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ON THE QUANTITY $\delta_s(g(z), f)$ OF GAPPY ENTIRE FUNCTIONS

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1. Introduction.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function. We denote by $\Lambda_f=\{\lambda_k\}$, $M_f=\{u_k\}$, $k=1, 2, \cdots$ the sequences of exponents n for which $a_n \neq 0$ and $a_n=0$ respectively, arranged in increasing order. Also let $g(z)$ be an arbitrary meromorphic function in the plane growing slowly compared with the function $f(z)$, i.e., $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$.

If $f(z)$ has finite order we define

$$\delta_s(g(z), f)=1-\underline{\lim}_{r\to\infty}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

If $f(z)$ has infinite order, let E be any set in $(1, \infty)$ having finite length. We define

$$\delta_s(g(z), f)=1-\sup_E \underline{\lim}_{r\to\infty, r\notin E}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}=\inf_E \overline{\lim}_{r\to\infty, r\notin E}\frac{m\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

Obviously

$$\delta(g(z), f)=1-\overline{\lim}_{r\to\infty}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}\leqq \delta_s(g(z), f).$$

In [1], we obtained the following

THEOREM A. *Let dn be the highest common factor of all the numbers $\lambda_{m+1}-\lambda_m$ for $m\geqq n$ and suppose that*

$$d_n \longrightarrow \infty \qquad \text{as } n\to\infty. \tag{1.1}$$

Then

$$\delta_s(g(z), f)=0 \tag{1.2}$$

for every entire function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$.

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In this paper we further prove that (1.2) is also valid for every meromorphic function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$. That is, we shall prove

THEOREM 1. *With the hypotheses of Theorem A, we have*

$$\delta_s(g(z), f)=0$$

for every meromorphic function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$.

From (1.1), we see that Λ_f is very 'thin' in a sense. For the gappy entire function $f(z)$ with 'thickish' Λ_f, we have

THEOREM 2. *Suppose that $b(\geqq 0)$, $d(\geqq 2)$ are integers such that $b+kd\in M_f$ for $k\geqq 1$. Then for every meromorphic function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$, we have*

$$\delta_s(g(z), f)\leqq 1-\frac{1}{d} \tag{1.3}$$

except possibly if, for $0<|z|<\varepsilon$, $g(z)=\frac{1}{z^m}\sum_{n=0}^{\infty} Cnz^n$ and $C_{b+m}=a_b$, where $m(\geqq 0)$ is an integer and ε is a positive number.

Remark 1. (1.3) can break down if $g(z)=\frac{1}{z^m}\sum_{n=0}^{\infty} Cnz^n$ and $C_{b+m}=a_b$. For

$$f(z)=\frac{1}{6}+\sum_{n=1}^{\infty}\frac{z^{2n}}{(n+3)!}$$

satisfies the hypotheses of Theorem 2 with $b=1$ and $d=2$. But for $g(z)=-(1/z^6)(1+z^2+z^4/2)$, with $m=6$ and $C_{b+m}=a_b=0$, we have

$$f(z)-g(z)=\frac{1}{z^6}\left(1+z^2+\frac{z^4}{2}+\frac{z^6}{6}+\sum_{n=1}^{\infty}\frac{z^{2(n+3)}}{(n+3)!}\right)=\frac{e^{z^2}}{z^6}$$

and so

$$\delta_s(g(z), f)=1.$$

Also for

$$f(z)=z+e^{z^2}$$

satisfies the hypotheses of Theorem 2 with $b=1$ and $d=2$. But for $g(z)=z$, with $m=0$ and $C_{b+m}=1=a_b$, we have

$$\delta_s(z, f)=1.$$

Remark 2. The inequality (1.3) is sharp. For $f(z)=-\frac{\sqrt{\pi}}{2}+\int_0^z e^{-t^2}dt$ satisfies the hypotheses of Theorem 2 with $b=0$ and $d=2$. Hence we have $\delta_s(0, f)\leqq 1-1/2=1/2$. Also as is shown by Nevenlinna [5], we have $\delta(0, f)=1/2$. Thus

$\delta_s(0, f) \geqq \delta(0, f)=1/2$. Therefere $\delta_s(0, f)=1/2$.

As a consequence of Theorem 2, we have

COROLLARY 1. *Suppose that* $b(>0)$, $d(\geqq 2)$ *are integers such that* $b \in \Lambda_f$ *but* $b+kd \in M_f$ *for* $k \geqq 1$. *Then*

$$\sum_{a \neq \infty} \delta(a, f) \leqq 1-\frac{1}{d}. \tag{1.4}$$

Remark 3. (1.4) is obviously better than the corresponding result of Hayman in [3, p. 330]: $\sum_{a \neq \infty} \delta(a, f) \leqq 1-\frac{1}{d(d-1)}$ with the hypotheses of Corollary 1.

