 3′. *We have*

$$\sigma_j\equiv 2\gamma^{\frac{j}{2}} \qquad (\mathfrak{l}^{v+1}) \qquad \text{if } j\equiv 0\ (2),$$
$$\sigma_j\equiv 2\gamma^{\frac{j-1}{2}}\pi^{-\frac{s-1}{2}} \qquad (\mathfrak{l}^{v+1}) \qquad \text{if } j\equiv 1\ (2).$$

PROOF. Because we have $\gamma=N_{K/k}\Pi=\pi^{-2d}\beta=\pi\frac{1-\mu}{\pi^s}$, we see $2\pi^{-jd}\beta^{[\frac{j}{2}]}$ $=2(\beta\pi^{1-s})^{\frac{j}{2}}=2\gamma^{\frac{j}{2}}$ if $j\equiv 0$ (2) and $2\pi^{-jd}\beta=2(\beta\pi^{1-s})^{\frac{j-1}{2}}=2\gamma^{\frac{j-1}{2}}\pi^{-\frac{s-1}{2}}$ if $j\equiv 1$ (2).

Therefore we obtain similarly

LEMMA 4. *For any element* $a\in\mathfrak{O}_{k_T}$, *we have*

$$N_{K/k}(1-a\Pi^j)\equiv(1+a\gamma^{\frac{j}{2}})^2 \qquad (\mathfrak{l}^{v+1}) \qquad \text{if } j\equiv 0\ (2),$$
$$N_{K/k}(1-a\Pi^j)\equiv 1+a^2\gamma^j+2a\gamma^{\frac{j-1}{2}}\pi^{-\frac{s-1}{2}} \ (\mathfrak{l}^{v+1}) \qquad \text{if } j\equiv 1\ (2).$$

LEMMA 5. *For any partition* $n=i_0+i_1+\cdots+i_p$, $i_j\geqq 0$, *of natural number* n, *it is necessary and sufficient for* $\frac{n!}{i_0!\cdots i_p!}\equiv 1$ (2) *that we have* $c_t=\sum_{j=0}^{p}c_t^{(j)}$,

$t=0,\cdots,r$, *with respect to the coefficients of* $n=\sum_{t=0}^{r} c_t 2^t$ *and* $i_j=\sum_{t=0}^{r} c_t^{(j)}2^t$, $0\leqq c_t\leqq 1$, $0\leqq c_t^{(j)}\leqq 1$.

PROOF. If $p=1$, we have always $c_t^{(0)}+c_t^{(1)}+d_{t-1}=c_t+2d_t$, with $d_t=0$ or 1 for $t=1,\cdots,r-1$, and $d_{-1}=d_r=0$.

Because $\mathrm{Ord}_2\frac{n!}{i_0!\,i_1!}=0$ is equivalent to $\sum_{t=0}^{r}c_t=\sum_{t=0}^{r}c_t^{(0)}+\sum_{t=0}^{r}c_t^{(1)}$, this also is equivalent to $d_t=0$ for all t, i.e., $c_t=c_t^{(0)}+c_t^{(1)}$ for all t. For the numbers $p>1$, the proposition can be easily verified by induction on p.

3. A formula of Hasse.

We can immediately verify one of the formulas of Hasse from Lemma 4.

Let ν be another principal unit in k, and denote by (ν,μ) the quadratic local Hilbert norm symbol, that is, $(\nu,\mu)=+1$ or -1 according to that ν is the norm of an element from K to k or not.

THEOREM 1 (Hasse [2], [3]). *Under the assumption* $s+v=2e$, $\mathrm{Ord}_{\mathfrak{l}}(1-\mu)=s$, $\mathrm{Ord}_{\mathfrak{l}}(1-\nu)=v$,

$$(\nu,\mu)=(-1)^{S_{k_T}\left(\frac{1-\nu}{2}\frac{1-\mu}{2}\right)}.$$

Here S_{k_T} *means the trace from* k_T *to* R_2.

PROOF. Put $2\pi^{-\frac{s-1}{2}}=\gamma^{\frac{v+1}{2}}\Phi(\gamma)$, $\Phi(\gamma)=\sum_{i=0}^{\infty}\xi_i\gamma^i$, $\xi_i\in\mathfrak{O}_{k_T}$, then by Lemma 4

$$N_{K/k}(1-a\Pi^v)=1+\{a^2+a\Phi(\gamma)\}\gamma^v \quad (\mathfrak{l}^{v+1}),$$

for any element $a\in\mathfrak{O}_{k_T}$.

Therefore $\nu\equiv 1+b\gamma^v$ $(\mathfrak{l}^{v+1})$, $b\in\mathfrak{O}_{k_T}$ is the norm of an element of K if and only if $b\equiv a^2+a\xi_0$ (2) is valid for some element a in $\mathfrak{O}_{k_T}$. This condition is equivalent to $S_{k_T}(\xi_0^{-2}b)\equiv 0$ (2). By making use of that $\xi_0^{-2}\equiv\frac{\pi^{s-1}}{2^2}\gamma^{v+1}\equiv\frac{1-\mu}{2^2}\gamma^v$ (2), this also equals to $S_{k_T}\left(\frac{1-\nu}{2}\frac{1-\mu}{2}\right)\equiv 0$ (2). Therefore we obtain our formula.

COROLLARY (Hasse [2], [3]). *If* $\nu\equiv\mu\equiv 1$ (2), *then*

$$(\nu,\mu)=(-1)^{S\left(\frac{1-\nu}{2}\frac{1-\mu}{2}\right)}.$$

Here S *denotes the trace from* k *to* R_2.

PROOF. If $\mathrm{Ord}_{\mathfrak{l}}(1-\nu)>v$, then $(\nu,\mu)=1$. On the other side we also have $S\left(\frac{1-\nu}{2}\frac{1-\mu}{2}\right)\equiv 0$ (2) and the proposition is valid. If $\mathrm{Ord}_{\mathfrak{l}}(1-\nu)=v$, we see $v\geqq e$, $s\geqq e$, $v+s=2e$ by the assumption, from which $v=s=e$, $e\equiv 1$ (2) follows. Thus we have $S(x)\equiv eS_{k_T}(x)\equiv S_{k_T}(x)$ (2). Our proposition is also valid from Theorem 1.

