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On the qualitative analysis of the fractional boundary value problem describing thermostat control model via ψ-Hilfer fractional operator

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Abstract

In this research study, we are concerned with the existence and stability of solutions of a boundary value problem (BVP) of the fractional thermostat control model with ψ-Hilfer fractional operator. We verify the uniqueness criterion via the Banach fixed-point principle and establish the existence by using the Schaefer and Krasnoselskii fixed-point results. Moreover, we apply the arguments related to the nonlinear functional analysis to discuss various types of stability in the format of Ulam. Finally, by several examples we demonstrate applications of the main findings.

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1 Introduction

Fractional differential and integral equations have demonstrated high visibility and capability in applications of various topics related to physics, signal processing, mechanics, electromagnetics, economics, biology, and many more [1, 2]. Particularly speaking, it has been recognized that fractional integro-differential equations, whose kernels allow much freedom to describe various processes involving memory and hereditary properties, often appear in different fractional models caused by many real-life processes such as phenomena related to electromagnetic waves and heat transfer. Yang et al. [3] implemented a discussion on the steady heat-transfer in the context of fractal media by invoking the local fractional nonlinear integro-differential equations of Volterra type. Furthermore, electromagnetic waves in a wide range of dielectric media including the susceptibility following a fractional power law are formulated in the framework of integro-differential equations [4]. We can see some recent advances and applications of fractional modelings in several newly published researches such as [5–8]. Also, in some new papers, the advantages and power of mathematical modeling based on fractional operators are illustrated, and that is why in recent years, many researchers prefer studying real processes and phenomena by applying newly defined versions of fractional operators (see, e.g., [9–18]).

Amongst important physio-electrical models, we are concerned with the thermostat control model. In this context a thermostat is a regulating instrument that measures the temperature of a given physical system and takes actions, provided that its temperature is maintained near a desired level. Thermostats are applied in any industrial system or controlling devices that cool or heat the temperature, examples including central heating, building heating, water heaters, air conditioners, water heaters, and also kitchen equipment such as refrigerators, ovens, and scientific and medical incubators. In [19] the authors demonstrated interest in the investigation of a thermostat control model insulated at $\varsigma = 0$ with controller at $\varsigma = 1$ and proposed the following boundary value problem (BVP) for the first time:

$$\begin{cases} \nu''(\varsigma) + h(\varsigma, \nu(\varsigma)) = 0, & \varsigma \in (0,1], \\ \nu'(0) = 0, & c\nu'(1) + \nu(k) = 0, \end{cases} \tag{1}$$

in which $k \in (0,1)$, and $c > 0$ is assumed to be an arbitrary parameter. Based on such a second-order mathematical model, the thermostat discharges or adds an amount of the heat with respect to the temperature detected by the existing sensor at $\varsigma = k$. They proved existence results for (1) by following the fixed point index theory in the context of integral equations of Hammerstein type.

Knowing the magnificent advantage of fractional derivatives, the authors in [20] studied the fractional-order thermostat control model

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha} x(\varsigma) + f(\varsigma, x(\varsigma)) = 0, & \varsigma \in (0,1], \\ x'(0) = 0, & \lambda\, {}^{C}\mathfrak{D}^{\alpha-1} x(1) + x(\eta) = 0, \end{cases} \tag{2}$$

where ${}^{C}\mathfrak{D}^{\alpha}$ is the Caputo derivative of order $\alpha \in (1,2]$. Based on the hypothesis that the nonlinearity f is assumed to be either superlinear or sublinear, the existence of positive solutions was proved by the help of the obtained Green's function and the Guo–Krasnoselskii fixed-point results. In their recent paper [21] the authors studied the fractional configuration of the thermostat control model subject to a convex–concave source term. They used the fixed point technique to prove the existence and uniqueness of positive solutions and provided an iterative scheme to approximate the obtained solutions. For more details on fractional thermostat control models, the reader can consult [22–26]. Exploring this literature, we can notice that the reported results were restricted to the existence of positive solutions and their properties. However, to the best of the authors' knowledge, no results were observed on the thermostat control model in the frame of generalized fractional operators. Further, the stability of solutions of fractional thermostat control models was not addressed yet.

Motivated by the above discussions, we consider a category of ψ-Hilfer nonlinear implicit fractional boundary value problems (FBVPs) describing the thermostat control model of the form

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(\varsigma) = f(\varsigma, x(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} x(\varepsilon\varsigma)), & \varsigma \in (0,T], \\ \sum_{i=1}^{m} \omega_i\, {}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} x(\xi_i) = A, & \sum_{j=1}^{n} \lambda_j\, {}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} x(\sigma_j) + \sum_{k=1}^{r} \delta_k x(\eta_k) = B, \end{cases} \tag{3}$$

where ${}^{H}\mathfrak{D}_{0^+}^{\nu,\rho;\psi}$ denotes the ψ-Hilfer derivative operators of order $\nu = \{\alpha, \beta_i, \mu_j\}$, $\alpha \in (1,2]$, β_i, $\mu_j \in (0,1]$, $A, B, \omega_i, \lambda_j, \delta_k \in \mathbb{R}$, ξ_i, σ_j, $\eta_k \in (0,T)$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, $k = 1,\dots,r$, $\rho \in [0,1]$, $\mathcal{I}_{0^+}^{q;\psi}$ is the ψ-RL-integral of order $q > 0$, θ, $\varepsilon \in (0,1]$, $f \in \mathcal{C}(\mathbb{J} \times \mathbb{R}^2, \mathbb{R})$, and $\mathbb{J} := [0,T]$ with $T > 0$. We establish existence and stability results for (3). We employ fixed point hypotheses to prove the existence results and we use the techniques of nonlinear functional analysis to study the stability in the Ulam sense. We present our results in a general platform, which covers many particular cases for specific values of ρ and ψ. For some relevant results, we refer the reader to recent papers [27–33].

The remaining parts of the research study adhere to the following plan. In Sect. 2, we define the norms, spaces, and other essential notions and lemmas related to the ψ-Hilfer fractional operator. Further, we derive the equivalent integral representation associated with the linear problem and state some fixed point theorems. We present the existence and uniqueness results in terms of three different fixed-point criteria in Sect. 3. In Sect. 4, we systematically present stabilization analysis of problem (3). In Sect. 5, we construct three particular examples, where the validity of the proposed results is verified. We terminate the investigation by conclusions.

2 Primitive notions

In this section, we give important basic definitions and primitive concepts of fractional calculus, which are useful throughout this paper.

We denote by $\mathbb{E} = \mathcal{C}(\mathbb{J}, \mathbb{R})$ the Banach space of continuous mappings on $\mathbb{J}$ with supnorm $\|x\| = \sup_{\varsigma\in\mathbb{J}}\{|x(\varsigma)|\}$ for $x \in \mathbb{E}$.

We also define the space of n times absolutely continuous functions

$$\mathcal{AC}^n[\mathbb{J}, \mathbb{R}] = \{f : \mathbb{J} \to \mathbb{R}; f^{(n-1)} \in \mathcal{AC}[\mathbb{J}, \mathbb{R}]\}.$$

Definition 2.1 ([34]) Let $\psi(\varsigma) \in \mathcal{C}^1([a,b], \mathbb{R})$ be an increasing function with $\psi'(\varsigma) \neq 0$ for each $\varsigma \in [a,b]$. The αth-ψ-RL-fractional integral of f depending on the function ψ is defined as

$$\mathcal{I}_{a^+}^{\alpha;\psi} f(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_a^\varsigma \psi'(s)\big(\psi(\varsigma) - \psi(s)\big)^{\alpha-1} f(s)\, ds, \quad \varsigma > a > 0, \alpha > 0,$$

where Γ is the gamma function.

Definition 2.2 ([34]) Let ψ be as above with $\psi'(\varsigma) \neq 0$. The αth-ψ-RL-fractional derivative of f depending on ψ is defined as

$$\begin{aligned}\mathfrak{D}_{a^+}^{\alpha;\psi} f(\varsigma) &= \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} f(\varsigma) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma}\right)^n \int_a^\varsigma \psi'(s)\big(\psi(\varsigma) - \psi(s)\big)^{n-\alpha-1} f(s)\, ds, \quad \alpha > 0,\end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3 ([35]) Let $\gamma = \alpha + \rho(n - \alpha)$, $\alpha \in (n-1, n)$, $f \in \mathcal{C}^n([a,b], \mathbb{R})$, and let $\psi(\varsigma) \in \mathcal{C}^1([a,b], \mathbb{R})$ be increasing with $\psi'(\varsigma) \neq 0$ for each $\varsigma \in [a,b]$. Then the αth-ψ-Hilfer derivative of f of type $\rho \in [0,1]$ depending on the function ψ is given by

$$ {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(\varsigma) = \mathcal{I}_{a^+}^{\rho(n-\alpha);\psi} \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(\varsigma) = \mathcal{I}_{a^+}^{\gamma-\alpha;\psi} \mathfrak{D}_{a^+}^{\gamma;\psi} f(\varsigma), $$

where $\mathfrak{D}_{a^+}^{\gamma;\psi} f(\varsigma) = \mathfrak{D}_{a^+}^{n;\psi} \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(\varsigma)$.

Lemma 2.4 ([34]) *Let $\alpha, \beta > 0$. In this case, we have the ψ-semigroup property*

$$ \mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} f(\varsigma) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} f(\varsigma), \quad \varsigma > a. $$

Next, we present the following properties.

Proposition 2.5 ([34, 35]) *Let $\varsigma > a$ and consider $\chi^{\upsilon}(\varsigma) = (\psi(\varsigma) - \psi(a))^{\upsilon-1}$. Then, for $\upsilon > 0$ and $\alpha \geq 0$, we have the following properties*:

(i) $\mathcal{I}_{a^+}^{\alpha;\psi} \chi^{\upsilon}(\varsigma) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)} \chi^{\upsilon+\alpha}(\varsigma)$;

(ii) $\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} \chi^{\upsilon}(\varsigma) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)} \chi^{\upsilon-\alpha}(\varsigma)$;

(iii) ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} \chi^{\upsilon}(\varsigma) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)} \chi^{\upsilon}(\varsigma), \upsilon > \gamma = \alpha + \rho(2-\alpha)$.

