MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 157
JANUARY 1982

On the Quasi-Optimality in L_∞ of the $\mathring{H}^1$-Projection into Finite Element Spaces*

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Abstract. The $\mathring{H}^1$-projection into finite element spaces based on quasi-uniform partitions of a bounded smooth domain in R^N, $N \geqslant 2$ arbitrary, is shown to be stable in the maximum norm (or, in the case of piecewise linear or bilinear functions, almost stable). It is *not* assumed that the mesh-domains coincide with the basic domain.

1. Introduction. Let u be a function on a bounded closed domain $\mathfrak{R}$ with smooth boundary in R^N, $N \geqslant 2$. With $0 < h < \frac{1}{2}$ a parameter, let $\mathfrak{R}_h = \bigcup_{i=1}^{I(h)} \bar{\tau}_i^h$ be mesh-domains partitioned into finite elements τ_i^h, and assume temporarily that $\mathfrak{R}_h \subseteq \mathfrak{R}$. (As will be seen in (1.6) et seq., the last restriction is easy to overcome when applying our result.) Denote by $W_\infty^1(\mathfrak{R}_h)$ the class of functions with essentially bounded first derivatives (in the distribution sense), and let S_h, $0 < h < \frac{1}{2}$, be finite-dimensional subspaces of $W_\infty^1(\mathfrak{R}_h)$, consisting of functions χ that vanish on $\partial\mathfrak{R}_h$ and are such that $\chi|_{\bar{\tau}_i^h} \in \mathcal{C}^2(\bar{\tau}_i^h)$.

Define $u_h \equiv Pu \in S_h$ as the $\mathring{H}^1$-projection of u; i.e.,

$$(1.1)\qquad \begin{aligned} \int_{\mathfrak{R}_h} \nabla u_h \cdot \nabla \chi &= \int_{\mathfrak{R}_h} \nabla u \cdot \nabla \chi \\ &= \sum_{i=1}^{I(h)} \left(-\int_{\tau_i^h} u \Delta \chi + \int_{\partial \tau_i^h} u \frac{\partial \chi}{\partial n} \right) \quad \text{for all } \chi \in S_h. \end{aligned}$$

Note that u_h is well defined for any continuous u. All integrals occurring are assumed to be exactly evaluated; hence, the influence of numerical quadrature is not considered, cf. Wahlbin [25].

Concerning the spaces S_h, certain further conditions, detailed in Section 3, are imposed. A brief summary of these is as follows: (i) The partitions of the $\mathfrak{R}_h$'s are quasi-uniform; (ii) With

$$(1.2)\qquad \delta \equiv \max_{x \in \partial \mathfrak{R}_h} \operatorname{dist}(x, \partial \mathfrak{R}),$$

we have $\delta \leqslant Ch^2$; (iii) For smooth functions v that vanish on $\partial\mathfrak{R}$, we can approximate v by functions in the spaces S_h to order $h^r + \delta$, $r \geqslant 2$ an integer. The exact conditions are easily verified in many concrete examples, including such with isoparametric modifications.

Received June 27, 1980; revised March 26, 1981.
1980 *Mathematics Subject Classification.* Primary 65N30, 65N15.
* This work was supported by the National Science Foundation.

Our main result, Theorem 5.1, is that

$$(1.3) \qquad \|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\mathfrak{R}_h)},$$

where

$$\bar{r} = \begin{cases} 1, & r = 2, \\ 0, & r \geqslant 3. \end{cases}$$

For $r \geqslant 3$, u_h is thus a quasi-optimal approximation to u.