4. Recursive formulas.

It is our task to obtain an explicit formula for the symbol (ν, μ) under no assumption with respect to the orders of $1-\nu$, $1-\mu$.

For convenience' sake, we put $2^{f-1}=\Delta$, and then of course we have $a^{2\Delta}=a^{2^f}\equiv a$ (2) for any element $a\in\mathfrak{O}_{k_T}$.

If two principal units A, B in k belong to the same coset with respect to the multiplicative group of the norms $N_{K/k}K^*$, i.e., there exists an element X in K such that $A=B\ N_{K/k}X$, which is equivalent to $A\equiv B\ N_{K/k}X'$ $(\mathfrak{l}^{v+1})$ for some element X' in K, then we shall write $A\sim B$.

By making use of this notation, Lemma 4 yields for a natural number m and $a\in\mathfrak{O}_{k_T}$.

$$(1)\qquad 1+a^{2\Delta}\gamma^{2m}\sim 1+\frac{a^{\Delta}\cdot 2\gamma^{m}}{1+a\gamma^{2m}}\,,$$

$$(2)\qquad 1+a^{2\Delta}\gamma^{2m+1}\sim 1+\frac{a^{\Delta}\cdot\gamma^{\frac{v+1}{2}+m}}{1+a\gamma^{2m+1}}\Phi(\gamma)\,.$$

Here $\Phi(\gamma)$ means an $\mathfrak{l}$-adic expansion of $2\pi^{-\frac{s-1}{2}}\gamma^{-\frac{v+1}{2}}$ with a prime element γ, and so a unit in k.

In order to simplify our computations, we assume $\mu=1-\beta=1-\lambda\pi^s$, $\lambda\in T$, where T means a system of representatives of Teichmüller, that is, the multiplicatively closed representative system of the residue class field of k.

If we denote by $2=\sum_{i=e}^{\infty}\eta_i\gamma^i$ the $\mathfrak{l}$-adic expansion of 2 with the coefficients contained in T, then we have $\Phi(\gamma)=2\lambda^{\frac{s-1}{2}}\gamma^{-e}=\sum_{i=e}^{\infty}\lambda^{\frac{s-1}{2}}\eta_i\gamma^{i-e}$, $\lambda^{\frac{s-1}{2}}\eta_i\in T$. Then also from (1), (2) follow after a short calculation,

$$(3)\qquad 1+a^{2\Delta}\gamma^{2m}\sim 1+\sum_{t=e+m}^{v}\Big(\sum_{(2i+1)m+j=t}(-1)^i a^{\Delta}a^i\eta_j\Big)\gamma^t\,,$$

$$(4)\qquad 1+a^{2\Delta}\gamma^{2m+1}\sim 1+\sum_{t=v^*+m}^{v}\Big(\sum_{m+(2m+1)i+j-\frac{s-1}{2}=t}(-1)^i a^{\Delta}a^i\eta_j\lambda^{\frac{s-1}{2}}\Big)\gamma^t\,.$$

Here we put $v^*=\dfrac{v+1}{2}$. In the second summation we mean $j\geqq e$ and $i\geqq 0$ and we fix these meaning in this and the next sections. Also the letter l means always an odd number.

We see

$$\sum_{(2i+1)m+j=t}(-1)^i a^{\Delta}a^i\eta_j\equiv\sum_{l\cdot 2m+2j=2t}(a^l\eta_j^2)^{\Delta}\qquad(\mathrm{mod}\ 2)\,,$$

$$\sum_{m+(2m+1)i+j-\frac{s-1}{2}=t}(-1)^i a^{\Delta}a^i\eta_j\lambda^{\frac{s-1}{2}}\equiv\sum_{l(2m+1)+2j-s=2t}(a^l\eta_j^2\lambda^{s-1})^{\Delta}\qquad(\mathrm{mod}\ 2)\,.$$

If we denote the coefficients of $F=\sum_{j=e}^{\infty}\eta_j^2\gamma^{2j}$ and $G=\sum_{j=e}^{\infty}\eta_j^2\lambda^{s-1}\gamma^{2j-s}$ respectively by D_mF and D_mG, we evidently obtain the following formulas.

$$1+a^{2\Delta}\gamma^{2m}\sim 1+\sum_{t=e+m}^{v}\Big(\sum_{l\cdot 2m+m_0=2t}a^l\cdot D_{m_0}F\Big)^{\Delta}\gamma^t\,, \tag{5}$$

$$1+a^{2\Delta}\gamma^{2m+1}\sim 1+\sum_{t=v^*+m}^{v}\Big(\sum_{l(2m+1)+m_0=2t}a^l\cdot D_{m_0}G\Big)^{\Delta}\gamma^t\,. \tag{6}$$

Remark that $D_pF=0$ for $p<2e$, and that our formulas are reduced to the trivial ones if $v<v^*+m$ occurs.

Now we put $H=-F+G$ and denote its coefficients by D_mH. Observing that $D_mH=-D_mF$ for $m\equiv 0$ (2) and $D_mH=D_mG$ for $m\equiv 1$ (2), we obtain for both cases $n=2m$ and $n=2m+1$,

$$1+a^{2\Delta}\gamma^n\sim 1+\sum_{t=\{\frac{v+n}{2}\}}^{v}\Big(\sum_{l\cdot n+m_0=2t}a^lD_{m_0}H\Big)^{\Delta}\gamma^t\,. \tag{7}$$

Here $\{x\}$ means the symbol indicating the least integer $\geqq x$ and so $\{x\}=-[-x]$.

REMARK.