Lemma 2.6 *Let $\alpha \in (m-1, m)$, $\beta \in (n-1, n)$, $n, m \in \mathbb{N}$, $n \leq m$, $\rho \in [0,1]$, and $\alpha > \beta + \rho(n-\beta)$. If $h \in \mathcal{C}_{1-\gamma,\psi}(\mathbb{J}, \mathbb{R})$, then*

$$ {}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} h(\varsigma) = \mathcal{I}_{0^+}^{\alpha-\beta;\psi} h(\varsigma). $$

Proof Letting $\xi = \beta + \rho(n-\beta)$ with $n-1 < \xi < n$, we get

$$ \begin{aligned} {}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} \left(\mathcal{I}_{a^+}^{\alpha;\psi} h(\varsigma) \right) &= \mathcal{I}_{a^+}^{\xi-\beta;\psi} \mathfrak{D}_{a^+}^{\xi;\psi} \left(\mathcal{I}_{a^+}^{\alpha;\psi} h(\varsigma) \right) \\ &= \mathcal{I}_{a^+}^{\xi-\beta;\psi} \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n \mathcal{I}_{a^+}^{n-\xi;\psi} \left(\mathcal{I}_{a^+}^{\alpha;\psi} h(\varsigma) \right) \\ &= \mathcal{I}_{a^+}^{\xi-\beta;\psi} \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n \mathcal{I}_{a^+}^{n-\xi+\alpha;\psi} h(\varsigma). \end{aligned} $$

By Definition 2.1 we obtain

$$ \begin{aligned} &\left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right) \mathcal{I}_{0^+}^{n-\xi+\alpha;\psi} h(\varsigma) \\ &\quad = \frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \left(\frac{1}{\Gamma(n-\xi+\alpha)} \int_a^{\varsigma} \psi'(s) \bigl(\psi(\varsigma) - \psi(s) \bigr)^{n+\alpha-\xi-1} h(s)\, ds \right) \\ &\quad = \frac{1}{\Gamma(n-\xi+\alpha)} \frac{1}{\psi'(\varsigma)} \\ &\qquad \times \left(\int_a^{\varsigma} (n+\alpha-\xi-1) \psi'(s) \psi'(\varsigma) \bigl(\psi(\varsigma) - \psi(s) \bigr)^{n+\alpha-\xi-2} h(s)\, ds \right) \\ &\quad = \frac{1}{\Gamma(n-\xi+\alpha-1)} \int_a^{\varsigma} \psi'(s) \bigl(\psi(\varsigma) - \psi(s) \bigr)^{n+\alpha-\xi-2} h(s)\, ds \\ &\quad = \mathcal{I}_{a^+}^{n-\xi+\alpha-1;\psi} h(\varsigma) \end{aligned} $$

and

$$\begin{aligned}
&\left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^2 \mathcal{I}_{0^+}^{n-\xi+\alpha;\psi} h(\varsigma) \\
&= \frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\left(\frac{1}{\Gamma(n-\xi+\alpha-1)}\int_a^\varsigma \psi'(s)\big(\psi(\varsigma)-\psi(s)\big)^{n+\alpha-\xi-2} h(s)\,ds\right) \\
&= \frac{1}{\Gamma(n-\xi+\alpha-1)}\frac{1}{\psi'(\varsigma)} \\
&\quad\times\left(\int_a^\varsigma (n+\alpha-\xi-2)\psi'(s)\psi'(\varsigma)\big(\psi(\varsigma)-\psi(s)\big)^{n+\alpha-\xi-3} h(s)\,ds\right) \\
&= \frac{1}{\Gamma(n-\xi+\alpha-2)}\int_a^\varsigma \psi'(s)\big(\psi(\varsigma)-\psi(s)\big)^{n+\alpha-\xi-3} h(s)\,ds \\
&= \mathcal{I}_{a^+}^{n-\xi+\alpha-2;\psi} h(\varsigma).
\end{aligned}$$

Repeating the above process, we have

$$\begin{aligned}
&\left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n \mathcal{I}_{0^+}^{n-\xi+\alpha;\psi} h(\varsigma) \\
&= \frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\left(\frac{1}{\Gamma(\alpha-\xi)}\int_a^\varsigma \psi'(s)\big(\psi(\varsigma)-\psi(s)\big)^{\alpha-\xi-1} h(s)\,ds\right) \\
&= \frac{1}{\Gamma(\alpha-\xi+1)}\frac{1}{\psi'(\varsigma)}\left(\int_a^\varsigma (\alpha-\xi)\psi'(s)\psi'(\varsigma)\big(\psi(\varsigma)-\psi(s)\big)^{\alpha-\xi-1} h(s)\,dx\right) \\
&= \frac{1}{\Gamma(\xi+\alpha)}\int_a^\varsigma \psi'(s)\big(\psi(\varsigma)-\psi(s)\big)^{\alpha-\xi-1} h(s)\,ds \\
&= \mathcal{I}_{a^+}^{\alpha-\xi;\psi} h(\varsigma),
\end{aligned}$$

which implies that

$${}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi}\big(\mathcal{I}_{a^+}^{\alpha;\psi} h(\varsigma)\big) = \mathcal{I}_{a^+}^{\xi-\beta;\psi}\mathcal{I}_{a^+}^{\alpha-\xi;\psi} h(\varsigma) = \mathcal{I}_{a^+}^{\alpha-\beta;\psi} h(\varsigma).$$

This completes the proof. □

Lemma 2.7 ([35]) *Let* $f \in \mathcal{C}^n(\mathbb{J},\mathbb{R})$, $\gamma = \alpha + \rho(n-\alpha)$ *with* $\alpha \in (n-1,n)$, *and* $\rho \in [0,1]$. *Then*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\,{}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(\varsigma) = f(\varsigma) - \sum_{k=1}^{n}\frac{(\psi(\varsigma)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]}\mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(a)$$

for any $\varsigma \in \mathbb{J}$, *so that* $f_\psi^{[n]} f(\varsigma) := (\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma})^n f(\varsigma)$.

Lemma 2.8 *Let* $\Omega \neq 0$, $\alpha \in (1,2]$, $\beta_1,\ldots,\beta_m,\mu_1,\ldots,\mu_n \in (0,1]$, $\rho \in [0,1]$, *and* $\gamma = \alpha + \rho(2-\alpha)$. *Suppose that* $h \in \mathbb{E}$. *Then* $x \in \mathcal{C}^2(\mathbb{J},\mathbb{R})$ *is a solution of*

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(\varsigma) = h(\varsigma), \quad \varsigma \in (0,T], \\ \sum_{i=1}^m \omega_i\,{}^H\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} x(\xi_i) = A, \qquad \sum_{j=1}^n \lambda_j\,{}^H\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} x(\sigma_j) + \sum_{k=1}^r \delta_k x(\eta_k) = B, \end{cases} \tag{4}$$

iff x fulfills the ψ-integral equation

$$x(\varsigma) = \mathcal{I}_{0^+}^{\alpha;\psi} h(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_4\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\xi_i)\Bigg) - \Omega_2\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_k)\Bigg)\Bigg] + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_1\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_k)\Bigg) - \Omega_3\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\xi_i)\Bigg)\Bigg], \tag{5}$$

where

$$\Omega_1 = \sum_{i=1}^{m} \frac{\omega_i(\psi(\xi_i)-\psi(0))^{\gamma-\beta_i-1}}{\Gamma(\gamma-\beta_i)}, \tag{6}$$

$$\Omega_2 = \sum_{i=1}^{m} \frac{\omega_i(\psi(\xi_i)-\psi(0))^{\gamma-\beta_i-2}}{\Gamma(\gamma-\beta_i-1)}, \tag{7}$$

$$\Omega_3 = \sum_{j=1}^{n} \frac{\lambda_j(\psi(\sigma_j)-\psi(0))^{\gamma-\mu_j-1}}{\Gamma(\gamma-\mu_j)} + \sum_{k=1}^{r} \frac{\delta_k(\psi(\eta_k)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}, \tag{8}$$

$$\Omega_4 = \sum_{j=1}^{n} \frac{\lambda_j(\psi(\sigma_j)-\psi(0))^{\gamma-\mu_j-2}}{\Gamma(\gamma-\mu_j-1)} + \sum_{k=1}^{r} \frac{\delta_k(\psi(\eta_k)-\psi(0))^{\gamma-2}}{\Gamma(\gamma-1)}, \tag{9}$$

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3. \tag{10}$$

Proof Let $x \in \mathbb{E}$ be a solution of problem (4). Taking the operator $\mathcal{I}_{0^+}^{\alpha;\psi}$ on both sides of (4) and using Lemma 2.7, we have

$$x(\varsigma) = \mathcal{I}_{0^+}^{\alpha;\psi} h(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}c_1 + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Gamma(\gamma-1)}c_2, \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary real constants.

Taking the operators ${}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi}$ and ${}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi}$ into (11), we obtain

$$\begin{aligned} {}^{H}\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\varsigma) &= \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-\beta_i-1}}{\Gamma(\gamma-\beta_i)}c_1 + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-\beta_i-2}}{\Gamma(\gamma-\beta_i-1)}c_2, \\ {}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} x(\varsigma) &= \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-\mu_j-1}}{\Gamma(\gamma-\mu_j)}c_1 + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-\mu_j-2}}{\Gamma(\gamma-\mu_j-1)}c_2. \end{aligned}$$

From the first and second boundary conditions in (3) we get the system

$$\begin{cases} \Omega_1 c_1 + \Omega_2 c_2 = A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\xi_i), \\ \Omega_3 c_1 + \Omega_4 c_2 = B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_k), \end{cases}$$

where Ω_1, Ω_2, Ω_3, and Ω_4 are given by (6), (7), (8), and (9), respectively. Solving the system, it follows that

$$c_1 = \frac{1}{\Omega}\Bigg[\Omega_4\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\xi_i)\Bigg) - \Omega_2\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_k)\Bigg)\Bigg],$$

$$c_2 = \frac{1}{\Omega}\Bigg[\Omega_1\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} h(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_k)\Bigg) - \Omega_3\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} h(\xi_i)\Bigg)\Bigg],$$

where Ω is given by (10). Hence the solution x follows by inserting c_1 and c_2 into (11). This implies that $x(\varsigma)$ satisfies (5).

On the contrary, it is easy to show by a straightforward procedure that $x(\varsigma)$, which is illustrated by (5), fulfills the given FBVP (3) in terms of supposed boundary conditions. Lemma 2.8 is proved. □

3 Existence results

Set $F_x(\varsigma) = f(\varsigma, x(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} x(\varepsilon\varsigma))$, where

$$\mathcal{I}_{0^+}^{u;\psi} F_x(c) = \frac{1}{\Gamma(u)}\int_0^c \psi'(\tau)\big(\psi(c) - \psi(\tau)\big)^{u-1} F_x(\tau)\,d\tau$$

with $u = \{q, \beta_j\}$ and $c = \{\varsigma, \sigma, \theta_j\}$ for $j = 1, 2, \dots, n$. According to Lemma 2.8, we define $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ as

$$\begin{aligned}(\mathcal{Q}x)(\varsigma) = {}& \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_4\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_x(\xi_i)\Bigg) \\ & - \Omega_2\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_x(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\eta_k)\Bigg)\Bigg] \\ & + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_1\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_x(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\eta_k)\Bigg) \\ & - \Omega_3\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_x(\xi_i)\Bigg)\Bigg]. \end{aligned} \tag{12}$$

Note that the proposed ψ-Hilfer FBVP describing thermostat control model (3) involves solutions if and only if $\mathcal{Q}$ possesses fixed points. For brevity, we denote

$$\Psi_1(X, u) = \frac{(\psi(X)-\psi(0))^u}{\Gamma(u+1)}, \tag{13}$$

$$\Psi_2(U, V) = \frac{1}{|\Omega|}\big(|U|\Psi_1(T, \gamma-2) + |V|\Psi_1(T, \gamma-1)\big), \tag{14}$$

$$\begin{aligned}\Lambda(U,V) &= U\Psi_1(T,\alpha) + V\Psi_1(T,q+\alpha) \\ &\quad + \Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\bigl[U\Psi_1(\xi_i,\alpha-\beta_i) + V\Psi_1(\xi_i,q+\alpha-\beta_i)\bigr] \\ &\quad + \Psi_2(\Omega_1,\Omega_2)\Biggl(\sum_{j=1}^{n}|\lambda_j|\bigl[U\Psi_1(\sigma_j,\alpha-\mu_j) + V\Psi_1(\sigma_j,q+\alpha-\mu_j)\bigr] \\ &\quad + \sum_{k=1}^{r}|\delta_k|\bigl[U\Psi_1(\eta_k,\alpha) + V\Psi_1(\eta_k,q+\alpha)\bigr]\Biggr). \qquad (15)\end{aligned}$$

3.1 Uniqueness property

In the forthcoming first theorem, we will prove the uniqueness of solution for the ψ-Hilfer FBVP describing thermostat control model (3) by invoking the Banach principle (Lemma 3.1).