One would wish to apply the above result when u is the solution of a model Dirichlet problem

$$(1.4) \qquad -\Delta u = f \quad \text{in } \mathfrak{R}, \qquad u = 0 \quad \text{on } \partial\mathfrak{R},$$

so that

$$(1.5) \qquad \int_{\mathfrak{R}_h} \nabla u_h \cdot \nabla \chi = \int_{\mathfrak{R}_h} f\chi \quad \text{for all } \chi \in S_h.$$

In general, one has $\mathfrak{R}_h \not\subseteq \mathfrak{R}$, unless: (i) $\mathfrak{R}$ is convex and the partitions of the $\mathfrak{R}_h$ are straight-edged, or: (ii) $\partial\mathfrak{R}$ is a polynomial curve and isoparametric modifications are used at the boundary. Hence, in general, f is not given on all of $\mathfrak{R}_h$, so that u_h is not well defined by (1.5) (this difficulty disappears with judicious choice of a numerical integration procedure). In the present analysis, it is assumed that f is suitably extended to $\tilde{f}$ and that $\tilde{f}$ is used in the definition (1.5) of u_h. Then u_h can be regarded as the $\mathring{H}^1$-projection of a function u^δ which solves the problem

$$-\Delta u^\delta = \tilde{f} \quad \text{in } \mathfrak{R}^\delta, \qquad u^\delta = 0 \quad \text{on } \partial\mathfrak{R}^\delta,$$

where $\mathfrak{R}^\delta$ is a domain with smooth boundary such that $\mathfrak{R}_h \cup \mathfrak{R} \subseteq \mathfrak{R}^\delta$. It is clearly possible, when h is small enough, to construct such domains with $\max_{x \in \partial\mathfrak{R}} \operatorname{dist}(x, \partial\mathfrak{R}^\delta) \leqslant C\delta$; compare (1.2) for notation.

By the maximum principle and (1.3), one has

$$(1.6) \qquad \begin{aligned} \|u - u_h\|_{L_\infty(\mathfrak{R}_h \cap \mathfrak{R})} &\leqslant \|u - u^\delta\|_{L_\infty(\mathfrak{R})} + \|u^\delta - u_h\|_{L_\infty(\mathfrak{R}_h)} \\ &\leqslant |u^\delta|_{L_\infty(\partial\mathfrak{R})} + C\left(\ln \frac{1}{h}\right)^{\bar{r}} \inf_{\chi \in S_h} \|u^\delta - \chi\|_{L_\infty(\mathfrak{R}_h)}, \end{aligned}$$

where C can be taken independent of δ (see the proof of (1.3)).

From the above (1.3), or (1.6) when $\mathfrak{R}_l \not\subseteq \mathfrak{R}$, it is possible to derive various convergence estimates for $u - u_h$ in terms of data f. Consider only the "isoparametric" situation; i.e., take $\delta \leqslant Ch^r$. (In general, the highest order that can be obtained is $\|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant C(f)(\ln 1/h)^{\bar{r}}(h^r + \delta)$.) Assume first that $\mathfrak{R}_h \subseteq \mathfrak{R}$. Using approximation theory, Schauder estimates, and interpolation of function spaces, one may establish, for a large class of finite element spaces, that

$$\|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant C_l h^{\min(l,r)} \left(\ln \frac{1}{h}\right)^{\bar{r}} \|f\|_{\mathcal{C}^{l-2}(\mathfrak{R})},$$

for $2 < l \neq r$. The method of analysis indicated gives constants C_l that tend to infinity as l tends to r from above or below.

For a sharper estimate when $f \in W_\infty^{r-2}$, one can proceed in many situations in the following way (which was pointed out to us by V. Thomée): Assume that for a suitable χ in S_h, typically an interpolant,

$$\|u - \chi\|_{L_\infty(\mathfrak{R}_h)} \leqslant Ch^{r-N/p}\|u\|_{W_p^r(\mathfrak{R})},$$

for any $p < \infty$ large enough, where C does not depend on p; cf. Ciarlet [6, Theorem 3.1.6]. Tracing constants in Agmon, Douglis, and Nirenberg [1], one finds that

$$\|u\|_{W_p^r(\mathfrak{R})} \leqslant Cp\|f\|_{W_p^{r-2}(\mathfrak{R})}.$$

Taking $p = \ln 1/h$ and combining with (1.3), we obtain

$$\|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant Ch^r\left(\ln \frac{1}{h}\right)^{\bar{r}+1}\|f\|_{W_\infty^{r-2}(\mathfrak{R})}.$$

A similar result has been obtained in the piecewise linear case by Rannacher [17].

