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ON THE QUASI-CONFORMAL CURVATURE TENSOR OF AN ALMOST KENMOTSU MANIFOLD WITH NULLITY DISTRIBUTIONS

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Abstract. The objective of the present paper is to characterize quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with (k,μ)-nullity and $(k,\mu)'$-nullity distributions, respectively. Also we characterize almost Kenmotsu manifolds with vanishing extended quasi-conformal curvature tensor and extended ξ-quasi-conformally flat almost Kenmotsu manifolds such that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution.
Keywords: Almost Kenmotsu manifold, Einstein manifold, Weyl conformal curvature tensor, Quasi-conformal curvature tensor, Extended quasi-conformal curvature tensor.

1. Introduction

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M,g) is defined by

$$g(QX,Y) = S(X,Y), \tag{1.1}$$

where S denotes the Ricci tensor of type $(0,2)$ on M and $X,Y \in T(M)$. The Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and scalar curvature of M, respectively.

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For a $(2n+1)$-dimensional Riemannian manifold, the quasi-conformal curvature tensor $\tilde{C}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y], \qquad (1.3)\end{aligned}$$

where a and b are two scalars. The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [21]. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.

A $(2n+1)$-dimensional Riemannian manifold will be called a manifold of the quasi-constant curvature if the Riemannian curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad +g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \qquad (1.4)\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, p, q are scalars and there exists a unit vector field ρ satisfying $g(X,\rho) = T(X)$. The notion of the quasi-constant curvature for Riemannian manfiolds was introduced by Chen and Yano [4].

At present, the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k-nullity distribution was introduced by Gray [10] and Tanno [15] in the study of Riemannian manifolds (M,g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \qquad (1.5)$$

for any X, $Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Blair, Koufogiorgos and Papantonio [1] introduced the generalized notion of k-nullity distribution, named (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1}$, ϕ, ξ, η, $g)$, which is defined for any $p \in M$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\mu[g(Y,Z)hX - g(X,Z)hY]\}, \qquad (1.6)\end{aligned}$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation.

In [7] Dileo and Pastore introduce the notion of $(k,\mu)'$-nullity distribution, another generalized notion of k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p \in M^{2n+1}$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu)' = \{Z \in T_pM : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \qquad (1.7)\end{aligned}$$

where $h' = h \circ \phi$.

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ, ξ, η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2],[3]),

$$\phi^2 = -I + \eta \otimes \xi, \; \eta(\xi) = 1, \tag{1.8}$$

where I denotes the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$ hold; both can be derived from (1.8) easily.
If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of T_pM^{2n+1}, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X, \Phi Y)$ for any X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [2]. Recently in ([7],[8],[9],[13],[14]), an almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also, Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X,Y. It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

At each point $p \in M$,we have

$$T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}$$

where $\{\xi_p\}$ is 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p. Then the Weyl conformal curvature tensor C is a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi\}.$$

Three particular cases can be considered as follows:
(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero .
(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M))$, that is, the projection of the image of C in $\{\xi\}$ is zero.
(3) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero, which is equivalent to $\phi^2 C(\phi X, \phi Y)\phi Z = 0$.

Definition 1.1. [22] A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be ξ-conformally flat if the linear operator $C(X,Y)$ is an endomorphism of $\phi(T(M))$, that is, if

$$C(X,Y)\phi(T(M)) \subset \phi(T(M)).$$

Then it follows immediately that

Proposition 1.1. *[22] On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following conditions are equivalent.*

(a) M^{2n+1} *is ξ-conformally flat,*

(b) $\eta(C(X,Y)Z) = 0,$

(c) $\phi^2 C(X,Y)Z = -C(X,Y)Z,$

(d) $C(X,Y)\xi = 0,$

where $X, Y, Z \in T(M)$.

Almost Kenmotsu manifolds have been studied by several authors such as Dileo and Pastore ([7]-[9]), Wang and Liu ([16]-[20]), De and Mandal([5], [6], [12]) and many others. In the present paper we like to study quasi-conformal curvature tensor of almost Kenmotsu manifolds with (k,μ) and $(k,\mu)'$-nullity distributions, respectively. Also, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with (k,μ)-nullity distribution.
The paper is organized as follows:
In Section 2, we give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution and ξ belonging to the $(k,\mu)'$-nullity distribution. Section 3 deals with quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. As a consequence of the main result, we obtain several corollaries. Section 4 is devoted to the study of quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution. In the final section, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with (k,μ)-nullity distribution.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [13]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.1}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

(2.3) $$\phi l\phi - l = 2(h^2 - \phi^2),$$

(2.4) $$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([7], [18])

(2.5) $$h = 0 \Leftrightarrow h' = 0, \;\; h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2).$$

3. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution

In this section we study quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution.
From (1.6) we obtain

(3.1) $$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 3.1. *[7] Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n+1)$. Suppose that the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 3.1 it follows from the equation (3.1),

(3.2) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(3.3) $$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X,$$

(3.4) $$S(X,\xi) = -2n\eta(X),$$

(3.5) $$Q\xi = -2n\xi,$$

for any vector fields X, Y on M^{2n+1}.