Lemma 3.1 ([36]) *Let S be a nonempty closed set contained in the Banach space $\mathbb{E}$. Then any contraction self-map $\mathcal{Q}$ on $\mathbb{E}$ has a unique fixed point.*

Theorem 3.2 *Let $f : \mathbb{J} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and*:

(H_1) *There exist $L_1, L_2 > 0$ such that*

$$\bigl|f(\varsigma,u_1,v_1) - f(\varsigma,u_2,v_2)\bigr| \le L_1|u_1-u_2| + L_2|v_1-v_2|$$

for any u_l, $v_l \in \mathbb{R}$, $l = 1,2$, and $\varsigma \in \mathbb{J}$.

If

$$\Lambda(L_1,L_2) < 1, \qquad (16)$$

where $\Lambda(\cdot,\cdot)$ is defined in (15), then the ψ-Hilfer FBVP describing the thermostat control model (3) has a unique solution x in $\mathbb{E}$.

Proof We convert the ψ-Hilfer FBVP describing the thermostat control model (3) into $x = \mathcal{Q}x$, where $\mathcal{Q}$ is given by (12). Obviously, the obtained fixed-points of $\mathcal{Q}$ are the possible solutions of the mentioned ψ-Hilfer FBVP (3). Following Lemma 3.1, we verify that $\mathcal{Q}$ admits a unique fixed point, which means that (3) involves exactly one solution.

Define a bounded, closed, convex, and nonempty subset $B_{r_1} := \{x \in \mathbb{E} : \|x\| \le r_1\}$ with

$$r_1 \ge \frac{\Lambda(M_1,0) + |A|\Psi_2(\Omega_3,\Omega_4) + |B|\Psi_2(\Omega_1,\Omega_2)}{1-\Lambda(L_1,L_2)}, \qquad \sup_{\varsigma\in\mathbb{J}}\bigl|f(\varsigma,0,0)\bigr| := M_1 < \infty, \qquad (17)$$

where Ω_i for $i = 1,2,3,4$, $\Psi_2(\cdot,\cdot)$, and $\Lambda(\cdot,\cdot)$ are given by (6)–(9), (14) and (15), respectively. We divide the proof into two steps.

Step I. $\mathcal{Q}B_{r_1} \subset B_{r_1}$.

Let $x \in B_{r_1}$ and $\varsigma \in \mathbb{J}$. Then

$$\begin{aligned}&\bigl|(\mathcal{Q}x)(\varsigma)\bigr| \\ &\quad \le \mathcal{I}_{0^+}^{\alpha;\psi}\bigl|F_x(s)\bigr|(T) + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\end{aligned}$$

$$
\begin{aligned}
&\times \Bigg[|\Omega_4| \Bigg(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \big|F_x(s)\big|(\xi_i) \Bigg) \\
&+ |\Omega_2| \Bigg(|B| + \sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \big|F_x(s)\big|(\sigma_j) + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \big|F_x(s)\big|(\eta_k) \Bigg) \Bigg] \\
&+ \frac{(\psi(T)-\psi(0))^{\gamma-2}}{|\Omega|\Gamma(\gamma-1)} \\
&\times \Bigg[|\Omega_1| \Bigg(|B| + \sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \big|F_x(s)\big|(\sigma_j) + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \big|F_x(s)\big|(\eta_k) \Bigg) \\
&+ |\Omega_3| \Bigg(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \big|F_x(s)\big|(\xi_i) \Bigg) \Bigg] \\
= {}& \mathcal{I}_{0^+}^{\alpha;\psi} \big|F_x(s)\big|(T) + \Psi_2(\Omega_3, \Omega_4) \Bigg(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \big|F_x(s)\big|(\xi_i) \Bigg) \\
&+ \Psi_2(\Omega_1, \Omega_2) \Bigg(|B| + \sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \big|F_x(s)\big|(\sigma_j) + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \big|F_x(s)\big|(\eta_k) \Bigg).
\end{aligned}
$$

By Proposition 2.5 we have

$$
\begin{aligned}
\mathcal{I}_{0^+}^{q;\psi} \big|x(\tau)\big|(s) &= \frac{1}{\Gamma(q)} \int_0^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{q-1} \big|x(\tau)\big| \, d\tau \\
&\le \frac{\|x\|}{\Gamma(q)} \int_0^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{q-1} \, d\tau \\
&= \frac{(\psi(s)-\psi(0))^q}{\Gamma(q+1)} \|x\|.
\end{aligned}
$$

From (H_1) we derive

$$
\begin{aligned}
\big|F_x(\varsigma)\big| &\le \big|f\big(\varsigma, x(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} x(\varepsilon\varsigma)\big) - f(\varsigma,0,0)\big| + \big|f(\varsigma,0,0)\big| \\
&\le L_1 \big|x(\theta\varsigma)\big| + L_2 \big|\mathcal{I}_{0^+}^{q;\psi} x(\varepsilon\varsigma)\big| + M_1 \\
&\le L_1 \|x\| + L_2 \frac{(\psi(\varepsilon\varsigma)-\psi(0))^q}{\Gamma(q+1)} \|x\| + M_1 \\
&\le \bigg(L_1 + L_2 \frac{(\psi(\varsigma)-\psi(0))^q}{\Gamma(q+1)} \bigg) \|x\| + M_1.
\end{aligned}
$$

Then we compute that

$$
\mathcal{I}_{0^+}^{\alpha;\psi} \big|F_x(s)\big|(T) \le \big[L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha)\big]\|x\| + M_1\Psi_1(T,\alpha), \tag{18}
$$

$$
\begin{aligned}
&\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \big|F_x(s)\big|(\xi_i) \\
&\quad \le \big[L_1\Psi_1(\xi_i,\alpha-\beta_i) + L_2\Psi_1(\xi_i,q+\alpha-\beta_i)\big]\|x\| + M_1\Psi_1(\xi_i,\alpha-\beta_i),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \big|F_x(s)\big|(\sigma_j) \\
&\quad \le \big[L_1\Psi_1(\sigma_j,\alpha-\mu_j) + L_2\Psi_1(\sigma_j,q+\alpha-\mu_j)\big]\|x\| + M_1\Psi_1(\sigma_j,\alpha-\mu_j),
\end{aligned} \tag{20}
$$

$$\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)\right|(\eta_k) \le \left[L_1\Psi_1(\eta_k,\alpha) + L_2\Psi_1(\eta_k,q+\alpha)\right]\|x\| + M_1\Psi_1(\eta_k,\alpha). \tag{21}$$

From (18), (19), (20), and (21) we obtain

$$\left|(\mathcal{Q}x)(\varsigma)\right| \le \Lambda(L_1,L_2)r_1 + \Lambda(M_1,0) + |A|\Psi_2(\Omega_3,\Omega_4) + |B|\Psi_2(\Omega_1,\Omega_2),$$

which implies that $\|\mathcal{Q}x\| \le r_1$. Hence $\mathcal{Q}B_{r_1} \subset B_{r_1}$.

Step II. The operator $\mathcal{Q}:\mathbb{E}\to\mathbb{E}$ is a contraction.

Let $x, y \in \mathbb{E}$. For each $\varsigma \in \mathbb{J}$, we get

$$\begin{aligned}
&\left|\big((\mathcal{Q}x)(\varsigma) - (\mathcal{Q}y)(\varsigma)\big)\right| \\
&\le \mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(T) + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Bigg[|\Omega_4|\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\left|F_x(s)-F_y(s)\right|(\xi_i) \\
&\quad + |\Omega_2|\Bigg(\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\left|F_x(s)-F_y(s)\right|(\sigma_j) + \sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(\eta_k)\Bigg)\Bigg] \\
&\quad + \frac{(\psi(T)-\psi(0))^{\gamma-2}}{|\Omega|\Gamma(\gamma-1)}\Bigg[|\Omega_1|\Bigg(\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\left|F_x(s)-F_y(s)\right|(\sigma_j) \\
&\quad + \sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(\eta_k)\Bigg) + |\Omega_3|\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\left|F_x(s)-F_y(s)\right|(\xi_i)\Bigg] \\
&= \mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(T) + \Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\left|F_x(s)-F_y(s)\right|(\xi_i) \\
&\quad + \Psi_2(\Omega_1,\Omega_2)\Bigg(\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\left|F_x(s)-F_y(s)\right|(\sigma_j) \\
&\quad + \sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(\eta_k)\Bigg).
\end{aligned} \tag{22}$$

From (H_1) we obtain

$$\begin{aligned}
\left|F_x(\varsigma)-F_y(\varsigma)\right| &= \left|f\big(\varsigma,x(\theta\varsigma),\mathcal{I}_{0^+}^{q;\psi}x(\varepsilon\varsigma)\big) - f\big(\varsigma,y(\theta\varsigma),\mathcal{I}_{0^+}^{q;\psi}y(\varepsilon\varsigma)\big)\right| \\
&\le L_1\left|x(\theta\varsigma)-y(\theta\varsigma)\right| + L_2\left|\mathcal{I}_{0^+}^{q;\psi}x(\varepsilon\varsigma) - \mathcal{I}_{0^+}^{q;\psi}x(\varepsilon\varsigma)\right| \\
&\le L_1\|x-y\| + L_2\frac{(\psi(\varepsilon\varsigma)-\psi(0))^q}{\Gamma(q+1)}\|x-y\| \\
&\le \left(L_1 + L_2\frac{(\psi(\varsigma)-\psi(0))^q}{\Gamma(q+1)}\right)\|x-y\|.
\end{aligned} \tag{23}$$

Then by substituting (23) into (22) we get

$$\begin{aligned}
&\left|(\mathcal{Q}x)(\varsigma) - (\mathcal{Q}y)(\varsigma)\right| \\
&\le \left[L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha)\right]\|x-y\| \\
&\quad + \Psi_2(\Omega_1,\Omega_2)\Bigg(\sum_{j=1}^{n}|\lambda_j|\left[L_1\Psi_1(\sigma_j,\alpha-\mu_j) + L_2\Psi_1(\sigma_j,q+\alpha-\mu_j)\right]\|x-y\|
\end{aligned}$$

$$+\sum_{k=1}^{r}|\delta_k|\big[L_1\Psi_1(\eta_k,\alpha)+L_2\Psi_1(\eta_k,q+\alpha)\big]\|x-y\|\Bigg)+\Psi_2(\Omega_3,\Omega_4)$$
$$\times\left(\sum_{i=1}^{m}|\omega_i|\big[L_1\Psi_1(\xi_i,\alpha-\beta_i)+L_2\Psi_1(\xi_i,q+\alpha-\beta_i)\big]\|x-y\|\right),$$

which illustrates that $\|\mathcal{Q}x-\mathcal{Q}y\|\leq\Lambda(L_1,L_2)\|x-y\|$. In view of the condition $\Lambda(L_1,L_2)<1$, we get that $\mathcal{Q}$ is a contraction. Hence by Lemma 3.1 we get that the solution $x\in\mathbb{E}$ is unique on $\mathbb{J}$ for the supposed ψ-Hilfer FBVP describing the thermostat control model (3). The proof of the theorem is completed. □

3.2 Existence property

The second result is obtained by invoking the Schaefer fixed-point theorem (Lemma 3.3).