2. Proof of Theorems.

From now on we denote by $S(r, F)$ any term which satisfies $S(r, F)= o\{T(r, F)\}$ as $r \notin E$, $r \to \infty$, where E is a set in $(1, +\infty)$ having finite length and in particular E is an empty set if $F(z)$ is of finite order.

In proving our theorems, we shall quote the following lemmas.

LEMMA 1. [2]: *If* $F(z)$ *is a transcendental entire function and* $g_1(z)$, $g_2(z)$, $\cdots$, $g_q(z)$ *are distinct meromorphic functions satisfying* $T(r, g_j)=o\{T(r, F)\}$ *as* $r \to \infty$ $(j=1, 2, \cdots, q)$, *then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{F-g_j(z)}\right) \leqq T(r, F)+S(r, F).$$

LEMMA 2 [4]: *If* $a_1, \cdots, a_q$ *are distinct finite complex numbers, then whenever* $F(z)$ *is transcendental entire function we have*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{F-a_j}\right) \leqq m\left(r, \frac{1}{F'}\right)+S(r, F).$$

Proof of Theorem 1.

By our Theorem A, we may assume that $g(z)$ is a meromorphic function which has at least a pole and satisfies

$$T(r, g)=o\{T(r, f)\} \qquad \text{as } r \to \infty \tag{2.1}$$

Let's discuss two cases separately.

Case (a). $g(z)$ has a pole $Zo \neq 0$.

Let $q(\geqq 2)$ be an arbitrary integer. We write

$$Wn=\exp\left(\frac{2\pi i}{d_n}\right),$$

$$P(z)=\sum_{v=0}^{n-1} a_{\lambda v} z^{\lambda v}, \tag{2.2}$$

$$F(z)=\sum_{v=n}^{\infty} a_{\lambda v} z^{\lambda v}. \tag{2.3}$$

Obviously, $\lim_{n\to\infty}(Wn)^j Zo=Zo$ $(1\leqq j\leqq q)$. Hence there exists a larger n such that $(Wn)^j Zo$ $(1\leqq j\leqq q)$ is not a pole of $g(z)$.

Noticing $d_n|(\lambda_{m+1}-\lambda_m)$ for $m\geqq n$, we have

$$d_n|(\lambda_v-\lambda_n) \qquad \text{for } v\geqq n,$$

where "$a|b$" denotes that a is a factor of b. Thus for $1\leqq j\leqq q$ we have, in view of (2.2) and (2.3),

$$\begin{aligned} f(W_n^j z)-g(W_n^j z) &= p(W_n^j z)-g(W_n^j z)+F(W_n^j z) \\ &= p(W_n^j z)-g(W_n^j z)+W_n^{j\lambda n}F(z) \\ &= W_n^{j\lambda n}\{F(z)-W_n^{-j\lambda n}(g(W_n^j z)-p(w_n^j z))\} \\ &= W_n^{j\lambda n}\{F(z)-g_j(z)\}, \end{aligned} \tag{2.4}$$

say, where $g_j(z)=W_n^{-j\lambda n}(g(W_n^j z)-p(W_n^j z))$.

Clearly, by (2.1) we have

$$T(r, g_j(z))=o\{T(r, f)\} \qquad \text{as } r\to\infty. \tag{2.5}$$

We assert that $g_1(z), g_2(z), \cdots, g_q(z)$ are distinct from each other. Otherwise, there must exist i, j $(1\leqq i<j\leqq q)$ such that

$$\Delta_{ij}(z)=Wn^{-j\lambda n}(g(Wn^j z)-p(Wn^j z))-Wn^{-i\lambda n}(g(Wn^i z)-p(Wn^i z))\equiv 0. \tag{2.6}$$

But by the definition of the integer n we easily conclude that

$$\Delta_{ij}(Wn^{-i}zo)=Wn^{-j\lambda n}(g(Wn^{j-i}zo)-p(Wn^{j-i}zo)-Wn^{i\lambda n}(g(zo)-p(zo))=\infty.$$

This contradicts (2.6).

Now using Lemma 1 to our functions $g_1(z), g_2(z), \cdots, g_q(z)$ and $F(z)$ and noticing (2.4) and (2.5), we deduce that

$$\sum_{j=1}^{q} m\left(r, \frac{1}{F-g_j(z)}\right)\leqq T(r, F)+S(r, F)$$

and so that

$$\begin{aligned} qm\left(r, \frac{1}{f-g(z)}\right) &\leqq T(r, F)+S(r, F) \\ &\leqq T(r, f)+S(r, f). \end{aligned}$$

Thus we have

$$\delta_s(g(z), f)\leqq\frac{1}{q}.$$

But q can be assumed any large. Hence we obtain

$$\delta_s(g(z), f)=0 .$$

Case (b). $Zo=0$ is an only pole of $g(z)$.