By (1.6), one has the corresponding estimates for $\|u - u_h\|_{L_\infty(\mathfrak{R}\cap\mathfrak{R}_h)}$ when $\mathfrak{R}_h \not\subseteq \mathfrak{R}$, and the domains differ by at most Ch^r; here the mean value theorem and elliptic regularity are used to handle the term $|u^\delta|_{L_\infty(\partial\mathfrak{R})}$ of (1.6).

We have chosen to treat the $\mathring{H}^1$-projection and the model problem (1.4) in this paper. This choice was made for notational simplicity. More general second-order elliptic Dirichlet problems, and the corresponding projections, can be analyzed by making appropriate modifications in our method.

Let us briefly list other work on quasi-optimal estimates for $u - u_h$ in various norms.

The question is trivial in the $\mathring{H}^1$-norm.

In the L_2-norm, Babuška and Aziz [2, Theorem 6.3.8] showed that when $S_h \subseteq H^2(\mathfrak{R})$ (and $\mathfrak{R}_h = \mathfrak{R}$), i.e., in practice when S_h consists of $\mathcal{C}^1$ elements, then

$$\|u - u_h\|_{L_2(\mathfrak{R})} \leqslant C \inf_{\chi\in S_h}\|u - \chi\|_{L_2(\mathfrak{R})}. \tag{1.7}$$

The result is false when $S_h \not\subseteq H^2(\mathfrak{R})$; see Babuška and Osborn [3, p. 58] for a simple counterexample. In the one-dimensional situation on an interval I for $\mathcal{C}^0$ piecewise polynomials, the estimate (1.7) holds provided the infimum is taken only over functions χ in S_h that interpolate u in $\mathcal{C}^0(I)$ at mesh-points x_j; cf. Eisenstat, Schreiber, and Schultz [9]. In a similar vein, in [3] the L_2-norm is replaced by a mesh-dependent norm,

$$\|v\|_{L_p(I,\{x_j\})} = \left(\int_I |v|^p + \sum_j \left(\frac{x_{j+1} - x_{j-1}}{2}\right)|v(x_j)|^p\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

and quasi-optimality in this norm is verified.

As noted also in [3], the estimate (1.3) in the maximum norm is true in one dimension, without the logarithm when $r = 2$; cf. Descloux [7], Douglas, Dupont, and Wahlbin [8], and Wheeler [26]. (It is also very easy to translate the methods of the present paper to the one-dimensional situation.)

Concerning estimates in the maximum norm in any number of space dimensions, much work has been devoted to showing quasi-optimality in the W_∞^1-norm (or the norm $\|\cdot\|_{L_\infty} + h\|\cdot\|_{W^1_\infty}$); cf. Natterer [14], Nitsche [15], Rannacher [17], and Scott [23]. A typical result is that (when $\mathcal{R}_h = \mathcal{R}$)

$$\|u - u_h\|_{W^1_\infty(\mathcal{R})} \leqslant C \inf_{\chi \in S_h} \|u - \chi\|_{W^1_\infty(\mathcal{R})}.$$

Note that there is no logarithmic factor for $r = 2$; this is a recent result of Rannacher and Scott [18]. (An example by Fried [10] and Jespersen [12] indicates that the logarithmic factor in (1.3) might be necessary for $r = 2$.)

In the maximum norm itself, quasi-optimality (modulo logarithmic factors or factors $h^{-\varepsilon}$, ε small) is previously known on plane polygonal domains, for meshes with or without refinements, and on convex polyhedral domains in R^3; see Schatz [19] and Schatz and Wahlbin [21].

It is frequently of interest to localize stability estimates of the form above. As an example, one has results of the type

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar r} \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega^1)} + C|||u - u_h|||_{\mathcal{R}_h},$$

where $\Omega \subset \Omega^1 \subset \mathcal{R}_h$ and $|||\cdot|||_{\mathcal{R}_h}$ denotes some weak norm measuring global effects; cf. Bramble, Nitsche, and Schatz [4], Bramble and Schatz [5], Nitsche and Schatz [16], and Schatz and Wahlbin [20], [22].