Lemma 3.3 ([36]) *Let $\mathcal{Q}:\mathbb{E}\to\mathbb{E}$ be a completely continuous self-map on the Banach space $\mathbb{E}$, and let $\mathbb{P}=\{x\in\mathbb{E}:x=\kappa\mathcal{Q}x,0<\kappa\leq1\}$ be bounded. Then a fixed-point exists for $\mathcal{Q}$ in $\mathbb{E}$.*

Theorem 3.4 *Let $f:\mathbb{J}\times\mathbb{R}^2\to\mathbb{R}$ be continuous. Suppose that:*

(H_2) *There exist nonnegative continuous functions $k_1,k_2,k_3\in\mathcal{C}(\mathbb{J},\mathbb{R}^+\cup\{0\})$ such that*

$$\big|f(\varsigma,u,v)\big|\leq k_1(\varsigma)+k_2(\varsigma)|u|+k_3(\varsigma)|v|,\quad u,v\in\mathbb{R},\varsigma\in\mathbb{J},$$

with $k_1^*=\sup_{\varsigma\in\mathbb{J}}\{k_1(\varsigma)\}$, $k_2^*=\sup_{\varsigma\in\mathbb{J}}\{k_2(\varsigma)\}$, and $k_3^*=\sup_{\varsigma\in\mathbb{J}}\{k_3(\varsigma)\}$.

Then a solution exists on $\mathbb{J}$ for the supposed ψ-Hilfer FBVP describing the thermostat control model (3).

Proof We divide the proof into four steps.

Step I. $\mathcal{Q}$ is continuous.

Let a sequence $\{x_n\}\subset\mathbb{E}$ be such that $x_n\to x$ in $\mathbb{E}$. Then, for every $\varsigma\in\mathbb{J}$, we obtain

$$\big|(\mathcal{Q}x_n)(\varsigma)-(\mathcal{Q}x)(\varsigma)\big|$$
$$\leq\mathcal{I}_{0^+}^{\alpha;\psi}\big|F_{x_n}(s)-F_x(s)\big|(T)+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Omega|\Gamma(\gamma)}$$
$$\times\Bigg[|\Omega_4|\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\big|F_{x_n}(s)-F_x(s)\big|(\xi_i)$$
$$+|\Omega_2|\Bigg(\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\big|F_{x_n}(s)-F_x(s)\big|(\sigma_j)$$
$$+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\big|F_{x_n}(s)-F_x(s)\big|(\eta_k)\Bigg)\Bigg]$$
$$+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{|\Omega|\Gamma(\gamma-1)}\Bigg[|\Omega_1|\Bigg(\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\big|F_{x_n}(s)-F_x(s)\big|(\sigma_j)$$
$$+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\big|F_{x_n}(s)-F_x(s)\big|(\eta_k)\Bigg)$$

$$
\begin{aligned}
&\quad + |\Omega_3| \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \bigl|F_{x_n}(s) - F_x(s)\bigr|(\xi_i) \Biggr] \\
&= \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_{x_n}(s) - F_x(s)\bigr|(T) + \Psi_2(\Omega_1,\Omega_2) \Biggl(\sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \bigl|F_{x_n}(s) - F_x(s)\bigr|(\sigma_j) \\
&\quad + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_{x_n}(s) - F_x(s)\bigr|(\eta_k) \Biggr) \\
&\quad + \Psi_2(\Omega_3,\Omega_4) \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \bigl|F_{x_n}(s) - F_x(s)\bigr|(\xi_i) \\
&\le \Biggl[\Psi_1(T,\alpha) + \Psi_2(\Omega_1,\Omega_2) \Biggl(\sum_{j=1}^{n} |\lambda_j| \Psi_1(\sigma_j, \alpha - \mu_j) + \sum_{k=1}^{r} |\delta_k| \Psi_1(\eta_k,\alpha) \Biggr) \\
&\quad + \Psi_2(\Omega_3,\Omega_4) \sum_{i=1}^{m} |\omega_i| \Psi_1(\xi_i, \alpha - \beta_i) \Biggr] \|F_{x_n} - F_x\|.
\end{aligned}
$$

Since the continuity of f implies the continuity of F_x, we obtain

$$
\|F_{x_n} - F_x\| \to 0 \quad \text{as } n \to \infty,
$$

and therefore $\mathcal{Q}$ is continuous.

Step II. $\mathcal{Q}$ maps bounded sets to bounded ones contained in $\mathbb{E}$.

For $r_2 > 0$, there is $N > 0$ such that, for every $x \in B_{r_2} = \{x \in \mathbb{E} : \|x\| \le r_2\}$, we have $\|\mathcal{Q}x\| \le N$.

Indeed, for any $\varsigma \in \mathbb{J}$ and $x \in B_{r_2}$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{Q}x)(\varsigma)\bigr| \\
&\quad \le \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_x(s)\bigr|(T) \\
&\qquad + \frac{(\psi(T) - \psi(0))^{\gamma-1}}{|\Omega|\Gamma(\gamma)} \Biggl[|\Omega_4| \Biggl(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \bigl|F_x(s)\bigr|(\xi_i) \Biggr) \\
&\qquad + |\Omega_2| \Biggl(|B| + \sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \bigl|F_x(s)\bigr|(\sigma_j) + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_x(s)\bigr|(\eta_k) \Biggr) \Biggr] \\
&\qquad + \frac{(\psi(T) - \psi(0))^{\gamma-2}}{|\Omega|\Gamma(\gamma-1)} \\
&\qquad \times \Biggl[|\Omega_1| \Biggl(|B| + \sum_{j=1}^{n} |\lambda_j| \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} \bigl|F_x(s)\bigr|(\sigma_j) + \sum_{k=1}^{r} |\delta_k| \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_x(s)\bigr|(\eta_k) \Biggr) \\
&\qquad + |\Omega_3| \Biggl(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \bigl|F_x(s)\bigr|(\xi_i) \Biggr) \Biggr] \\
&\quad = \mathcal{I}_{0^+}^{\alpha;\psi} \bigl|F_x(s)\bigr|(T) + \Psi_2(\Omega_3,\Omega_4) \Biggl(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} \bigl|F_x(s)\bigr|(\xi_i) \Biggr) \\
&\qquad + \Psi_2(\Omega_1,\Omega_2)
\end{aligned}
$$

$$\times\left(|B|+\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\left|F_x(s)\right|(\sigma_j)+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)\right|(\eta_k)\right). \tag{24}$$

It follows from (H_2) that

$$\begin{aligned}\left|F_x(\varsigma)\right| &\leq k_1(\varsigma)+k_2(\varsigma)\left|x(\theta\varsigma)\right|+k_3(\varsigma)\left|\mathcal{I}_{0^+}^{q;\psi}x(\varepsilon\varsigma))\right| \\ &\leq k_1^*+\left(k_2^*+k_3^*\frac{(\psi(\varsigma)-\psi(0))^q}{\Gamma(q+1)}\right)\|x\|.\end{aligned} \tag{25}$$

Substituting (25) into (24), we get

$$\begin{aligned}&\left|(\mathcal{Q}x)(\varsigma)\right| \\ &\quad\leq\left[k_2^*\Psi_1(T,\alpha)+k_3^*\Psi_1(T,q+\alpha)\right]\|x\| \\ &\qquad+\Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\left[k_2^*\Psi_1(\xi_i,\alpha-\beta_i)+k_3^*\Psi_1(\xi_i,q+\alpha-\beta_i)\right]\|x\| \\ &\qquad+\Psi_2(\Omega_1,\Omega_2)\left(\sum_{j=1}^{n}|\lambda_j|\left[k_2^*\Psi_1(\sigma_j,\alpha-\mu_j)+k_3^*\Psi_1(\sigma_j,q+\alpha-\mu_j)\right]\|x\|\right. \\ &\qquad\left.+\sum_{k=1}^{r}|\delta_k|\left[k_2^*\Psi_1(\eta_k,\alpha)+k_3^*\Psi_1(\eta_k,q+\alpha)\right]\|x\|\right) \\ &\qquad+|A|\Psi_2(\Omega_3,\Omega_4)+|B|\Psi_2(\Omega_1,\Omega_2) \\ &\qquad+k_1^*\Psi_1(T,\alpha)+k_1^*\Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\Psi_1(\xi_i,\alpha-\beta_i) \\ &\qquad+k_1^*\Psi_2(\Omega_1,\Omega_2)\left(\sum_{j=1}^{n}|\lambda_j|\Psi_1(\sigma_j,\alpha-\mu_j)+\sum_{k=1}^{r}|\delta_k|\Psi_1(\eta_k,\alpha)\right),\end{aligned}$$

from which we get

$$\|\mathcal{Q}x\|\leq\Lambda\left(k_2^*,k_3^*\right)r_2+\Lambda\left(k_1^*,0\right)+|A|\Psi_2(\Omega_3,\Omega_4)+|B|\Psi_2(\Omega_1,\Omega_2):=N.$$

Step III. $\mathcal{Q}$ maps bounded sets to equicontinuous ones contained in $\mathbb{E}$.

For $0\leq\varsigma_1<\varsigma_2\leq T$ and $x\in B_{r_2}$, since f is bounded on the compact set $\mathbb{J}\times B_{r_2}$, we have

$$\begin{aligned}&\left|(\mathcal{Q}x)(\varsigma_2)-(\mathcal{Q}x)(\varsigma_1)\right| \\ &\quad\leq\frac{|(\psi(\varsigma_2)-\psi(0))^{\gamma-1}-(\psi(\varsigma_1)-\psi(0))^{\gamma-1}|}{|\Omega|\Gamma(\gamma)} \\ &\qquad\times\left[|\Omega_4|\left(|A|+\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\left|F_x(s)\right|(\xi_i)\right)\right. \\ &\qquad\left.+|\Omega_2|\left(|B|+\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\left|F_x(s)\right|(\sigma_j)+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)\right|(\eta_k)\right)\right] \\ &\qquad+\frac{|(\psi(\varsigma_2)-\psi(0))^{\gamma-2}-(\psi(\varsigma_1)-\psi(0))^{\gamma-2}|}{|\Omega|\Gamma(\gamma-1)}\end{aligned}$$

$$\times\Bigg[|\Omega_1|\Bigg(|B|+\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\big|F_x(s)\big|(\sigma_j)$$
$$+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\big|F_x(s)\big|(\eta_k)\Bigg)+|\Omega_3|\Bigg(|A|+\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\big|F_x(s)\big|(\xi_i)\Bigg)\Bigg]$$
$$+\big|\mathcal{I}_{0^+}^{\alpha;\psi}F_x(s)(\varsigma_2)-\mathcal{I}_{0^+}^{\alpha;\psi}F_x(s)(\varsigma_1)\big|.$$

By setting $\sup_{(\varsigma,u,v)\in\mathbb{J}\times B_{r_2}^2}|f(\varsigma,u,v)|=\widehat{f}<\infty$ it follows that

$$\begin{aligned}
&\big|(\mathcal{Q}x)(\varsigma_2)-(\mathcal{Q}x)(\varsigma_1)\big|\\
&\quad\le\frac{\widehat{f}}{\Gamma(\alpha+1)}\\
&\qquad\times\Big[\big(\psi(\varsigma_2)-\psi(\varsigma_1)\big)^{\alpha}+\Big|\big(\psi(\varsigma_2)-\psi(0)\big)^{\alpha}-\big(\psi(\varsigma_2)-\psi(\varsigma_1)\big)^{\alpha}-\big(\psi(\varsigma_1)-\psi(0)\big)^{\alpha}\Big|\Big]\\
&\qquad+\frac{|(\psi(\varsigma_2)-\psi(0))^{\gamma-1}-(\psi(\varsigma_1)-\psi(0))^{\gamma-1}|}{|\Omega|\Gamma(\gamma)}\\
&\qquad\times\Bigg[|\Omega_4|\Bigg(|A|+\widehat{f}\sum_{i=1}^{m}\frac{|\omega_i|(\psi(\xi_i)-\psi(0))^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Bigg)\\
&\qquad+|\Omega_2|\Bigg(|B|+\widehat{f}\sum_{j=1}^{n}\frac{|\lambda_j|(\psi(\sigma_j)-\psi(0))^{\alpha-\mu_j}}{\Gamma(\alpha-\mu_j+1)}+\widehat{f}\sum_{k=1}^{r}\frac{|\delta_k|(\psi(\eta_k)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\Bigg]\\
&\qquad+\frac{(\psi(\varsigma_2)-\psi(0))^{\gamma-2}-(\psi(\varsigma_1)-\psi(0))^{\gamma-2}}{|\Omega|\Gamma(\gamma-1)}\\
&\qquad\times\Bigg[|\Omega_1|\Bigg(|B|+\widehat{f}\sum_{j=1}^{n}\frac{|\lambda_j|(\psi(\sigma_j)-\psi(0))^{\alpha-\mu_j}}{\Gamma(\alpha-\mu_j+1)}\\
&\qquad+\widehat{f}\sum_{k=1}^{r}\frac{|\delta_k|(\psi(\eta_k)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\\
&\qquad+|\Omega_3|\Bigg(|A|+\widehat{f}\sum_{i=1}^{m}\frac{|\omega_i|(\psi(\xi_i)-\psi(0))^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Bigg)\Bigg].
\end{aligned}\tag{26}$$

Clearly, the right-hand side of (26) is independent of the unknown variable x and approaches 0 as $\varsigma_2\to\varsigma_1$. Then the operator $\mathcal{Q}$ is equicontinuous. So, the operator $\mathcal{Q}$ admits the relative compactness on B_{r_2}, and the Arzelá–Ascoli theorem gives the complete continuity of $\mathcal{Q}$.