(I). In the case where $n\geqq v^*$, we particularly obtain from (7) the following formula.

$$1+a\gamma^n\sim 1+\sum_{t=\{\frac{v+n}{2}\}}^{v}(aD_{2t-n}H)^{\Delta}\gamma^t\,. \tag{8}$$

For, $l\geqq 3$ yields $nl+m\geqq 3v^*+m>v+1+m$ and we see by our definition $D_{m_0}H=0$ for $m_0<v$. Therefore $2t=ln+m_0>v+1+v=2v+1$, i.e., $t\geqq v+1$. Thus the terms with the summation $2t=ln+m_0$, $l\geqq 3$ makes no contribution in (7). On the other hand we have trivially $1+a^{2\Delta}\gamma^n\equiv 1+a\gamma^n$ $(\mathfrak{p}^{v+1})$ for $n\geqq v^*$. Thus (8) follows from (7).

(II). Our definition evidently yields

$$H=-F+G=-F+\lambda^{s-1}\gamma^{-s}F=(-1+\lambda^{s-1}\gamma^{-s})F$$
$$=\Big(-1+\frac{1}{1-\mu}\Big)F=\frac{\mu}{1-\mu}F\,.$$

By making use of the formula (7), we can transform $1+a\gamma^t$, $a\in\mathfrak{O}_{k_T}$ as follows.

$$1+a\gamma^t\sim 1+\alpha\gamma^v\,,\qquad \alpha\in\mathfrak{O}_{k_T}\,,$$

in which α is not uniquely determined, but $1+\alpha\gamma^v\sim 1+\alpha'\gamma^v$, $\alpha,\alpha'\in\mathfrak{O}_{k_T}$ is, from Theorem 1, equivalent to

$$S_{k_T}\Big(\frac{\alpha}{D_vG}\Big)\equiv S_{k_T}\Big(\frac{\alpha'}{D_vG}\Big)\quad(2)\,. \tag{9}$$

Here we shall denote by $\alpha\approx\alpha'$, that two elements α, α' satisfy the relation (9), and by $\varphi(a,t)$ a representative of the equivalence class of α, determined by $1+a\gamma^t$.

Evidently we have the following propositions.

(I) If $\varphi(a,t)\equiv\beta$ (2), then $\varphi(a,t)\approx\beta$.

(II) If $1+a\gamma^t\sim 1+a'\gamma^{t'}$, then $\varphi(a,t)\approx\varphi(a',t')$.

(III) $\varphi(a+a',n)\approx\varphi(a,n)+\varphi(a',n)$ for $n\geqq v^*$.

Now we shall obtain a recursive formula for $\varphi(a,t)$ from the formula (7).

Set $\delta^t_{t_1}a\equiv(\sum_{lt+m_0=2t_1} a^l\cdot D_{m_0}H)^{\Delta}$ (mod 2) $(t\geqq t_1)$. Then from (7) and the remark we see

$$1+a^{2\Delta}\gamma^t\sim 1+\sum_{t_1=\{\frac{v+t}{2}\}}^{v}\delta^t_{t_1}a\,\gamma^{t_1}\sim\prod_{t_1=\{\frac{v+t}{2}\}}^{v}(1+\delta^t_{t_1}a\,\gamma^{t_1})$$

$$\sim\prod_{t_1=\{\frac{v+t}{2}\}}^{v}(1+\varphi(\delta^t_{t_1}a,t_1)\gamma^v)\sim 1+\sum_{t_1=\{\frac{v+t}{2}\}}^{v}\varphi(\delta^t_{t_1}a,t_1)\gamma^v.$$

Consequently we obtain

$$\varphi(a^{2\Delta},t)\approx\sum_{t_1=\{\frac{v+t}{2}\}}^{v}\varphi(\delta^t_{t_1}a,t_1),\tag{10}$$

in particular for $t\geqq v^*$,

$$\varphi(a,t)\approx\sum_{t_1=\{\frac{v+t}{2}\}}^{v}\varphi(\delta^t_{t_1}a,t_1).\tag{11}$$

Here note that $\varphi(a,v)\approx a\approx(aD_vH)^{\Delta}$.

By making repeatedly use of (10), (11) with (I), (II), (III),

$$\varphi(a^{2\Delta},t)\approx\sum_{t_2=\{\frac{3v+t}{2^2}\}}^{v}\varphi(\sum_{t_1=\{\frac{v+t}{2}\}}^{2t_2-v}\delta^{t_1}_{t_2}\delta^t_{t_1}a,t_2).$$

Generally we obtain

$$\varphi(a^{2\Delta},t)\approx\varphi(\sum_{t_i=u_i}^{v}\sum_{t_{i-1}=u_{i-1}}^{2t_i-v}\cdots\sum_{t_1=u_1}^{2t_2-v}\delta^{t_{i-1}}_{t_i}\delta^{t_{i-2}}_{t_{i-1}}\cdots\delta^t_{t_1}a,t_i),$$

in which we put $u_i=\left\{\frac{(2^i-1)v+s}{2^i}\right\}$.

If we select a sufficiently large i such that $u_i=v$ holds, our remark after (11) shows

$$\varphi(a^{2\Delta},t)\approx\sum_{t_{i-1}=u_{i-1}}^{v}\cdots\sum_{t_1=u_1}^{2t_2-v}\delta^{t_{i-1}}_{v}\delta^{t_{i-2}}_{t_{i-1}}\cdots\cdots\delta^t_{t_1}a.\tag{12}$$

5. An explicit expression for $\varphi(a, t)$.

Our next concern is to give an explicit expression of the member of the right hand side of (12) by means of $1+a^{2\Delta}\gamma^t$.