Our technique of analysis in the present paper does not distinguish between different dimensions N and requires no relations between r and N; for $r = N = 2$, however, a shorter proof is possible; see Remark 5.3. In a broad outline our argument is a simplification of that in [20], but additional and lengthy details are needed to take into account the discrepancy between $\mathcal{R}$ and $\mathcal{R}_h$.

We shall use standard notation for the Sobolev spaces $W^k_p(\Omega)$ and $H^k(\Omega) = W^k_2(\Omega)$, k a nonnegative integer, $1 \leqslant p \leqslant \infty$, and for the Hölder spaces $\mathcal{C}^l(\Omega)$. We also set $\|v\|_{\dot H^1(\Omega)} = \|\nabla v\|_{L_2(\Omega)}$ with a slight abuse of the norm notation. Generic constants C and c will be independent of h and of essential variables and functions involved; these essential quantities are separately indicated. Two important constants which are not generic are c' and C_*.

We thank K. Eriksson and V. Thomée for many valuable suggestions in connection with this paper.

2. Preliminaries. Consider the problem of finding w such that, with η given,

$$\begin{cases} -\Delta w = \eta & \text{in } \mathcal{R}, \\ w = 0 & \text{on } \partial\mathcal{R}, \end{cases} \tag{2.1}$$

where, for simplicity, the boundary $\partial\mathcal{R}$ is infinitely differentiable. It is well known that $\|w\|_{H^2(\mathcal{R})} \leqslant C\|\eta\|_{L_2(\mathcal{R})}$, a result we shall use many times. Also,

$$w(x) = \int_{\operatorname{supp} \eta} G^x(y)\eta(y)\, dy,$$

where $G^x(y)$ is the Green's function for (2.1). It is known (see, e.g., Krasovskiĭ [13]) that, for x, y in $\mathcal{R}$,

$$|D^\alpha_x G^x(y)| \leqslant \begin{cases} C(1 + |\ln|x - y||) & \text{for } |\alpha| = 0, N = 2, \\ C_{|\alpha|}|x - y|^{2-N-|\alpha|} & \text{otherwise.} \end{cases} \tag{2.2}$$

Our most common use of this will be the following: Assume that $\operatorname{dist}(\Omega, \operatorname{supp} \eta) = d > 0$. Then, for $l \neq 0$,

$$
\begin{aligned}
\|w\|_{W^l_\infty(\Omega)} &\leq C d^{2-N-l} \int_{\operatorname{supp} \eta} |\eta(y)|\, dy \\
&\leq C d^{2-N-l} (\operatorname{diam}(\operatorname{supp} \eta))^{N/2} \|\eta\|_{L_2(\mathfrak{R})}.
\end{aligned}
\tag{2.3}
$$

3. The Finite Element Spaces. In A.1–A.6 we collect the assumptions that we shall need on the finite element spaces. *We phrase these assumptions so that they can be readily verified in many concrete situations.*

Let $0 < h < \frac{1}{2}$ be a parameter and $\mathfrak{R}_h$, with $\mathfrak{R}_h \subseteq \mathfrak{R}$, mesh-domains made up of closures of disjoint open elements τ_i^h, $i = 1, \ldots, I(h)$,

$$
\mathfrak{R}_h = \bigcup_{1}^{I(h)} \overline{\tau_i^h}.
$$

Denote by $\delta = \delta_h$ the maximal distance between $\partial \mathfrak{R}_h$ and $\partial \mathfrak{R}$,

$$
\delta = \max_{x \in \partial \mathfrak{R}_h} \operatorname{dist}(x, \partial \mathfrak{R}).
$$

We let the notation $W_p^{k,h}(\Omega)$, for $\Omega \subseteq \mathfrak{R}_h$, stand for the piecewise norms relative to the partitions above.

We assume the following two properties of the partitions.

A.1. $\mathfrak{R}_h \subseteq \mathfrak{R}$, where $\partial \mathfrak{R}$ is infinitely differentiable. The boundaries $\partial \mathfrak{R}_h$ are sectionally smooth and uniformly Lipschitz for $0 < h < \frac{1}{2}$, and there exists a constant C such that $\delta \leq C h^2$.