Step IV. $\mathbb{P}=\{x\in\mathbb{E}:x=\tau\mathcal{Q}x,\tau\in(0,1]\}$ is bounded.

Let $x\in\mathbb{P}$. Then $x=\tau\mathcal{Q}x$ for some $\tau\in(0,1]$. From (H_2), for each $\varsigma\in\mathbb{J}$, we get the estimate

$$x(\varsigma)=\varsigma\Bigg(\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\varsigma)+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_4\Bigg(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_x(\xi_i)\Bigg)$$
$$-\Omega_2\Bigg(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_x(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\eta_k)\Bigg)\Bigg]$$

$$+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_1\Bigg(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_x(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\eta_k)\Bigg)$$
$$-\Omega_3\Bigg(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_x(\xi_i)\Bigg)\Bigg]\Bigg).$$

From Step II, for any $\varsigma\in\mathbb{J}$, we get that $\|\mathcal{Q}x\|\le N<\infty$. Hence the set $\mathbb{P}$ is bounded.

Using Theorem 3.4, we see that we can find $N>0$ such that $\|x\|\le N<\infty$. By Lemma 3.3 we find at least one fixed-point for $\mathcal{Q}$, which is the corresponding solution of the suggested ψ-Hilfer FBVP describing the thermostat control model (3). □

Finally, by applying the Krasnoselskii fixed-point criterion we will prove the existence of a solution.

Lemma 3.5 ([37]) *Let $\mathcal{M}$ be a closed convex bounded nonempty set in a Banach space. Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be such that (i) $\mathcal{Q}_1x+\mathcal{Q}_2y\in\mathcal{M}$ for $x,y\in\mathcal{M}$; (ii) $\mathcal{Q}_1$ is compact and continuous; and (iii) $\mathcal{Q}_2$ is a contraction. Then there exists $z\in\mathcal{M}$ such that $z=\mathcal{Q}_1z+\mathcal{Q}_2z$.*

Theorem 3.6 *Let $f:\mathbb{J}\times\mathbb{R}^2\to\mathbb{R}$ be a continuous functions satisfying (H_1). Moreover, suppose that:*

(H_3) $f(\varsigma,u,v)\le\Theta(\varsigma)$, $(\varsigma,u,v)\in\mathbb{J}\times\mathbb{R}^2$, *and* $\Theta(\varsigma)\in\mathcal{C}(\mathbb{J},\mathbb{R}^+)$.

If

$$\Lambda(L_1,L_2)-L_1\Psi_1(T,\alpha)-L_2\Psi_1(T,q+\alpha)<1, \tag{27}$$

where $\Lambda_1(\cdot,\cdot)$ and $\Psi_1(\cdot,\cdot)$ are given by (15) and (13), respectively, then the ψ-Hilfer FBVP describing the thermostat control model (3) admits a solution on $\mathbb{J}$.

Proof Let $\sup_{\varsigma\in\mathbb{J}}|\Theta(\varsigma)|=\Theta^*$ and set $B_{r_3}:=\{x\in\mathbb{E}:\|x\|\le r_3\}$, where

$$r_3\ge|A|\Psi_2(\Omega_3,\Omega_4)+|B|\Psi_2(\Omega_1,\Omega_2)+\Lambda_1\big(\Theta^*,0\big).$$

We define $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_{r_3} by

$$(\mathcal{Q}_1x)(\varsigma)=\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\varsigma),\quad\varsigma\in\mathbb{J},$$
$$(\mathcal{Q}_2x)(\varsigma)=\frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_4\Bigg(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_x(\xi_i)\Bigg)$$
$$-\Omega_2\Bigg(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_x(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\eta_k)\Bigg)\Bigg]$$
$$+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_1\Bigg(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_x(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\eta_k)\Bigg)$$
$$-\Omega_3\Bigg(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_x(\xi_i)\Bigg)\Bigg],\quad(\varsigma\in\mathbb{J}).$$

Note that $\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2$.

For any $x, y \in B_{r_3}$, we have

$$\begin{aligned}
&\bigl|(\mathcal{Q}_1x)(\varsigma)+(\mathcal{Q}_2y)(\varsigma)\bigr| \\
&\quad\le \mathcal{I}_{0^+}^{\alpha;\psi}\bigl|F_x(s)\bigr|(T)+\Psi_2(\Omega_3,\Omega_4)\Biggl(|A|+\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\bigl|F_y(s)\bigr|(\xi_i)\Biggr) \\
&\qquad+\Psi_2(\Omega_1,\Omega_2)\Biggl(|B|+\sum_{j=1}^{n}|\lambda_j|\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\bigl|F_y(s)\bigr|(\sigma_j)+\sum_{k=1}^{r}|\delta_k|\mathcal{I}_{0^+}^{\alpha;\psi}\bigl|F_y(s)\bigr|(\eta_k)\Biggr) \\
&\quad\le |A|\Psi_2(\Omega_3,\Omega_4)+|B|\Psi_2(\Omega_1,\Omega_2) \\
&\qquad+\Theta^*\Psi_1(T,\alpha)+\Theta^*\Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\Psi_1(\xi_i,\alpha-\beta_i) \\
&\qquad+\Theta^*\Psi_2(\Omega_1,\Omega_2)\Biggl(\sum_{j=1}^{n}|\lambda_j|\Psi_1(\sigma_j,\alpha-\mu_j)+\sum_{k=1}^{r}|\delta_k|\Psi_1(\eta_k,\alpha)\Biggr) \\
&\quad=|A|\Psi_2(\Omega_3,\Omega_4)+|B|\Psi_2(\Omega_1,\Omega_2)+\Lambda\bigl(\Theta^*,0\bigr)\le r_3.
\end{aligned}$$

This implies that $\mathcal{Q}_1x+\mathcal{Q}_2x\in B_{r_3}$, which satisfies Lemma 3.5(i).

Next, we show that Lemma 3.5(ii) is fulfilled.

Let $\{x_n\}\subset\mathbb{E}$ be such that $x_n\to x$ in $\mathbb{E}$. For each $\varsigma\in\mathbb{J}$, we have

$$\bigl|(\mathcal{Q}_1x_n)(\varsigma)-(\mathcal{Q}_1x)(\varsigma)\bigr|\le\mathcal{I}_{0^+}^{\alpha;\psi}\bigl|F_{x_n}(s)-F_x(s)\bigr|(T)\le\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|F_{x_n}-F_x\|.$$

Due to the continuity of f, this implies the continuity of F_x. Hence we obtain

$$\|F_{x_n}-F_x\|\to 0\quad\text{as } n\to\infty.$$

Thus Q_1 is continuous. Also, the set $Q_1B_{r_3}$ is uniformly bounded since

$$\|\mathcal{Q}_1x\|\le\Theta^*\Psi_1(T,\alpha).$$

Now we prove the compactness of $\mathcal{Q}_1$. Setting $\sup_{(\varsigma,u,v)\in\mathbb{J}\times B_{r_3}^2}|f(\varsigma,u,v)|=f^*<\infty$, for all $\varsigma_1,\varsigma_2\in\mathbb{J}$ with $0\le\varsigma_1<\varsigma_2\le T$, we have (see Step III of Theorem 3.4)

$$\begin{aligned}
&\bigl|(\mathcal{Q}_1x)(\varsigma_2)-(\mathcal{Q}_1x)(\varsigma_1)\bigr| \\
&\quad\le\bigl|\mathcal{I}_{0^+}^{\alpha;\psi}F_x(s)(\varsigma_2)-\mathcal{I}_{0^+}^{\alpha;\psi}F_x(s)(\varsigma_1)\bigr| \\
&\quad\le\frac{f^*}{\Gamma(\alpha+1)}\Bigl[\bigl(\psi(\varsigma_2)-\psi(\varsigma_1)\bigr)^{\alpha} \qquad (28)\\
&\qquad+\bigl|\bigl(\psi(\varsigma_2)-\psi(0)\bigr)^{\alpha}-\bigl(\psi(\varsigma_2)-\psi(\varsigma_1)\bigr)^{\alpha}-\bigl(\psi(\varsigma_1)-\psi(0)\bigr)^{\alpha}\bigr|\Bigr].
\end{aligned}$$

Thus the right-hand side of (28) (independently of the unknown variable x) approaches 0 as $\varsigma_2\to\varsigma_1$. Therefore $\mathcal{Q}_1$ is equicontinuous. Thus by the Arzelà–Ascoli theorem $\mathcal{Q}_1$ is relatively compact.

Furthermore, it is easy to compute, utilizing (27), that $\mathcal{Q}_2$ is a contraction, and so Lemma 3.5(iii) holds. Therefore Lemma 3.5 is fulfilled, and thus a solution exists on $\mathbb{J}$ for the ψ-Hilfer FBVP describing the thermostat control model (3). □

4 The Ulam stability analysis

In this section, we prove that the ψ-Hilfer FBVP describing the thermostat control model (3) is Ulam–Hyers ($\mathbb{UH}$) stable, generalized Ulam–Hyers ($\mathbb{GUH}$) stable, Ulam–Hyers–Rassias ($\mathbb{UHR}$) stable, and generalized Ulam–Hyers–Rassias ($\mathbb{GUHR}$) stable.