From our definition we see immediately

$$\delta_{t_2}^{t_1}\delta_{t_1}^{t}a \equiv \sum_{tl+m_0=2t_1} (a^l \cdot D_{m_0}H)^{\Delta^2}(D_{2t_1-t_2}H)^{\Delta} \qquad (\text{mod } 2).$$

Generally we have

$$\delta_v^{t_{i-1}}\delta_{t_{i-1}}^{t_{i-2}} \cdots \delta_{t_1}^{t}a \approx \sum_l a^{l\Delta^i}(D_{2t_1-tl}H)^{\Delta^i}(D_{2t_2-t_1}H)^{\Delta^{i-1}} \cdots (D_{2v-t_{i-1}}H)^{\Delta}.$$

Therefore we obtain from (12),

$$(13) \quad \varphi(a^{2\Delta}, t) \approx \sum_{t_{i-1}=u_{i-1}}^{v} \cdots \sum_{t_1=u_1}^{2t_2-v} \sum_l a^{l\Delta^i}(D_{2t_1-tl}H)^{\Delta^i} \cdots (D_{2v-t_{i-1}}H)^{\Delta}$$

$$\approx \sum_l a^{l\Delta^i} \sum_{t_{i-1}=u_{i-1}}^{v} \cdots \sum_{t_1=u_1}^{2t_2-v} (D_{2t_1-tl}H)^{\Delta^i} \cdots (D_{2v-t_{i-1}}H)^{\Delta}$$

$$\approx \sum_{2^iv=tl+m} a^{l\Delta^i} \sum_{m=m_0+2m_1+\cdots+2^{i-1}m_{i-1}} (D_{m_0}H)^{\Delta^i}(D_{m_1}H)^{\Delta^{i-1}} \cdots (D_{m_{i-1}}H)^{\Delta}.$$

Now we define a formal power series $H(x)=\sum_{n=0}^{\infty}(-1)^{n-1}D_nHx^n=\frac{\mu(x)}{1-\mu(x)}F(x)$ by taking an indeterminate x instead of γ in the I-adic expansion of H, indicated before. The $\mu(x)$ denotes the polynomial $1-\lambda^{1-s}x^s$ and the $F(x)$ a formal power series defined by replacement of γ in F. Then

$$(14) \qquad H(x)^{2^i-1} = \sum_{t=(2^i-1)v}^{\infty} \sum_{\alpha} (-1)^{t-1}\frac{(2^i-1)!}{i_0!\,i_1!\cdots}(D_0H)^{i_0}(D_1H)^{i_1}\cdots x^t.$$

where the second summation runs over all partition α: $i_0+i_1+\cdots=2^i-1$, $i_1+2i_2+\cdots=t$.

Lemma 5 shows for all $t \geqq (2^i-1)v$,

$$(15) \quad \{D_t(H(x)^{2^i-1})\}^{\Delta^i} \equiv \sum_{t=m_0+2m_1+\cdots+2^{i-1}m_{i-1}} (D_{m_0}H)^{\Delta^i}(D_{m_1}H)^{\Delta^{i-1}} \cdots (D_{m_{i-1}}H)^{\Delta} \qquad (\text{mod } 2).$$

Here $D_t(H(x)^{2^i-1})$ denotes the coefficient of degree t of the series $H(x)^{2^i-1}$.

If we select for a an element belonging to T, we have $a^{2\Delta}=a$. So we have

$$(16) \qquad \varphi(a, t) \approx \sum_{2^iv=tl+m} \{a^lD_m(H(x)^{2^i-1})\}^{\Delta^i},$$

in which the suffix m runs over all integers satisfying $2^iv=tl+m$, $(l,2)=1$, $2^iv-v \leqq m \leqq 2^iv-tl \leqq 2^iv-l$.

On the other hand

$$(17) \quad D_m(H(x)^{2^i-1}) = D_m(H(x)^{2^i}H(x)^{-1}) = \sum_{m=n_1+n_2} D_{n_1}H(x)^{2^i} \cdot D_{n_2}H(x)^{-1}.$$

If $2[x]$ denotes the ideal consisting of power series with the coefficients divisible by 2 in the ring of formal power series of x over $\mathfrak{O}_{kT}$, then

$$H(x)^{2^i} \equiv \sum_{n=v}^{\infty}(D_nH)^{2^i} \cdot x^{2^in} \qquad (\text{mod } 2[x]).$$

Therefore

$$D_{n_1}H(x)^{2^i} \equiv \begin{cases} 0 & (\text{mod } 2) \quad \text{if} \quad n \not\equiv 0 \ (2^i), \\ (D_{\frac{n_1}{2^i}}H)^{2^i} & (\text{mod } 2) \quad \text{if} \quad n \equiv 0 \ (2^i). \end{cases}$$

The only remaining term mod 2 of (17) is that of $n_1=2^i v$, since $n_1 \geqq 2^i(v+1)$ with $n_2 \geqq -v$ yields $n_1+n_2 \geqq 2^i(v+1)-v=2^i v+(2^i-v)>2^i v-t \geqq m$.

Thus

$$D_m(H(x)^{2^i-1}) \equiv (D_v H)^{2^i} \cdot D_{m-2^i v}H(x)^{-1} \equiv (D_v G)^{2^i} D_{m-2^i v}H(x)^{-1} \qquad (\text{mod } 2).$$

And we obtain finally

$$(18) \qquad \varphi(a,t) \approx \sum_{2^i v=tl+m} \{a^l \cdot (D_v G)^{2^i} \cdot D_{m-2^i v}H(x)^{-1}\}^{\Delta^i},$$

$$\approx (D_v G) \sum_{2^i v=tl+m} \{a^l \cdot D_{m-2^i v}H(x)^{-1}\}^{\Delta^i},$$

$$\approx (D_v G) \sum_{2^i v=tl+m} \{a^l \cdot D_{m-2^i v} \frac{1-\mu(x)}{\mu(x)} F(x)^{-1}\}^{\Delta^i}.$$

Now we define a polynomial $\nu_t(x)=1+ax^t$ for $1+a\gamma^t$, and then for $(l,2)=1$,

$$a^l = D_{tl}\{(1-\nu_t(x))^l\} = D_{tl}\{\sum_{(k,2)=1}(1-\nu_t(x))^k\} \equiv D_{tl}\left(\frac{1-\nu_t(x)}{\nu_t(x)^2}\right) \qquad (\text{mod } 2).$$