Theorem 3.1. *An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (k,μ)-nullity distribution is quasi-conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Let us first consider the manifold M^{2n+1} which is quasi-conformally flat, that is,

(3.6) $$\tilde{C}(X,Y)Z = 0,$$

for any vector fields X, Y, Z on M^{2n+1}.
From (1.3) we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &+S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &+\frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \qquad (3.7)\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (3.2) and (3.4) we get

$$\begin{aligned}\eta(X)g(Y,W) - \eta(Y)g(X,W) &= \frac{b}{a}[-2n\eta(X)g(Y,W) + 2n\eta(Y)g(X,W) \\ &+S(Y,W)\eta(X) - S(X,W)\eta(Y)] \\ &+\frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(X,W)\eta(Y) \\ &-g(Y,W)\eta(X)]. \qquad (3.8)\end{aligned}$$

Putting $Y = \xi$ in the above equation we obtain after simplification

$$S(X,W) = \alpha g(X,W) + \beta\eta(X)\eta(W), \qquad (3.9)$$

where $\alpha = \frac{a}{b}[\frac{2bn}{a} + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] + 1]$ and $\beta = \frac{a}{b}[-\frac{4bn}{a} - \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] - 1]$.
Therefore, we have $\alpha + \beta = -2n$.
Now using the above relation, (3.9) implies

$$r = 2n(\alpha - 1). \qquad (3.10)$$

In [7], Dileo and Pastore proved that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$. Using this value of r we obtain from (3.10), $\alpha = -2n$. This implies $\beta = 0$.
Hence (3.9) reduces to

$$S(X,W) = -2ng(X,W). \qquad (3.11)$$

From (3.7) we obtain

$$\begin{aligned}aR(X,Y)Z &= -b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+\frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y]. \qquad (3.12)\end{aligned}$$

Using the value of r and (3.11) in (3.12) yields

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y], \qquad (3.13)$$

which implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.
Conversely, suppose that the manifold is locally isometric to the hyperbolic space

$\mathbb{H}^{2n+1}(-1)$. That is, (3.13) holds.
Contracting X in (3.13) yields

$$S(Y,Z) = -2ng(Y,Z). \tag{3.14}$$

Hence (3.13) and (3.14) together implies $\tilde{C}(X,Y)Z = 0$. That is, the manifold is quasi-conformally flat.
Hence the theorem is proved.

Now, if $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Hence we can state the following:

Corollary 3.1. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

The above corollary has been proved by De and Mandal [5].

Theorem 3.2. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)- nullity distribution is ξ-quasi-conformally flat if and only if the manifold is an Einstein manifold.*

Proof: Let us consider a manifold that is ξ-quasi-conformally flat. That is,

$$\tilde{C}(X,Y)\xi = 0,$$

which implies

$$\begin{aligned} aR(X,Y)\xi &= -b[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad + \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.15}$$

Using (3.2) and (3.4) and $r = -2n(2n+1)$ we get from the above equation

$$\eta(Y)QX - \eta(X)QY = -2n[\eta(Y)X - \eta(X)Y], \tag{3.16}$$

Putting $Y = \xi$ in the above equation we obtain

$$QX = -2nX, \tag{3.17}$$

which implies $S(X,Y) = -2ng(X,Y)$. That is, the manifold is Einstein.
Conversely, assume that the manifold is Einstein. Then there exists a scalar λ such that

$$S(X,Y) = \lambda g(X,Y). \tag{3.18}$$

In an almost Kenmotsu manifold with (k,μ)-nullity distribution, the scalar curvature $r = -2n(2n+1)$. This implies $\lambda = -2n$. Now

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.19}$$

Using (3.18) we get

$$\tilde{C}(X,Y)Z = a[R(X,Y)Z + (g(Y,Z)X - g(X,Z)Y)]. \tag{3.20}$$

Putting $Z=\xi$ in the above equation and using (3.2) we obtain

$$\tilde{C}(X,Y)\xi = 0,$$

which implies that the manifold is ξ-quasi-conformally flat.