Definition 4.1 ([38]) The ψ-Hilfer FBVP describing the thermostat control model (3) is said to be $\mathbb{UH}$ stable if there exists $M_f \in \mathbb{R}^+$ such that for any $\epsilon > 0$ and for every solution $z \in \mathbb{E}$ of

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) - f\big(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)\big)\right| \le \epsilon, \tag{29}$$

there is a solution $x \in \mathbb{E}$ of the ψ-Hilfer FBVP describing the thermostat control model (3) such that

$$\left|z(\varsigma) - x(\varsigma)\right| \le M_f \epsilon, \quad \varsigma \in \mathbb{J}. \tag{30}$$

Definition 4.2 ([38]) The ψ-Hilfer FBVP describing the thermostat control model (3) is said to be $\mathbb{GUH}$ stable if there exists $\mathcal{B} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ subject to $\mathcal{B}(0) = 0$ such that, for every solution $z \in \mathbb{E}$ of

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) - f\big(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)\big)\right| \le \epsilon\mathcal{B}(\varsigma), \tag{31}$$

there is a solution $x \in \mathbb{E}$ of ψ-Hilfer FBVP describing the thermostat control model (3) such that

$$\left|z(\varsigma) - x(\varsigma)\right| \le \mathcal{B}(\epsilon), \quad \varsigma \in \mathbb{J}. \tag{32}$$

Definition 4.3 ([38]) The ψ-Hilfer FBVP describing the thermostat control model (3) is said to be $\mathbb{UHR}$ stable by terms of $\mathcal{B} \in \mathcal{C}(\mathbb{J}, \mathbb{R}^+)$ if there exists $M_{f,\mathcal{B}} \in \mathbb{R}^+$ such that for each $\epsilon > 0$ and for every solution $z \in \mathbb{E}$ of (31), there is a solution $x \in \mathbb{C}$ of the ψ-Hilfer FBVP describing the thermostat control model (3) such that

$$\left|z(\varsigma) - x(\varsigma)\right| \le M_{f,\mathcal{B}}\epsilon\mathcal{B}(\varsigma), \quad \varsigma \in \mathbb{J}. \tag{33}$$

Definition 4.4 ([38]) The ψ-Hilfer FBVP describing the thermostat control model (3) is said to be $\mathbb{GUHR}$ stable by terms of $\mathcal{B} \in \mathcal{C}(\mathbb{J}, \mathbb{R}^+)$ if there is $M_{f,\mathcal{B}} \in \mathbb{R}^+$ such that for every solution $z \in \mathbb{E}$ of

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) - f\big(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)\big)\right| \le \mathcal{B}(\varsigma), \tag{34}$$

there is a solution $x \in \mathbb{E}$ of the ψ-Hilfer FBVP describing the thermostat control model (3) such that

$$\left|z(\varsigma) - x(\varsigma)\right| \le M_{f,\mathcal{B}}\mathcal{B}(\varsigma), \quad \varsigma \in \mathbb{J}. \tag{35}$$

Remark 4.5 We easily see that:

(a_1) Def. 4.1 ⇒ Def. 4.2;

(a_2) Def. 4.3 ⇒ Def. 4.4;

(a_3) Def. 4.3 for $\mathcal{B}(\varsigma) = 1$ ⇒ Def. 4.1.

Remark 4.6 $z \in \mathbb{E}$ is a solution of (29) if there exists f $u \in \mathbb{E}$ (where u depends on z) such that:

(i) $|\nu(\varsigma)| \le \epsilon$, $\forall \varsigma \in \mathbb{J}$.

(ii) ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) = f(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)) + u(\varsigma)$, $\varsigma \in \mathbb{J}$.

Remark 4.7 $z \in \mathbb{E}$ is a solution of (31) if there exists f $\nu \in \mathbb{E}$ (depending on z) such that:

(i) $|\nu(\varsigma)| \le \epsilon\mathcal{B}(\varsigma)$, $\forall \varsigma \in \mathbb{J}$.

(ii) ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) = f(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)) + \nu(\varsigma)$, $\varsigma \in \mathbb{J}$.

Remark 4.8 There exist an increasing function $\mathcal{B} \in \mathcal{C}(\mathbb{J}, \mathbb{R}^+)$ and a constant $n_{\mathcal{B}} > 0$ such that for each $\varsigma \in \mathbb{J}$, we have the following integral inequality:

$$\mathcal{I}_{0^+}^{\alpha;\psi} \mathcal{B}(\varsigma) \le n_{\mathcal{B}} \mathcal{B}(\varsigma). \tag{36}$$

4.1 The Ulam–Hyers stability

First, we give the following lemma, which will be utilized in the arguments on $\mathbb{UH}$ and $\mathbb{GUH}$ stability of problem (3).

Lemma 4.9 *Let $\rho \in [0,1]$ and $\alpha \in (1,2]$, and let $z \in \mathbb{E}$ be the solution of (29). Then $z \in \mathbb{E}$ fulfills the integral inequality*

$$\left|z(\varsigma) - \mathcal{X}_z(\varsigma) - \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\varsigma)\right| \le \Lambda(1,0)\epsilon, \tag{37}$$

where

$$\begin{aligned} \mathcal{X}_z(\varsigma) = {}& \frac{(\psi(\varsigma) - \psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)} \left[\Omega_4 \left(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_z(\xi_i) \right) \right. \\ & \left. - \Omega_2 \left(B - \sum_{j=1}^{n} \lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_z(\sigma_j) - \sum_{k=1}^{r} \delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\eta_k) \right) \right] \\ & + \frac{(\psi(\varsigma) - \psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)} \left[\Omega_1 \left(B - \sum_{j=1}^{n} \lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_z(\sigma_j) - \sum_{k=1}^{r} \delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\eta_k) \right) \right. \\ & \left. - \Omega_3 \left(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_z(\xi_i) \right) \right] \end{aligned} \tag{38}$$

with Ω_1, Ω_2, Ω_3, Ω_4, Ω, and $\Lambda(\cdot,\cdot)$ given by (6), (7) (8), (9), (10), and (15), respectively.

Proof Let z satisfy (29). By Remark 4.6(ii) and Lemma 2.8 we obtain

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(\varsigma) = f(\varsigma, z(\theta\varsigma), \mathcal{I}_{0^+}^{q;\psi} z(\varepsilon\varsigma)) + u(\varsigma), \quad \varsigma \in (0,T], \\ \sum_{i=1}^{m} \omega_i {}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} z(\xi_i) = A, \qquad \sum_{j=1}^{n} \lambda_j {}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} z(\sigma_j) + \sum_{k=1}^{r} \delta_k z(\eta_k) = B. \end{cases} \tag{39}$$

Then the solution of (39) can be written in the form

$$\begin{aligned}
z(\varsigma) &= \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_4\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_z(\xi_i)\Bigg) \\
&\quad - \Omega_2\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_z(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\eta_k)\Bigg)\Bigg] \\
&\quad + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_1\Bigg(B - \sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_z(\sigma_j) - \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\eta_k)\Bigg) \\
&\quad - \Omega_3\Bigg(A - \sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_z(\xi_i)\Bigg)\Bigg] \\
&\quad + \mathcal{I}_{0^+}^{\alpha;\psi} u(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_2\Bigg(\sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} u(\sigma_j) + \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} u(\eta_k)\Bigg) \\
&\quad - \Omega_4\sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} u(\xi_i)\Bigg] + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_3\sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} u(\xi_i) \\
&\quad - \Omega_1\Bigg(\sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} u(\sigma_j) + \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} u(\eta_k)\Bigg)\Bigg].
\end{aligned}$$

Remark 4.6(i) implies that

$$\begin{aligned}
&\left|z(\varsigma) - \mathcal{X}_z(\varsigma) - \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\varsigma)\right| \\
&= \Bigg|\mathcal{I}_{0^+}^{\alpha;\psi} u(\varsigma) + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_2\Bigg(\sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} u(\sigma_j) + \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} u(\eta_k)\Bigg) \\
&\quad - \Omega_4\sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} u(\xi_i)\Bigg] + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_3\sum_{i=1}^{m}\omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} u(\xi_i) \\
&\quad - \Omega_1\Bigg(\sum_{j=1}^{n}\lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} u(\sigma_j) + \sum_{k=1}^{r}\delta_k \mathcal{I}_{0^+}^{\alpha;\psi} u(\eta_k)\Bigg)\Bigg]\Bigg| \\
&\le \Bigg\{\Psi_1(T,\alpha) + \frac{1}{|\Omega|}\big(|\Omega_4|\Psi_1(T,\gamma-1) + |\Omega_3|\Psi_1(T,\gamma-2)\big)\sum_{i=1}^{m}|\omega_i|\Psi_1(\xi_i,\alpha-\beta_i) \\
&\quad + \frac{1}{|\Omega|}|\Omega_2|\big(\Psi_1(T,\gamma-1) + |\Omega_1|\Psi_1(T,\gamma-2)\big)\sum_{j=1}^{n}|\lambda_j|\Psi_1(\sigma_j,\alpha-\mu_j) \\
&\quad + \frac{1}{|\Omega|}\big(|\Omega_2|\Psi_1(T,\gamma-1) + |\Omega_1|\Psi_1(T,\gamma-2)\big)\sum_{k=1}^{r}|\delta_k|\Psi_1(\eta_k,\alpha)\Bigg\}\epsilon \\
&= \Bigg\{\Psi_1(T,\alpha) + \Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|\Psi_1(\xi_i,\alpha-\beta_i) \\
&\quad + \Psi_2(\Omega_1,\Omega_2)\Bigg(\sum_{j=1}^{n}|\lambda_j|\Psi_1(\sigma_j,\alpha-\mu_j) + \sum_{k=1}^{r}|\delta_k|\Psi_1(\eta_k,\alpha)\Bigg)\Bigg\}\epsilon
\end{aligned}$$

$$= \Lambda(1,0)\epsilon.$$

Inequality (37) is achieved. □

Next, we prove the $\mathbb{UH}$ and $\mathbb{GUH}$ stability of solution to problem (3).

Theorem 4.10 *Let $f : \mathbb{J} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, and let (H_1) be satisfied subject to*

$$L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha) < 1.$$

Then the ψ-Hilfer FBVP describing the thermostat control model (3) is $\mathbb{UH}$ and $\mathbb{GUH}$ stable in $\mathbb{E}$.

Proof Let $z \in \mathbb{E}$ satisfy (29), and let $x \in \mathbb{E}$ be a unique solution of the ψ-Hilfer FBVP describing the thermostat control model (3) with nonlocal conditions of the form

$$\sum_{i=1}^{m} \omega_i {}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} x(\xi_i) = A \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j {}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} x(\sigma_j) + \sum_{k=1}^{r} \delta_k x(\eta_k) = B.$$

By using Lemma 2.8 we have

$$x(\varsigma) = \mathcal{X}_x(\varsigma) + \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\varsigma),$$

where

$$\begin{aligned}
\mathcal{X}_x(\varsigma) = {}& \frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)} \Bigg[\Omega_4\Bigg(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_x(\xi_i)\Bigg) \\
& - \Omega_2\Bigg(B - \sum_{j=1}^{n} \lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_x(\sigma_j) - \sum_{k=1}^{r} \delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\eta_k)\Bigg)\Bigg] \\
& + \frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)} \Bigg[\Omega_1\Bigg(B - \sum_{j=1}^{n} \lambda_j \mathcal{I}_{0^+}^{\alpha-\mu_j;\psi} F_x(\sigma_j) - \sum_{k=1}^{r} \delta_k \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\eta_k)\Bigg) \\
& - \Omega_3\Bigg(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha-\beta_i;\psi} F_x(\xi_i)\Bigg)\Bigg].
\end{aligned} \tag{40}$$

On other hand, if ${}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} x(\xi_i) = {}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi} z(\xi_i)$, ${}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} x(\sigma_j) = {}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi} z(\sigma_j)$, and $x(\eta_k) = z(\eta_k)$, then $\mathcal{X}_x(\varsigma) = \mathcal{X}_z(\varsigma)$. By applying the triangle inequality $|x+y| \le |x| + |y|$ and Lemma 4.9, for $\varsigma \in \mathbb{J}$, we obtain

$$\begin{aligned}
&\big|z(\varsigma) - x(\varsigma)\big| \\
&\quad = \big|z(\varsigma) - \mathcal{X}_x(\varsigma) - \mathcal{I}_{0^+}^{\alpha;\psi} F_x(\varsigma)\big| \\
&\quad \le \big|z(\varsigma) - \mathcal{X}_z(\varsigma) - \mathcal{I}_{0^+}^{\alpha;\psi} F_z(\varsigma)\big| + \mathcal{I}_{0^+}^{\alpha;\psi} \big|F_z(s) - F_x(s)\big|(\varsigma) + \big|\mathcal{X}_z(\varsigma) - \mathcal{X}_x(\varsigma)\big| \\
&\quad \le \Lambda(1,0)\epsilon + \big(L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha)\big)\big|z(\varsigma) - x(\varsigma)\big|,
\end{aligned}$$

which implies that

$$\left|z(\varsigma)-x(\varsigma)\right| \le \frac{\Lambda(1,0)}{1-(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha))}\epsilon.$$

By setting

$$M_f = \frac{\Lambda(1,0)}{1-(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha))}$$

we get $|z(\varsigma)-x(\varsigma)| \le M_f\epsilon$. Therefore problem (3) is $\mathbb{UH}$ stable. Moreover, if we take $\mathcal{B}(\epsilon) = M_f\epsilon$ with $\mathcal{B}(0) = 0$, then the ψ-Hilfer FBVP describing the thermostat control model (3) is $\mathbb{GUH}$ stable. □

4.2 The Ulam–Hyers–Rassias stability

This lemma will is helpful in the arguments on $\mathbb{UHR}$ and $\mathbb{GUHR}$ stability of our results.