A.2. There exists a constant C such that, for any $f \in W_1^1(\tau_i^h)$, $0 < h < \frac{1}{2}$, $i = 1, \ldots, I(h)$,

$$
\int_{\partial \tau_i^h} |f| \leq C \left\{ h^{-1} \|f\|_{L_1(\tau_i^h)} + \|f\|_{W_1^1(\tau_i^h)} \right\}.
$$

The assumption A.2 is easy to verify for quasi-uniform partitions occurring in practice.

Let $S_h = S_h(\mathfrak{R}_h)$ be a finite-dimensional subspace of $W^1_\infty(\mathfrak{R}_h) \cap W^{2,h}_\infty(\mathfrak{R}_h)$, and let furthermore the functions in S_h vanish on $\partial \mathfrak{R}_h$. Here $W_p^{l,h}(\mathfrak{R}_h)$ is defined by the norm

$$
\|v\|_{W_p^{l,h}(\mathfrak{R}_h)} = \left(\sum_i \|v\|^p_{W_p^l(\tau_i^h)} \right)^{1/p},
$$

with the appropriate modifications for $p = \infty$. Also, $H^{l,h} = W_2^{l,h}$.

After extension by zero, we can regard functions in S_h as being in $W^1_\infty(\mathfrak{R})$.

For the spaces S_h we first assume an inverse property:

A.3 (*Inverse Property*). There exist constants C and $c' > 0$ such that, for any χ in S_h and $\tau = \tau_i^h$,

$$
\left(\sum_{|\alpha| = l} \|D^\alpha \chi\|^p_{L_p(\tau)} \right)^{1/p} \leq C h^{m-l-N(1/q-1/p)} \|\chi\|_{W_q^m(\tau')},
$$

for $0 \leq m \leq l \leq 2$, $1 \leq q \leq p \leq \infty$, where $\tau' = \{x \in \tau : \operatorname{dist}(x, \partial\tau) > c'h\}$.

This assumption is like a well-known one valid for quasi-uniform partitions, except for the smaller domain τ' on the right. Its proof, however, would be the same in all concrete cases.

We shall finally list three different approximation hypotheses:

A.4 (*High Order Local Approximation*). There exist integers $r \geqslant 2$ and M, and constants C and $c > 0$ such that the following holds.

For any $v \in W_\infty^r(\mathfrak{R})$ with v vanishing on $\partial\mathfrak{R}$, there exists χ in S_h with the following property.

Let $B = B(y, d)$ and $B' = B(y, 2d)$ be concentric balls of radii d and $2d$, respectively, where $d \geqslant ch$, and set $D_h = B \cap \mathfrak{R}_h$, $D' = B' \cap \mathfrak{R}$. Then

$$\begin{aligned} h^{-1}\|v - \chi\|_{L_\infty(D_h)} &+ \|v - \chi\|_{W_\infty^1(D_h)} + h\|v - \chi\|_{W_\infty^{2,h}(D_h)} \\ &\leqslant Ch^{r-1}\|v\|_{W_\infty^r(D')} + Ch^{-1}\delta \sum_{m=1}^{M} d^{m-1}\|v\|_{W_\infty^m(D')}. \end{aligned} \tag{3.1}$$

We have phrased this assumption in terms of certain concentric balls, but it is easily extended to more general domains.

The last term on the right of (3.1) merits some elucidation: For concreteness, consider a space S_h which comes from a larger finite element space $\tilde{S}_h$ by restricting functions to be zero on $\partial\mathfrak{R}_h$. Assume that $\tilde{S}_h$ admits an interpolant $\tilde{\chi} = \tilde{\chi}(v)$ such that

$$h^{-1}\|v - \tilde{\chi}\|_{L_\infty(D_h)} + \|v - \tilde{\chi}\|_{W_\infty^1(D_h)} + h\|v - \tilde{\chi}\|_{W_\infty^{2,h}(D_h)} \leqslant Ch^{r-1}\|v\|_{W_\infty^r(D')}.$$

Such an estimate can often be derived, e.g., by use of the Bramble-Hilbert lemma.