Thus it follows from the above formula that

$$(19) \qquad \varphi(a,t) \approx (D_v G) \sum_{0=n_0+n_1} \left\{D_{n_0}\left(\frac{1-\nu_t(x)}{\nu_t(x)^2}\right) D_{n_1}\left(\frac{1-\mu(x)}{\mu(x)} \frac{1}{F(x)}\right)\right\}^{\Delta^i}$$

$$\approx (D_v G) \cdot D_0\left(\frac{1-\nu_t(x)}{\nu_t(x)^2} \frac{1-\mu(x)}{\mu(x)} \frac{1}{F(x)}\right)^{\Delta^i}.$$

If $2(x)=\sum_{i=e}^{\infty}\eta_i x^i$ denotes the power series attached to the $\mathfrak{l}$-adic expansion $2=\sum_{i=e}^{\infty}\eta_i\gamma^i$ of 2, then we readily see

$$F(x) \equiv 2(x)^2 \qquad (\text{mod } 2[x]).$$

Consequently an explicit expression of $\varphi(a,t)$ follows from the formula (19) as follows.

$$(20) \qquad \varphi(a,t) \approx (D_v G) \cdot \left\{D_0\left(\frac{1-\nu_t(x)}{\nu_t(x)^2} \frac{1-\mu(x)}{\mu(x)} \frac{1}{2(x)^2}\right)\right\}^{\Delta^i}.$$

For an odd number t, we need only the terms of odd degree of $\frac{1-\mu(x)}{\mu(x)} \frac{1}{2(x)^2}$ and then we may replace $\frac{1-\mu(x)}{\mu(x)} \frac{1}{2(x)^2}$ by $\frac{1-\mu(x)}{\mu(x)^2} \frac{1}{2(x)^2}$. Therefore for $(t,2)=1$,

$$(21) \qquad \varphi(a,t) \approx (D_v G)\left\{D_0\left(\frac{1-\nu_t(x)}{\nu_t(x)^2} \frac{1-\mu(x)}{\mu(x)^2} \frac{1}{2(x)^2}\right)\right\}^{\Delta^i}.$$

The formulas (20) and (21) give us a complete solution for our problem.

6. A local law of reciprocity.

We shall select a system of generators of principal unit group in k such as $1-a_t\gamma^t$, $a_t \in T$, $t=1,2,\cdots$. Then we have to modify slightly the formulas (20), (21) as follows.

By making use of $\mathrm{Ord}_{\mathfrak{l}}(2\gamma^t)=e+t \geqq \frac{v+s}{2}+t \geqq v^*$, a short calculation shows

$$(22) \qquad 1-a\gamma^t \equiv (1+a\gamma^t)\Big(1+\frac{2a\gamma^t}{1+a\gamma^t}\Big) \qquad (\mathfrak{l}^{v+1})$$

$$\equiv (1+a\gamma^t)\prod_{k=e+t}^{v}(1+\sum_{m+(1+j)t=k} a^{1+j}\eta_m)\gamma^k) \qquad (\mathfrak{l}^{v+1})$$

where in the case of $t \geqq v^*$, the second products of the number of the right hand side become the unity.
Therefore

$$1-a\gamma^t \sim 1+\{\varphi(a,t)+\sum_{k=e+t}^{v}\varphi(\sum_{m+(1+j)t=k} a^{1+j}\eta_m, k)\}\gamma^v .$$

In this equivalence we see by (20), (21), for some large number i,

$$(23) \quad \sum_{k=e+t}^{v}\varphi(\sum_{m+(1+j)t=k} a^{1+j}\eta_m, k) \approx (D_v G)\Big\{D_0\Big(\frac{2(x)ax^t}{1+ax^t}\cdot\frac{1-\mu(x)}{\mu(x)}\cdot\frac{1}{2(x)^2}\Big)\Big\}^{\Delta^t}.$$

Put $\nu(x)=1-ax^t$ for the principal unit $\nu=1-a\gamma^t$, then

$$(24) \quad 1-a\gamma^t \sim 1+\Big[(D_v G)\Big\{D_0\Big(\frac{1}{2(x)^2}\,\frac{1-\nu(x)}{\nu(x)^2}\,\frac{1-\mu(x)}{\mu(x)}+\frac{1}{2(x)}\,\frac{1-\nu(x)}{\nu(x)}\,\frac{1-\mu(x)}{\mu(x)}\Big)\Big\}^{\Delta^i}\Big]\gamma^v .$$

From Theorem 1 we obtain the following fundamental formula.

$$(25) \quad (\nu,\mu)=(-1)^{S_{k_T}(C)}, \quad C=D_0\Big\{\frac{1}{2(x)^2}\,\frac{1-\nu(x)}{\nu(x)^2}\,\frac{1-\mu(x)}{\mu(x)}+\frac{1}{2(x)}\,\frac{1-\nu(x)}{\nu(x)}\,\frac{1-\mu(x)}{\mu(x)}\Big\}.$$

Here $\nu=1-a\gamma^t$, $\mu=1-b\gamma^s$, $a,b \in T$ and $\nu(x)=1-ax^t$, $\mu(x)=1-bx^s$.
Particularly under the condition $(t,2)=1$ we have

$$(26) \quad C=D_0\Big\{\frac{1}{2(x)^2}\,\frac{1-\nu(x)}{\nu(x)^2}\,\frac{1-\mu(x)}{\mu(x)^2}+\frac{1}{2(x)}\,\frac{1-\nu(x)}{\nu(x)}\,\frac{1-\mu(x)}{\mu(x)}\Big\}$$

$$=D_0\Big\{\frac{1}{2(x)^2}\,\frac{1-\nu(x)}{\nu(x)}\,\frac{1-\mu(x)}{\mu(x)}\Big\}.$$

In this case our formula shows that the norm symbol is symmetric with respect to ν and μ.

Now we shall take as a system of generators of the principal unit group in k the principal units expressed by means of the Artin-Hasse-Šafarevič functions. By this system we can more simply express our local reciprocity law.