Lemma 4.11 *Let $\rho \in [0,1]$ and $\alpha \in (1,2]$, and let $z \in \mathbb{E}$ be the solution of* (31). *Then $z \in \mathbb{E}$ fulfills the integral inequality*

$$\left|z(\varsigma)-\mathcal{X}_z(\varsigma)-\mathcal{I}_{0^+}^{\alpha;\psi}F_z(\varsigma)\right| \le \Theta\epsilon n_{\mathcal{B}}\mathcal{B}(\varsigma), \tag{41}$$

where

$$\Theta = 1+\Psi_2(\Omega_1,\Omega_2)\left(\sum_{j=1}^{n}|\lambda_j|+\sum_{k=1}^{r}|\delta_k|\right)+\Psi_2(\Omega_3,\Omega_4)\sum_{i=1}^{m}|\omega_i|, \tag{42}$$

and $\mathcal{X}_z(\varsigma)$ is defined by (38).

Proof Let z satisfy (31). By Remark 4.7(ii) and Lemma 2.8 the solution of the problem

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}z(\varsigma) = f(\varsigma,z(\theta\varsigma),\mathcal{I}_{0^+}^{q;\psi}z(\varepsilon\varsigma))+\nu(\varsigma), & \varsigma\in(0,T], \\ \sum_{i=1}^{m}\omega_i{}^H\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi}z(\xi_i) = A, \qquad \sum_{j=1}^{n}\lambda_j{}^H\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi}z(\sigma_j)+\sum_{k=1}^{r}\delta_k z(\eta_k) = B, \end{cases} \tag{43}$$

can be written in the following form:

$$\begin{aligned} z(\varsigma) = {}& \mathcal{I}_{0^+}^{\alpha;\psi}F_z(\varsigma)+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\left[\Omega_4\left(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_z(\xi_i)\right)\right. \\ &\left.-\Omega_2\left(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_z(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_z(\eta_k)\right)\right] \\ &+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\left[\Omega_1\left(B-\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}F_z(\sigma_j)-\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}F_z(\eta_k)\right)\right. \\ &\left.-\Omega_3\left(A-\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}F_z(\xi_i)\right)\right] \\ &+\mathcal{I}_{0^+}^{\alpha;\psi}\nu(\varsigma)+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\left[\Omega_2\left(\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\nu(\sigma_j)+\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}\nu(\eta_k)\right)\right. \end{aligned}$$

$$-\Omega_4\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\nu(\xi_i)\Bigg]+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_3\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\nu(\xi_i)$$
$$-\Omega_1\Bigg(\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\nu(\sigma_j)+\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}u\nu(\eta_k)\Bigg)\Bigg].$$

From Remarks 4.7(i) and 4.8 we obtain the following estimate:

$$\begin{aligned}
&\left|z(\varsigma)-\mathcal{X}_z(\varsigma)-\mathcal{I}_{0^+}^{\alpha;\psi}F_z(\varsigma)\right| \\
&=\Bigg|\mathcal{I}_{0^+}^{\alpha;\psi}\nu(\varsigma)+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)}\Bigg[\Omega_2\Bigg(\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\nu(\sigma_j)+\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}\nu(\eta_k)\Bigg) \\
&\quad-\Omega_4\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\nu(\xi_i)\Bigg]+\frac{(\psi(\varsigma)-\psi(0))^{\gamma-2}}{\Omega\Gamma(\gamma-1)}\Bigg[\Omega_3\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha-\beta_i;\psi}\nu(\xi_i) \\
&\quad-\Omega_1\Bigg(\sum_{j=1}^{n}\lambda_j\mathcal{I}_{0^+}^{\alpha-\mu_j;\psi}\nu(\sigma_j)+\sum_{k=1}^{r}\delta_k\mathcal{I}_{0^+}^{\alpha;\psi}\nu(\eta_k)\Bigg)\Bigg]\Bigg| \\
&\le\Bigg\{1+\frac{\Psi_1(T,\gamma-1)}{|\Omega|}\Bigg[|\Omega_2|\Bigg(\sum_{j=1}^{n}|\lambda_j|+\sum_{k=1}^{r}|\delta_k|\Bigg)+|\Omega_4|\sum_{i=1}^{m}|\omega_i|\Bigg] \\
&\quad+\frac{\Psi_1(T,\gamma-2)}{|\Omega|}\Bigg[|\Omega_3|\sum_{i=1}^{m}|\omega_i|+|\Omega_1|\Bigg(\sum_{j=1}^{n}|\lambda_j|+\sum_{k=1}^{r}|\delta_k|\Bigg)\Bigg]\Bigg\}\epsilon n_{\mathcal{B}}\mathcal{B}(\varsigma) \\
&=\Theta\epsilon n_{\mathcal{B}}\mathcal{B}(\varsigma),
\end{aligned}$$

from which we get (41). □

Next, we check $\mathbb{UHR}$ and $\mathbb{GUHR}$ stability of solution to problem (3).

Theorem 4.12 *Let $f:\mathbb{J}\times\mathbb{R}^2\to\mathbb{R}$ be a continuous mapping under the assumption (H_1), and let (36) be satisfied. If $(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha))<1$, then the ψ-Hilfer FBVP describing the thermostat control model (3) is $\mathbb{UHR}$ and $\mathbb{GUHR}$ stable in $\mathbb{E}$.*

Proof Let $z\in\mathbb{E}$ satisfy (31), and let x be a unique solution of (3). By Lemma 4.11 we get

$$x(\varsigma)=\mathcal{X}_x(\varsigma)+\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\varsigma),$$

where $\mathcal{X}_x(\varsigma)$ is given by (40). Similarly, if ${}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi}x(\xi_i)={}^{H}\mathfrak{D}_{0^+}^{\beta_i,\rho;\psi}z(\xi_i)$, ${}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi}x(\sigma_j)={}^{H}\mathfrak{D}_{0^+}^{\mu_j,\rho;\psi}z(\sigma_j)$, and $x(\eta_k)=z(\eta_k)$, then $\mathcal{X}_x(\varsigma)=\mathcal{X}_z(\varsigma)$. Applying the triangle inequality with Lemma 4.11, for $\varsigma\in\mathbb{J}$, we estimate

$$\begin{aligned}
\left|z(\varsigma)-x(\varsigma)\right|&=\left|z(\varsigma)-\mathcal{X}_x(\varsigma)-\mathcal{I}_{0^+}^{\alpha;\psi}F_x(\varsigma)\right| \\
&\le\left|z(\varsigma)-\mathcal{X}_z(\varsigma)-\mathcal{I}_{0^+}^{\alpha;\psi}F_z(\varsigma)\right|+\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_z(s)-F_x(s)\right|(\varsigma)+\left|\mathcal{X}_z(\varsigma)-\mathcal{X}_x(\varsigma)\right| \\
&\le\Theta\epsilon n_{\mathcal{B}}\mathcal{B}(\varsigma)+\left(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha)\right)\left|z(\varsigma)-x(\varsigma)\right|,
\end{aligned}$$

where Θ and $\Psi_1(\cdot,\cdot)$ are defined by (42) and (13), respectively. Then

$$\left|z(\varsigma)-x(\varsigma)\right| \leq \frac{\Theta n_{\mathcal{B}}}{1-(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha))}\mathcal{B}(\varsigma)\epsilon.$$

By setting

$$M_{f,\mathcal{B}} = \frac{\Theta n_{\mathcal{B}}}{1-(L_1\Psi_1(T,\alpha)+L_2\Psi_1(T,q+\alpha))}$$

we get the estimate

$$\left|z(\varsigma)-x(\varsigma)\right| \leq M_{f,\mathcal{B}}\mathcal{B}(\varsigma)\epsilon. \tag{44}$$

This proves that the ψ-Hilfer FBVP describing the thermostat control model (3) is $\mathbb{UHR}$ stable. Furthermore, taking $\epsilon = 1$ in (44) with $\mathcal{B}(0) = 0$, the ψ-Hilfer FBVP describing the thermostat control model (3) is $\mathbb{GUHR}$ stable. □

5 Example

In this section, we provide some examples compatible to the exactitude and applicability of our main results.

Example 5.1 Consider the ψ-Hilfer FBVP describing the thermostat control model

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{3};e^{\frac{\sqrt{\varsigma}}{2}}} x(\varsigma) = f(\varsigma, x(\theta\varsigma), \mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(\varepsilon\varsigma)), \quad \varsigma \in (0,3/2], \\ \sum_{i=1}^{m}(\frac{\tan^{-1}(i)}{(i+1)^2}){}^H\mathfrak{D}_{0^+}^{\frac{\pi}{2i+1},\frac{1}{3};e^{\frac{\sqrt{\varsigma}}{2}}} x(\frac{i}{i+1}) = \frac{\sqrt{3}}{2}, \\ \sum_{j=1}^{n}(\sin(\frac{\pi j}{j+1})){}^H\mathfrak{D}_{0^+}^{\frac{\sqrt{j}}{j+1},\frac{1}{3};e^{\frac{\sqrt{\varsigma}}{2}}} x(\frac{j+1}{2j+3}) + \sum_{k=1}^{r}(e^{-\sqrt{k\pi}})x(\frac{\sqrt{k}}{k+1}) = -\frac{1}{3}. \end{cases} \tag{45}$$

Here $\alpha = 3/2$, $q = \sqrt{\pi}$, $\rho = 1/3$, $\psi(\varsigma) = \exp(\sqrt{\varsigma}/2)$, $T = 3/2$, $m = 2$, $n = 2$, $r = 2$, $\beta_i = \pi/(2i+1)$, $\mu_j = \sqrt{j}/(j+1)$, $\omega_i = (\tan^{-1}(i))/((i+1)^2)$, $\lambda_j = \sin((\pi j)/(j+1))$, $\delta_k = \exp(-\sqrt{k\pi})$, $\xi_i = i/(i+1)$, $\sigma_j = (j+1)/(2j+3)$, $\eta_k = \sqrt{k}/(k+1)$, $i = j = k = 1,2$, $A = \sqrt{3}/2$, and $B = -1/3$. From this information we can calculate that $\Omega_1 \approx 0.2318892799$, $\Omega_2 \approx -0.2621694875$, $\Omega_3 \approx 0.2107768913$, $\Omega_4 \approx 0.1675500424$, and $\Omega \approx 0.09411232825 \neq 0$.