It is easily seen that there exists an integer $m\geqq 1$ such that

$$g(z)=\frac{1}{z^m} g^*(z) ,$$

where $g^*(z)$ is an entire function satisfying

$$T(r, g^*(z))=o\{T(r, f)\} \qquad \text{as } r\to\infty . \tag{2.7}$$

we set

$$f^*(z)=z^m f(z)$$

$$=z^m \sum_{n=1}^{\infty} anz^{\lambda_n}=\sum_{n=1}^{\infty} a_{\lambda_n} z^{\eta_n} ,$$

say, where $\eta_n=\lambda_n+m$.

By (2.7) we have

$$T(r, g^*(z))=o\{T(r, f^*)\} \qquad \text{as } r\to\infty . \tag{2.8}$$

Let dn^* be the highest common factor of all the numbers $\eta_{s+1}-\eta_s$ for $s\geqq n$. Apparently we have

$$dn^* \longrightarrow \infty \qquad \text{as } n\to\infty . \tag{2.9}$$

Using Theorem A to our functions $f^*(z)$ and $g^*(z)$ and noticing (2.8) and (2.9), we get

$$\delta_s(g^*(z), f^*(z))=0 .$$

But

$$f(z)-g(z)=\frac{1}{z^m}(f^*(z)-g^*(z)) .$$

Therefore

$$\delta s(g(z), f)=\delta s(g^*(z), f^*)=0 .$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

Suppose that $g(z)$ is any meromorphic function satisfying

$$T(r, g)=o\{T(r, f)\} \qquad \text{as } r\to\infty \tag{2.10}$$

we write

$$W=\exp\left(\frac{2\pi i}{d}\right) ,$$

$$h_v(z)=W^{-vb}\{f(W^v z)-g(W^v z)\} \qquad (1\leqq v\leqq d) , \tag{2.11}$$

$$h(z)=\sum_{v=1}^{d} h_v(z) . \tag{2.12}$$

We then have, for any integer $n(\geqq 0)$, that $\sum_{v=1}^{d} W^{v(n-b)}=d$ if $d|(n-b)$ and $\sum_{v=1}^{d} W^{v(n-b)}=0$ if $d \nmid (n-b)$.

So by $b+kd \in M_f$ for $k \geqq 1$ we deduce that

$$\begin{aligned} h(z) &= \sum_{v=1}^{d} h_v(z) \\ &= \sum_{v=1}^{d} W^{-vb} \sum_{n=0}^{\infty} a_n W^{vn} z^n - \sum_{v=1}^{d} W^{-vb} g(W^v z) \\ &= d \sum_{k=k_0}^{0} a_{b+kd} z^{b+kd} - \sum_{v=1}^{d} W^{-vb} g(W^v z), \end{aligned}$$

where ko is the minimum of integers k $(-\infty < k \leqq 0)$ satisfying $b+kd \geqq 0$.

By (2.10), it is easily deduced that

$$T(r, h(z)) = o\{T(r, f)\} \qquad \text{as } r \to \infty. \tag{2.13}$$

We claim that

$$h(z) \not\equiv 0, \tag{2.14}$$

if the exceptional case mentioned in Theorem 2 doesn't occur.

In fact, we may write

$$g(z) = \frac{1}{z^m} \sum_{n=0}^{\infty} Cnz^n \qquad \text{as } 0 < |z| < \varepsilon,$$

where $m(\geqq 0)$ is an integer and ε is a small positive number.

Hence for $0 < |z| < \varepsilon$ we have

$$\begin{aligned} h(z) &= d \sum_{k=k_0}^{0} a_{b+kd}\, z^{b+kd} - \sum_{n=0}^{\infty} Cn\, z^{n-m} \sum_{v=1}^{d} W^{v(n-m-b)} \\ &= d\left(\sum_{k=k_0}^{0} a_{b+kd}\, z^{b+kd} - \sum_{k=k_1}^{\infty} Cm+b+kd\, z^{b+kd}\right), \end{aligned} \tag{2.15}$$

where k_1 is the minimum of integer k $(-\infty < k \leqq 0)$ satisfying $m+b+kd \geqq 0$.

In (2.15), if $z=0$ is a pole of $h(z)$, then (2.14) is already correct. Or $h(z)$ is holomorphic in $\{|z|<\varepsilon\}$ and (2.15) is its power-series expansion in $|z|<\varepsilon$. But the coefficient of z^b in $h(z)$ is $d(a_b - C_{m+b}) \neq 0$. Thus (2.14) is also correct.