We shall here recall the Artin-Hasse function $E(a,\gamma^t)$ for $a \in T$. If $\mu(m)$

denotes the Möbius' function, then

$$E(a, \gamma^t)=\prod_{(m,2)=1}(1-a^m\gamma^{mt})^{\frac{\mu(m)}{m}}=e^{-L(a,\gamma^t)}, \tag{27}$$

$$L(a, \gamma^t)=\sum_{j=0}^{\infty}\frac{1}{2^j}a^{2^j}\gamma^{2^jt}. \tag{28}$$

Since our norm symbol has evidently the property $(\nu_1\nu_2, \mu)=(\nu_1, \mu)(\nu_2, \mu)$ from the definition, an explicit expression of $(E(a, \gamma^t), \mu)$ can be obtained as follows.

$$\frac{a^m x^{mt}}{(1-a^m x^{mt})^2}\equiv\sum_{(n,2)=1}a^{mn}x^{mnt} \qquad (\text{mod } 2[x]), \tag{29}$$

$$\frac{a^m x^{mt}}{1-a^m x^{mt}}\equiv\sum_{k=1}^{\infty}a^{mk}x^{mkt} \qquad (\text{mod } 2[x]), \tag{30}$$

$$((1-a^m\gamma^{mt})^{\frac{\mu(m)}{m}}, \mu)=((1-a^m\gamma^{mt}), \mu) \qquad \text{for } \frac{\mu(m)}{m}\equiv 1\ (2). \tag{31}$$

Our fundamental formula (25) with (29), (30), (31) yields

$$(E(a, \gamma^t), \mu)=(-1)^{S_{k_T}(C)},$$

$$C=D_0\Big\{\sum_{(m,2)=1}\frac{\mu(m)}{m}\sum_{(n,2)=1}a^{mn}x^{mnt}\frac{1-\mu(x)}{\mu(x)^2}\frac{1}{2(x)^2}$$

$$+\sum_{(m,2)=1}\frac{\mu(m)}{m}\sum_{k=1}^{\infty}a^{mk}x^{mkt}\frac{1-\mu(x)}{\mu(x)}\frac{1}{2(x)}\Big\}.$$

On the other hand, if $\varphi(l)$ means Euler's function, then

$$\sum_{(m,2)=1}\frac{\mu(m)}{m}\sum_{(n,2)=1}a^{mn}x^{mnt}=\sum_{(l,2)=1}\frac{1}{l}\varphi(l)a^l x^{lt}\equiv ax^t \qquad (\text{mod } 2[x]),$$

and similarly

$$\sum_{(m,2)=1}\frac{\mu(m)}{m}\sum_{k=1}^{\infty}a^{mk}x^{mkt}\equiv\sum_{j=0}^{\infty}a^{2^j}x^{2^jt} \qquad (\text{mod } 2[x]).$$

Here we define $\tilde{L}(a, x^t)=\sum_{j=0}^{\infty}a^{2^j}x^{2^jt}$. Then

$$(E(a, \gamma^t), \mu)=(-1)^{S_{k_T}(C)},$$

$$C=D_0\Big\{ax^t\frac{1-\mu(x)}{\mu(x)}\frac{1}{2(x)^2}+\tilde{L}(a, x^t)\frac{1-\mu(x)}{\mu(x)}\frac{1}{2(x)}\Big\}.$$

Our symbol has the property of symmetry which is easily verified, so that $(\nu, \mu_1\mu_2)=(\nu, \mu_1)(\nu, \mu_2)$ holds. Therefore the similar discussion as above shows for $(s, 2)=1$,

(32) $$(E(a,\gamma^t),\ E(b,\gamma^s))=(-1)^{S_{k_T}(C)},$$

$$C=D_0\Big\{ax^t\tilde{L}(b,x^s)\frac{1}{2(x)^2}+\tilde{L}(a,x^t)\tilde{L}(b,x^s)\frac{1}{2(x)}\Big\}.$$

The exponent $S_{k_T}(C)$ can be slightly simplified as follows.

(33) $$S_{k_T}(C)\equiv S_{k_T}\Big(D_0\Big\{ax^t\tilde{L}(b,x^s)\frac{1}{2(x)^2}+\tilde{L}(a^2,x^{2t})\tilde{L}(b^2,x^{2s})\frac{1}{2(x)^2}\Big\}\Big)\quad(\mathrm{mod}\ 2),$$

$$\equiv S_{k_T}\Big(D_0\Big\{\frac{1}{2(x)^2}\tilde{L}(a,x^t)\tilde{L}(b,x^s)\Big\}\Big)\quad(\mathrm{mod}\ 2).$$

Thus we have obtained our main theorem.

THEOREM 2. *Under the assumption* $(s,2)=1$, $a,b\in T$, *we have*

$$(E(a,\gamma^t),\ E(b,\gamma^s))=(-1)^R,\quad R\equiv S_{k_T}\Big(D_0\Big\{\frac{\tilde{L}(a,x^t)}{2(x)}\ \frac{\tilde{L}(b,x^s)}{2(x)}\Big\}\Big),$$

where $2(x)$ *means, as before, the formal power series defined by replacement of* γ *by* x *in the* $\mathfrak{l}$*-adic expansion of* 2 *by* γ, *whose coefficients belong to the Teichmüller representative system* T.

Note that we can regard γ as an arbitrary prime element in k in this theorem.

7. The case of ramification constant $2e$.

In this section we consider an extension field $K=k(\sqrt{\pi})$ over k, and an explicit formula of the symbol $(E(a,\pi^t),\pi)$ can be derived by the same method as before.