(i) Consider the function

$$\begin{aligned} & f\big(\varsigma, x(0.2\varsigma), \mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(0.5\varsigma)\big) \\ & \quad = \frac{\varsigma}{\varsigma+1} + \frac{2\varsigma+1}{2-\cos^2\pi\varsigma}\cdot\frac{|x(0.2\varsigma)|}{4+|x(0.2\varsigma)|} + \frac{(4\varsigma-1)^2}{4}\cdot\frac{|\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(0.5\varsigma)|}{25+|\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(0.5\varsigma)|} \end{aligned}$$

with $\theta = 1/5$ and $\varepsilon = 1/2$. For $x_i, y_i \in \mathbb{R}$, $i = 1,2$, and $\varsigma \in [0,3/2]$, we can find that

$$\left|f(\varsigma,x_1,y_1) - f(\varsigma,x_2,y_2)\right| \leq \frac{1}{2}|x_1-x_2| + \frac{1}{4}|y_1-y_2|.$$

Assumption (H_1) is satisfied with $L_1 = \frac{1}{2}$ and $L_2 = \frac{1}{4}$. Hence

$$\Lambda(1/2, 1/4) \approx 0.7369944526 < 1.$$

All assumptions of Theorem 3.2 are fulfilled; thus the ψ-Hilfer FBVP describing the thermostat control model (45) has exactly one solution on $[0, 3/2]$. Further,

$$M_f = \frac{\Lambda(1,0)}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 1.066442242 > 0.$$

By Theorem 4.10 the ψ-Hilfer FBVP describing the thermostat control model (45) is both $\mathbb{UH}$ and $\mathbb{GUH}$ stable on $[0, 3/2]$. Setting $\mathcal{B}(\varsigma) = \psi(\varsigma) - \psi(0)$, by Proposition 2.5(i) we easily compute that

$$\mathcal{I}_{0^+}^{\alpha;\psi}\mathcal{B}(\varsigma) = \frac{1}{\Gamma(\frac{7}{2})}\big(\psi(\varsigma) - \psi(0)\big)^{\frac{5}{2}}\Theta(\varsigma) \le \frac{8(e^{\frac{\sqrt{6}}{4}} - 1)^{\frac{5}{2}}}{15\sqrt{\pi}}\Theta(\varsigma).$$

Then inequality (36) is satisfied with $n_\Theta = \frac{8(e^{\frac{\sqrt{6}}{4}}-1)^{\frac{5}{2}}}{15\sqrt{\pi}} > 0$ and $\Theta \approx 2.901034837$. It follows that

$$M_{f,\mathcal{B}} = \frac{\Theta n_{\mathcal{B}}}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 0.4142937053 > 0.$$

Therefore by Theorem 4.12 the ψ-Hilfer FBVP describing the thermostat control model (45) is both $\mathbb{UHR}$ and $\mathbb{GUHR}$ stable on $[0, 3/2]$.

(ii) Consider the function

$$\begin{aligned}
&f\big(\varsigma, x(\sqrt{2}\varsigma/2), \mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}}x(\sqrt{3}\varsigma/2)\big) \\
&\quad= \frac{\sqrt{2}e^{-\varsigma}}{(\varsigma+2)^2} + \frac{\sqrt{2\varsigma+1}}{\varsigma+5}\cdot\sin^{-1}\big|x(\sqrt{2}\varsigma/2)\big| \\
&\qquad+ \frac{4\cos^2(\pi\varsigma)}{e^{\varsigma}+1}\cdot\frac{|\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}}x(\sqrt{3}\varsigma/2)|}{4 + |\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}}x(\sqrt{3}\varsigma/2)|}
\end{aligned}$$

with $\theta = \sqrt{2}/2$ and $\varepsilon = \sqrt{3}/2$. For $x_i, y_i \in \mathbb{R}$, $i = 1, 2$, and $\varsigma \in [0, 3/2]$, we can find that

$$\begin{aligned}
&\big|f(\varsigma, x_1, y_1) - f(\varsigma, x_2, y_2)\big| \\
&\quad\le \frac{\sqrt{2\varsigma+1}}{\varsigma+5}|x_1 - x_2| + \frac{\cos^2(\pi\varsigma)}{e^{\varsigma}+1}|y_1 - y_2| \le \frac{2}{5}|x_1 - x_2| + \frac{1}{2}|y_1 - y_2|.
\end{aligned}$$

This means that assumption (H_1) is satisfied with $L_1 = 2/5$, $L_2 = 1/2$, and $\Lambda(2/5, 1/2) \approx 0.6261447320 < 1$. Therefore, for any $x, y \in \mathbb{R}$ and $\varsigma \in [0, 3/2]$, we can estimate

$$\big|f(\varsigma, x, y)\big| \le \frac{\sqrt{2}e^{-\varsigma}}{(\varsigma+2)^2} + \frac{\sqrt{2\varsigma+1}}{\varsigma+5}|x| + \frac{\cos^2(\pi\varsigma)}{e^{\varsigma}+1}|y|.$$

Assumption (H_2) is also valid with $k_1 = (\sqrt{2}e^{-\varsigma})/((\varsigma + 2)^2)$, $k_2(\varsigma) = (\sqrt{2\varsigma + 1})/(\varsigma + 5)$, $k_3(\varsigma) = (\cos^2(\pi\varsigma))/(e^{\varsigma} + 1)$, and $k_1^* = \sqrt{2}/4$, $k_2^* = 2/5$ $k_3^* = 1/2$. Therefore the assumptions of Theorem 3.4 are fulfilled, and thus the ψ-Hilfer FBVP describing the thermostat control model (45) has at least one solution on $[0, 3/2]$. Moreover, we get

$$M_f = \frac{\Lambda(1,0)}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 0.8546613342 > 0.$$

Therefore by Theorem 4.10 problem (45) is both $\mathbb{UH}$ and $\mathbb{GUH}$ stable on $[0, 3/2]$. Furthermore, if we set $\mathcal{B}(\varsigma) = (\psi(\varsigma) - \psi(0))^{3/2}$, then by Proposition 2.5(i) we easily compute that

$$\mathcal{I}_{0^+}^{\alpha;\psi}\mathcal{B}(\varsigma) = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}+\alpha)}\big(\psi(\varsigma) - \psi(0)\big)^{\frac{3}{2}+\alpha}\Theta(\varsigma) \le \frac{(e^{\frac{\sqrt{6}}{4}} - 1)^3}{8\sqrt{\pi}}\Theta(\varsigma).$$

Then inequality (36) is satisfied with $n_\Theta = \frac{(e^{\frac{\sqrt{6}}{4}}-1)^3}{8\sqrt{\pi}} > 0$ and $\Theta \approx 2.901034837$. It follows that

$$M_{f,\mathcal{B}} = \frac{\Theta n_{\mathcal{B}}}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 0.1683734132 > 0.$$

Therefore by Theorem 4.12 problem (45) is both $\mathbb{UHR}$ and $\mathbb{GUHR}$ stable on $[0, 3/2]$.

(iii) Consider the function

$$\begin{aligned} &f\big(\varsigma, x(\varsigma/3), \mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(\varsigma/4)\big) \\ &\quad= \frac{e^{-3\varsigma}}{\varsigma+1} + \frac{3}{2\varsigma+5}\cdot\tan^{-1}\big|x(\varsigma/3)\big| + \frac{2}{(2\varsigma+3)^2}\cdot\frac{|\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(\varsigma/4)|}{2 + |\mathcal{I}_{0^+}^{\sqrt{\pi};e^{\frac{\sqrt{\varsigma}}{2}}} x(\varsigma/4)|}. \end{aligned}$$

with $\theta = 1/3$ and $\varepsilon = 1/4$. For $x_i, y_i \in \mathbb{R}$, $i = 1, 2$, and $\varsigma \in [0, 3/2]$, we can find that

$$\big|f(\varsigma, x_1, y_1) - f(\varsigma, x_2, y_2)\big| \le \frac{3}{5}|x_1 - x_2| + \frac{1}{9}|y_1 - y_2|.$$

This means that assumption (H_1) is satisfied with $L_1 = 3/5$ and $L_2 = 1/9$. We have

$$\Lambda(L_1, L_2) - L_1\Psi_1(T,\alpha) - L_2\Psi_1(T,q+\alpha) \approx 0.5034171857 < 1$$

and

$$\big|f(\varsigma, x, y)\big| \le \frac{e^{-3\varsigma}}{\varsigma+1} + \frac{3\pi}{2(2\varsigma+5)} + \frac{1}{(2\varsigma+3)^2},$$

which satisfy assumption (H_3) and (27), respectively. Applying Theorem 3.6, we find a solution for the ψ-Hilfer FBVP describing the thermostat control model (45) on $[0, 3/2]$. Moreover, we can also calculate that

$$M_f = \frac{\Lambda(1,0)}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 1.341638979 > 0.$$

Therefore by Theorem 4.10 problem (45) is both $\mathbb{UH}$ and $\mathbb{GUH}$ stable on $[0,3/2]$. In addition, if we set $\mathcal{B}(\varsigma) = (\psi(\varsigma) - \psi(0))^3$, then by Proposition 2.5(i) we easily compute that

$$\mathcal{I}_{0^+}^{\alpha;\psi}\mathcal{B}(\varsigma) = \frac{\Gamma(4)}{\Gamma(4+\alpha)}\big(\psi(\varsigma) - \psi(0)\big)^{3+\alpha}\Theta(\varsigma) \leq \frac{64(e^{\frac{\sqrt{6}}{4}} - 1)^{\frac{9}{2}}}{315\sqrt{\pi}}\Theta(\varsigma).$$

Then inequality (36) is satisfied with $n_\Theta = \frac{64(e^{\frac{\sqrt{6}}{4}} - 1)^{\frac{9}{2}}}{315\sqrt{\pi}} > 0$ and $\Theta \approx 2.901034837$. It follows that

$$M_{f,\mathcal{B}} = \frac{\Theta n_{\mathcal{B}}}{1 - (L_1\Psi_1(T,\alpha) + L_2\Psi_1(T,q+\alpha))} \approx 0.2424862463 > 0.$$

Therefore by Theorem 4.12 the ψ-Hilfer FBVP describing the thermostat control model (45) is both $\mathbb{UHR}$ and $\mathbb{GUHR}$ stable on $[0,3/2]$.

6 Conclusions

Oriented by the recent trend that supports considering some well-known physical models in the frame of generalized fractional operators, we study the model of thermostat that controls the heating or cooling sources. The model is described using integro-differential equation in the context of the generalized ψ-Hilfer fractional operator. Unlike previous research works, we proved the existence and uniqueness of solutions for a generalized category of the ψ-Hilfer FBVP describing the thermostat control model. The fixed-point approaches due to Banach, Schaefer, and Krasnoselskii are used to establish the relevant results. Different kinds of the Ulam stability, such as $\mathbb{UH}$, $\mathbb{GUH}$, $\mathbb{UHR}$, and $\mathbb{GUHR}$ stability are also investigated. Moreover, the results are well confirmed by several numerical examples. Note that we can continue such a research by extending it to coupled systems of fractional thermostat BVPs in terms of newly defined fractional operators with nonsingular kernels. Also, to obtain the exact solutions of such a coupled system, there are different numerical algorithms, which we can implement on these generalized fractional models in the next studies.

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