Next we assume without loss of generality that $h_1(z), h_2(z), \cdots, h_t(z)$ $(t \leqq d)$ are linearly independent. Then there exist complex numbers $e_1, e_2, \cdots, e_f$ such that

$$\begin{aligned} h(z) &= e_1 h_1(z) + e_2 h_2(z) + \cdots + e_t h_t(z) \\ &= H_1(z) + H_2(z) + \cdots + H_t(z), \end{aligned} \tag{2.16}$$

say, where $Hj(z) = ejhj(z)$ $(1 \leqq j \leqq t)$.

Also we may assume that $e_j \neq 0$ for $1 \leqq j \leqq t$, which doesn't, lose generality.

Hence if we set

$$\Delta=\begin{vmatrix} H_1(z) & H_2(z) & \cdots & H_t(z) \\ H_1'(z) & H_2'(z) & \cdots & H_t'(z) \\ \cdots & \cdots & \cdots & \cdots \\ H_1^{(t-1)}(z) & H_2^{(t-1)}(z) & \cdots & H_t^{(t-1)}(z) \end{vmatrix},$$

then we have $\Delta\not\equiv 0$.

By (2.16), we conclude that

$$\Delta=\begin{vmatrix} h(z) & H_2(z) & \cdots & H_t(z) \\ h'(z) & H_2'(z) & \cdots & H_t'(z) \\ \cdots & \cdots & \cdots & \cdots \\ h^{(t-1)}(z) & H_2^{(t-1)}(z) & \cdots & H_t^{(t-1)}(z) \end{vmatrix}$$

$$=h(z)\Delta_0(z)+h'\Delta_1(z)+\cdots+h^{(t-1)}(z)\Delta_{t-1}(z), \tag{2.17}$$

where $\Delta_j(z)$ $(0\leqq j\leqq t-1)$ is the algebraic complement with respect to $h^{(j)}(z)$ $(h^{(0)}(z)\equiv h(z))$.

It is well known that $m(r, F^{(k)}/F)=S(r, F)$ for $k\geqq 1$ whenever $F(z)$ ($\not\equiv$constant) is a meromorphic function [4]. Hence we deduce that, in view of (2.11), (2.17) and (2.13),

$$\begin{aligned} T(r, f) &\leqq T(r, H_1(z))+o\{T(r, f)\} \\ &\leqq N(r, H_1(z))+m\left(r, \frac{\sum_{v=0}^{t-1} h^{(v)}\Delta_v}{\prod_{v=2}^{t} H_v}\right)+m\left(r, \frac{\prod_{v=1}^{t} H_v}{\Delta}\right)+o\{T(r, f)\} \\ &\leqq N(r, H_1(z))+\sum_{v=0}^{t-1} m(r, h^{(v)})+\sum_{v=0}^{t-1} m\left(r, \frac{\Delta_v}{\prod_{j=2}^{t} H_j}\right)+m\left(r, \frac{\Delta}{\prod_{v=1}^{t} H_v}\right) \\ &\quad +N\left(r, \frac{\Delta}{\prod_{v=1}^{t} H_v}\right)+o\{T(r, f)\} \\ &\leqq N(r, H_1(z))+\sum_{v=0}^{t-1} m\left(r, \frac{h^{(v)}}{h}\right)+tm(r, h)+\sum_{v=0}^{t-1} m\left(r, \frac{\Delta_v}{\prod_{j=2}^{t} H_j}\right) \\ &\quad +m\left(r, \frac{\Delta}{\prod_{v=1}^{t} H_v}\right)+N(r, \Delta)+\sum_{v=1}^{t} N\left(r, \frac{1}{H_v}\right)+o\{T(r, f)\} \\ &\leqq o\{T(r, f)\}+S(r, h)+o\{T(r, f)\}+\sum_{j=2}^{t} S(r, H_j)+\sum_{j=1}^{t} S(r, H_j) \\ &\quad +o\{T(r, f)\}+tN\left(r, \frac{1}{f-g(z)}\right)+o\{T(r, f)\} \end{aligned}$$

$$\leqq tN\left(r, \frac{1}{f-g(z)}\right)+S(r, f).$$

That is,

$$T(r, f)\leqq tN\left(r, \frac{1}{f-g(z)}\right)+S(r, f).$$

Therefore we conclude that

$$\delta_s(g(z), f)\leqq 1-\frac{1}{t}\leqq 1-\frac{1}{d}.$$

The proof of Theorem 2 is thus completed.

Using Theorem 2 to the functions $f'(z)$ and $g(z)=0$ and Lemma 2 to the function f, we easily get the conclusion of Corollary 1. We here omit its details.

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