LEMMA 6. *Put* $\Pi=\sqrt{\pi}$ *and* $N_{K/k}\Pi=\pi$, *then for* $a\in\mathfrak{O}_{k_T}$,

$$N_{K/k}(1-a\Pi^t)=1+a^2\pi^t-2a\pi^{\frac{t}{2}}\quad\text{if }\ t\equiv 0\ (2),$$

$$N_{K/k}(1-a\Pi^t)=1+a^2\pi^t\quad\text{if }\ t\equiv 1\ (2).$$

The proof is trivial. Moreover it is evident that the ramification constant of K/k is $v=2e$. From Lemma 6 an analogous calculation to (8) gives us for a natural number t,

(34) $$1+a^{2\Delta}\pi^{2t}\sim 1+\sum_{t_1=\{\frac{e+t}{2}\}}^{e}\ \sum_{lt+j=2t_1}(a^l\eta_j^2)^{\Delta}\pi^{2t_1},$$

i. e.,

(35) $$1+a^{2\Delta}\pi^{2t}\sim 1+\sum_{t_1=\{\frac{e+t}{2}\}}^{e}\delta_{t_1}^t a\ \pi^{2t_1}.$$

Here of course our equivalence relation $\sim$ means that with respect to the multiplicative norm group $N_{K/k}K^*$, and we put $\delta_{t_1}^t a\equiv\sum_{\substack{lt+j=2t_1\\(l,2)=1,\ j\geqq e}}(a^l\eta_j^2)^{\Delta}$ (mod 2), as before.

Thus we have the following known theorem.

THEOREM 3. *If a principal unit* ν *has the order* $\mathrm{Ord}_{\mathfrak{l}}(1-\nu)=2e$, *we have*

$$(\nu, \pi)=(-1)^{S_{k_T}\left(\frac{\nu-1}{2^2}\right)}.$$

PROOF. By making use of above notations,

$$N_{K/k}(1-a\Pi^{2e})\equiv 1+(a^2-\eta_e a)\pi^{2e} \qquad (\mathfrak{l}^{2e+1}).$$

Therefore it is necessary and sufficient for $\nu=1+b\pi^{2e}$, $b\in\mathfrak{O}_{k_T}$, to be the norm of an element from K that there exists an element $a\in\mathfrak{O}_{k_T}$ satisfying $b\equiv a^2-\eta_e a$ (2), that is, equivalent to $S_{k_T}(b\eta_e{}^{-2})\equiv 0$ (2), in other word, $S_{k_T}\left(\frac{\nu-1}{2^2}\right)\equiv 0$ (2).

We define $\varphi(a^{2\Delta}, 2t)$ the same as before and then (35) becomes

$$\varphi(a^{2\Delta}, 2t)\approx \sum_{t_1=\left\{\frac{e+t}{2}\right\}}^{e} \varphi(\delta_{t_1}^{t} a, 2t_1).$$

In general, we also have

$$\varphi(a^{2\Delta}, 2t)\approx \sum_{t_i=u_i}^{e}\cdots\sum_{t_1=u_1}^{2t_2-e}\varphi(\delta_{t_i}^{t_{i-1}}\cdots\delta_{t_1}^{t}a, 2t_i),$$

where $u_j=\left\{\frac{(2^j-1)e+t}{2^j}\right\}$ and $m_j\geqq e$. For a sufficiently large number i,

$$\begin{aligned}\varphi(a^{2\Delta}, 2t)&\approx \sum_{t_{i-1}=u_{i-1}}^{e}\cdots\sum_{t_1=u_1}^{2t_2-e}\delta_e^{t_{i-1}}\cdots\delta_{t_1}^{t}a\\ &\approx \sum_{2^ie=tl+m} a^{l\Delta^i}\sum_{m=m_0+2m_1+\cdots+2^{i-1}m_{i-1}}\eta_{m_0}^{\Delta^{i-1}}\eta_{m_1}^{\Delta^{i-2}}\cdots\eta_{m_{i-1}}.\end{aligned}$$

A quite similar discussion as in previous sections yields for $a\in T$,

$$\begin{aligned}\varphi(a, 2t)&\approx \sum_{2^ie=tl+m}(a^{l\Delta}\cdot D_m(2(x)^{2^i-1}))^{\Delta^{i-1}}\\ &\approx \eta_e{}^2\sum_{2^ie=tl+m}(a^l\cdot D_{m-2^ie}(2(x)^{-1})^2)^{\Delta^i}\\ &\approx \eta_e{}^2\cdot\left\{D_0\left(\frac{1-\nu(x)}{\nu(x)^2}\,\frac{1}{2(x)^2}\right)\right\}^{\Delta^i}.\end{aligned}$$

Here $\nu(x)$ denotes the polynomial $\nu(x)=1+ax^{2t}$ for $\nu=1+a\pi^{2t}$, and $2(x)$ has the same meaning as before, but of course we must use π instead of γ.

Thus for an element $a\in T$,

$$(1+a\pi^{2t}, \pi)=(-1)^{S_{k_T}\left(D_0\left\{\frac{1}{2(x)^2}\,\frac{1-\nu(x)}{\nu(x)^2}\right\}\right)}.$$

Now we easily see

$$1-a\pi^{2t}\equiv(1+a\pi^{2t})\prod_{k=e+t}^{2e}\Big(1+\Big(\sum_{\substack{2t(1+i)+j=k\\ j\geqq e,\ i\geqq 0}}a^{1+i}\eta_j\Big)\gamma^k\Big) \qquad (\mathfrak{l}^{2e+1}),$$

that is,

$$1-a\pi^{2t} \sim (1+a\pi^{2t}) \prod_{k=\{\frac{e+t}{2}\}}^{e} (1+(\sum_{\substack{2t(1+i)+j=2k \\ j\geqq e,\ i\geqq 0}} a^{1+i}\eta_j)\gamma^{2k}).$$

Therefore for an element $a \in T$,

$$(1-a\pi^{2t}, \pi) = (-1)^{S_{k_T}(C)},$$

$$C \equiv D_0\Big(\frac{1-\nu(x)}{\nu(x)^2}\frac{1}{2(x)^2} + \frac{ax^{2t}}{1+ax^{2t}}\frac{2(x)}{2(x)^2}\Big) \qquad (\mathrm{mod}\ 2),$$

$$\equiv D_0\Big(\frac{1-\nu(x)}{\nu(x)^2}\frac{1}{2(x)^2} + \frac{1-\nu(x)}{\nu(x)}\frac{1}{2(x)}\Big) \qquad (\mathrm{mod}\ 2),$$

$$\equiv D_0\Big(\frac{1}{2(x)^2}\frac{1-\nu(x)}{\nu(x)}\Big) \qquad (\mathrm{mod}\ 2).$$

Here the same approach as (33) from (26) gives us the following theorem.
THEOREM 4.

$$(E(a, \pi^{2t}), \pi) = (-1)^R, \quad R \equiv S_{k_T}\Big(D_0\Big\{\frac{1}{2(x)^2}\tilde{L}(a, x^{2t})\Big\}\Big) \qquad (\mathrm{mod}\ 2).$$

8. The Šafarevič-Hasse-Kneser formula.

We shall show in this section that the Šafarevič-Hasse-Kneser formula [5], [7] in the quadratic case can be derived from our Theorem 2, Theorem 4.