If $a=1$ and $b=-\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.
Thus we are in a position to state the following:

Corollary 3.2. *An almost Kenmotsu manifold with (k,μ)-nullity distribution is ξ-conformally flat if and only if it is Einstein.*

4. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution

In this section we study ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \le -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Before presenting our main theorem we recall some results:

Lemma 4.1. *(Prop. 4.1 and Prop. 4.3 of [7]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0,\lambda,-\lambda\}$, with 0 as a simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvatures are given by the following:*

- *(a) $K(X,\xi) = k-2\lambda$ if $X \in [\lambda]'$ and*
  $K(X,\xi) = k+2\lambda$ if $X \in [-\lambda]'$,
- *(b) $K(X,Y) = k-2\lambda$ if $X,Y \in [\lambda]'$;*
  $K(X,Y) = k+2\lambda$ if $X,Y \in [-\lambda]'$ and
  $K(X,Y) = -(k+2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,
- *(c) M^{2n+1} has a constant negative scalar curvature $r = 2n(k-2n)$.*

Lemma 4.2. *(Lemma 3 of [16]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution . If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta\otimes\xi - 2nh'. \tag{4.1}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

From (1.7) we have,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{4.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (4.2)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{4.3}$$

Contracting X in (4.2), we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{4.4}$$

Moreover, in an almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{4.5}$$

and

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y) \tag{4.6}$$

holds.

Theorem 4.1. *A $(2n+1)$-dimensional$(n>1)$ quasi-conformally flat almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution is either conformally flat or of a quasi-constant curvature.*

Proof: Let us assume that the manifold M^{2n+1} is quasi-conformally flat, that is,

$$\tilde{C}(X,Y)Z = 0, \tag{4.7}$$

for any vector fields X, Y, Z on M^{2n+1}.
From (1.3) we have

$$\begin{aligned} a\tilde{R}(X,Y,Z,W) &= b[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &\quad + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &\quad + \frac{r}{(2n+1)}[\frac{a}{2n} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.8}$$

Putting $Z = \xi$ in the above equation and using (4.2) and (4.4) we have

$$\begin{aligned} &ak[\eta(Y)g(X,W) - \eta(X)g(Y,W)] + a\mu[\eta(Y)g(h'X,W) - \eta(X)g(h'Y,W)] \\ &= b[2nk\eta(X)g(Y,W) - 2nk\eta(Y)g(X,W) - \eta(Y)S(X,W) + \eta(X)S(Y,W)] \\ &\quad + \frac{r}{(2n+1)}[\frac{a}{2n} + 2b][\eta(Y)g(X,W) - \eta(X)g(Y,W)]. \end{aligned} \tag{4.9}$$

Putting $Y = \xi$ in the above equation and using (4.4) we get after simplifying

$$\begin{aligned} S(X,W) &= [-2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b}]g(X,W) \\ &\quad +[4nk - \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] + \frac{ak}{b}]\eta(X)\eta(W) - \frac{a\mu}{b}g(h'X,W). \end{aligned} \tag{4.10}$$

Let us denote

$$A = -2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b} \tag{4.11}$$

and

$$B = 4nk - \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] + \frac{ak}{b}. \tag{4.12}$$

Then, we see that

$$A + B = 2nk. \tag{4.13}$$

Putting $X = W = e_i$ in (4.10), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $i = 1, 2, 3...., (2n+1)$, we get

$$r = A(2n+1) + B. \tag{4.14}$$

From (4.13) and (4.14) we get

$$A = \frac{r}{2n} - k. \tag{4.15}$$

From (4.11) and (4.15), it follows that

$$-2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b} = \frac{r}{2n} - k.$$

The above relation gives

$$(a + 2nb - b)(r - 2nk(2n+1)) = 0. \tag{4.16}$$

Hence, either $a + 2nb - b = 0$ or $r = 2nk(2n+1)$.

Let us suppose that $a + 2nb - b = 0$. Then we see that $b = -\frac{a}{2n-1}$. Hence, from (1.3), it follows that $\tilde{C}(X,Y)Z = aC(X,Y)Z$, where $C(X,Y)Z$ is the Weyl conformal curvature tensor. So, in this case, the quasi-conformally flat manifold is conformally flat.

Now, if $r = 2nk(2n+1)$, then from (4.10) we obtain

$$S(X,W) = 2nkg(X,W) - \frac{a\mu}{b}g(h'X,W). \tag{4.17}$$

Using (4.17) in (4.8) yields

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= k[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad -\mu[g(h'X,Z)g(Y,W) - g(h'Y,Z)g(X,W) \\ &\quad +g(h'Y,W)g(X,Z) - g(h'X,W)g(Y,Z)].\end{aligned} \tag{4.18}$$

From (4.1) and (4.17), it follows that

$$g(h'X,W) = l[g(X,W) - \eta(X)\eta(W)], \tag{4.19}$$

where $l = \frac{2nb(k+1)}{a\mu-2nb} = -\frac{nb(k+1)}{a+nb}$, by Lemma 4.1.
Using (4.19) in (4.18) we get

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) \\ &\quad +g(Y,Z)\eta(X)\eta(W) - g(Y,W)\eta(X)\eta(Z)],\end{aligned} \tag{4.20}$$

where $p = k - 4l$ and $q = 2l$.
This completes the proof.