The Šafarevic-Hasse-Kneser formula reads:

$$(E(a, \pi^t),\ E(b, \pi^s)) = (\pi, E(sab, \pi^{t+s}) \prod_{i,j=1}^{\infty} E((2^{i-1}t+2^{j-1}s)a^{2^i}b^{2^j}, \pi^{2^it+2^js})),$$

in which a, b need not belong to the Teichmüller representative system T, but in the quadratic case we may assume that they belong to T, and furthermore $(t, 2) = (s, 2) = 1$.

From Theorem 4 follows

$$(\pi, E(sab, \pi^{t+s}) \prod_{i,j=1}^{\infty} E((2^{i-1}t+2^{j-1}s)a^{2^i}b^{2^j}, \pi^{2^it+2^js})) = (-1)^{S_{k_T}(C)},$$

$$C = D_0\Big\{\frac{1}{2(x)^2}\tilde{L}(ab, x^{t+s}) + \frac{1}{2(x)^2}\sum_{j=1}^{\infty}\tilde{L}(a^2b^{2^j}, x^{2t+2^js}) + \frac{1}{2(x)^2}\sum_{i=1}^{\infty}\tilde{L}(a^{2^i}b^2, x^{2^it+2s})\Big\}.$$

$$= D_0\frac{1}{2(x)^2}\{\tilde{L}(a, x^t)\tilde{L}(b, x^s) - ax^t\tilde{L}(b, x^s) - bx^s\tilde{L}(a, x^t)\},$$

$$= D_0\Big\{\frac{\tilde{L}(a, x^t)}{2(x)}\frac{\tilde{L}(b, x^s)}{2(x)}\Big\}.$$

Thus by Theorem 2, the Šafarevič-Hasse-Kneser formula has been verified.

Now by the canonical decomposition theorem of Šafarevič [8], [5], [7], we can decompose two principal units ν, μ in k:

$$\nu \equiv \prod_{\substack{1 \leq i \leq 2e \\ (i,2)=1}} E(a_i, \pi^i) \quad (\mathfrak{l}^{2e}), \qquad \mu \equiv \prod_{\substack{1 \leq i \leq 2e \\ (i,2)=1}} E(b_i, \pi^i) \quad (\mathfrak{l}^{2e}).$$

The elements a_i, b_i in this form belong to $\mathfrak{O}_{k_T}$, but in the quadratic case we can assume that they belong to the Teichmüller representative system T. For the square $E(2a, \pi^i) = E(a, \pi^i)^2$ is a trivial factor in our case.

Let $L(\nu, x)$, $L(\mu, x)$ denote the power series defined by $L(\nu, x) = \frac{1}{2(x)} \sum_{\substack{1 \leq i \leq 2e \\ (i,2)=1}} \widetilde{L}(a_i, x^i)$, $L(\mu, x) = \frac{1}{2(x)} \sum_{\substack{1 \leq i \leq 2e \\ (i,2)=1}} \widetilde{L}(b_i, x^i)$, then our main theorem shows

$$(\nu, \mu) = (-1)^{S_{k_T}(D_0\{L(\nu,x)\cdot L(\mu,x)\})},$$

which gives us a slightly more explicit formula of the norm symbol than Šafarevič-Hasse-Kneser symbol.

General Education Department,
Kyushu University

References

[1] B. Dwork, Norm residue symbol in local number fields, Abh. Math. Sem. Univ. Hamburg, 22 (1958), 180–190.

[2] H. Hasse, Bericht über neuere Untersuchungen und Probleme aus der Theorie algebraischen Zahlkörper, Berlin, Teil I, Ia, II, 1927, 1930.

[3] H. Hasse, Das allgemeine Reziprozitätsgesetz und seine Ergänzungssätze in beliebigen algebraischen Zahlkörpern für gewisse nicht-primäre Zahlen, J. Reine Angew. Math., 153 (1924), 192–207.

[4] H. Hasse, Die Gruppe der p^n-primären Zahlen für einen Primteiler von p. J. Reine Angew. Math., 176 (1937), 174–183.

[5] H. Hasse, Zur Arbeit von I. R. Šafarevič über das allgemeine Reziprozitätsgesetz, Math. Nachr., Berlin 5 (1951), 301–327.

[6] K. Hensel und H. Hasse, Über die Normenreste eines relativ-zyklischen Körpers vom Primzahlgrad l nach einem Primteiler $\mathfrak{l}$ von l, Math. Ann., 90 (1923), 262–278.

[7] M. Kneser, zum expliziten Reziprozitätsgesetz von I. R. Šafarevič, Math. Nachr., 6 (1951), 89–96.

[8] I. R. Šafarevič, A general reciprocity law, Amer. Math. Soc. Transl., Ser. 2, 4 (1956), 13–106.

[9] K. Shiratani, Note on the Kummer-Hilbert reciprocity law, J. Math. Soc. Japan, 12 (1960), 412–421.

[10] C. L. Siegel, Über das quadratische Reziprozitätsgesetz in algebraischen Zahlkörpern, Göttingen Nachrichten, II, Nr. 1, 1960.

[11] K. Yamamoto, On the Kummer-Hilbert reciprocity law, Mem. Faculty Scie. Kyushu Univ., Ser. A, Vol. III, 2 (1959), 85–95.