5. Extended quasi-conformal curvature tensor of an almost Kenmotsu manifold with (k,μ)-nullity distribution

In this section we study vanishing extended quasi-conformal curvature tensor and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to (k,μ)-nullity distribution.
The extended form of quasi-conformal curvature tensor can be written as

$$\begin{aligned}\tilde{C}_e(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \\ &\quad -\eta(X)\tilde{C}(\xi,Y)Z - \eta(Y)\tilde{C}(X,\xi)Z - \eta(Z)\tilde{C}(X,Y)\xi.\end{aligned} \tag{5.1}$$

Theorem 5.1. *In an almost Kenmotsu manifold with ξ belonging to (k,μ)-nullity distribution, the extended quasi-conformal curvature tensor vanishes if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Putting $Y = Z = \xi$ and supposing that the extended quasi-conformal tensor vanishes, we get from (5.1)

$$\begin{aligned}&aR(X,\xi)\xi + b[S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi] + (a+4nb)(X - \eta(X)\xi) \\ &\quad -\eta(X)\tilde{C}(\xi,\xi)\xi - \tilde{C}(X,\xi)\xi - \tilde{C}(X,\xi)\xi = 0.\end{aligned} \tag{5.2}$$

Now, using (3.4) and (3.5) the above equation reduces to

$$bQX = -2nbX + 2\tilde{C}(X,\xi)\xi. \tag{5.3}$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$\tilde{C}(X,\xi)\xi = 2nbX + bQX. \tag{5.4}$$

Putting the value of $\tilde{C}(X,\xi)\xi$ in (5.3) we get

$$QX = -2nX, \tag{5.5}$$

which implies

$$S(X,Y) = -2ng(X,Y). \tag{5.6}$$

This shows that the manifold is Einstein. Since, the extended quasi-conformal curvature tensor vanishes, we have from (5.1)

$$\begin{aligned} aR(X,Y)Z &= -b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -(a+4nb)[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\eta(X)\tilde{C}(\xi,Y)Z + \eta(Y)\tilde{C}(X,\xi)Z + \eta(Z)\tilde{C}(X,Y)\xi. \end{aligned} \tag{5.7}$$

Now, making use of (3.3), (3.4), (3.5) and (5.5) we obtain

$$\tilde{C}(\xi,Y)Z = 0,\ \tilde{C}(X,\xi)Z = 0.$$

Again since the manifold is Einstein, we have from Theorem 3.2

$$\tilde{C}(X,Y)\xi = 0.$$

Putting these values in (5.7) and using (5.6) we get

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]. \tag{5.8}$$

This implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

Conversely, suppose that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. That is, (5.8) holds.
Contracting X in (5.8) yields

$$S(Y,Z) = -2ng(Y,Z). \tag{5.9}$$

Now, as shown earlier in this theorem

$$\tilde{C}(\xi,Y)Z = \tilde{C}(X,\xi)Z = \tilde{C}(X,Y)\xi = 0.$$

Then, making use of (5.8), (5.9) and the above values, we obtain from (5.1) that

$$\tilde{C}_e(X,Y)Z = 0.$$

Hence the theorem is proved.

Theorem 5.2. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is extended ξ-quasi-conformally flat if and only if the manifold is Einstein.*

Proof: Suppose $\tilde{C}_e(X,Y)\xi = 0$ and putting $Y = \xi$, we get from (5.1)

$$aR(X,\xi)\xi + b[S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi] + (a+4nb)(X - \eta(X)\xi) \\ -\eta(X)\tilde{C}(\xi,\xi)\xi - \tilde{C}(X,\xi)\xi - \tilde{C}(X,\xi)\xi = 0. \tag{5.10}$$

Now, using (3.4) and (3.5) the above equation reduces to

$$bQX = -2nbX + 2\tilde{C}(X,\xi)\xi. \tag{5.11}$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$\tilde{C}(X,\xi)\xi = 2nbX + bQX. \tag{5.12}$$

Putting the value of $\tilde{C}(X,\xi)\xi$ in (5.11) we get

$$QX = -2nX, \tag{5.13}$$

which implies that the manifold is Einstein.

Conversely, if the manifold is Einstein then obviously $\tilde{C}_e(X,Y)\xi = 0$.
Hence the theorem is established.

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