To obtain χ in S_h, $\tilde{\chi}$ is cut down to be zero on $\partial\mathfrak{R}_h$. Often then χ and $\tilde{\chi}$ differ only in a boundary layer L_h of width approximately h and by the inverse property

$$\begin{aligned} h^{-1}\|\chi - \tilde{\chi}\|_{L_\infty(L_h)} + \|\chi - \tilde{\chi}\|_{W_\infty^1(L_h)} &+ h\|\chi - \tilde{\chi}\|_{W_\infty^{2,h}(L_h)} \\ &\leqslant Ch^{-1}\|\chi - \tilde{\chi}\|_{L_\infty(L_h)} \leqslant Ch^{-1}|\tilde{\chi}|_{L_\infty(\partial\mathfrak{R}_h \cap B)}. \end{aligned}$$

The last inequality would often be true in practical situations. If the interpolation process uses only point values of v, and not derivatives, then the above estimates can often be continued as

$$\leqslant Ch^{-1}|v|_{L_\infty(\partial\mathfrak{R}_h \cap B')} \leqslant Ch^{-1}\delta\,\|v\|_{W_\infty^1((\mathfrak{R}\setminus\mathfrak{R}_h)\cap B')},$$

where the last step used the mean value theorem. Therefore, (3.1) would obtain with $M = 1$ (and D' replaced by $(\mathfrak{R} \setminus \mathfrak{R}_h) \cap B'$ in the last term). Higher M are needed for interpolation processes that involve derivatives of v, and where consequently tangential derivatives along $\partial\mathfrak{R}_h$ are cut down to zero. Most often, the last part of (3.1) could be improved to

$$Ch^{-1}\delta \sum_{m=1}^{M} h^{m-1}\|v\|_{W_\infty^m((\mathfrak{R}\setminus\mathfrak{R}_h)\cap B')},$$

but we shall have no use for such an improvement.

A.5 (*Low-Order Global Approximation*). There exists a constant C such that, for v in $H^2(\mathfrak{R})$ and vanishing on $\partial\mathfrak{R}$, there exists χ in S_h such that

$$h^{-1}\|v - \chi\|_{L_2(\mathfrak{R})} + \|v - \chi\|_{H^1(\mathfrak{R})} + h\|v - \chi\|_{H^{2,h}(\mathfrak{R}_h)} \leqslant Ch\|v\|_{H^2(\mathfrak{R})}.$$

Let us briefly comment on how one would check A.5 in concrete cases. Since $\|v\|_{L_2(\mathfrak{R}\setminus\mathfrak{R}_h)} \leqslant C\delta\|v\|_{H^1(\mathfrak{R})}$ and $\|v\|_{H^1(\mathfrak{R}\setminus\mathfrak{R}_h)} \leqslant C\delta^{1/2}\|v\|_{H^2(\mathfrak{R})}$, by A.1 it suffices to consider the mesh-domain $\mathfrak{R}_h$ on the left. For N high one has to apply a preliminary smoothing argument since an interpolant, requiring point values, cannot immediately be used; see Hilbert [11] and Strang [24]. In our low-order case, this preliminary smoothing of v can be arranged to preserve the boundary condition $v = 0$ on $\partial\mathfrak{R}$. For, first flatten the boundary patchwise, then extend v *oddly* over the boundary, thus preserving H^2, and then employ an *even* smoothing kernel. The analysis of [11], [24], combined with ideas outlined in the comment after A.4, could then be carried through in many practical examples.

A.6 ("*Superapproximation*"). There exist constants C and $c > 0$, and an integer K, such that the following holds:

Let $B_i = B(y, id)$ with $d \geqslant ch$, and set $D_h^i = B_i \cap \mathfrak{R}_h$. Let ω be an infinitely differentiable function with support in B_3 and such that

$$\|\omega\|_{W_\infty^k(R^N)} \leqslant Ld^{-k}, \qquad k = 0, \ldots, K, \text{ and } \omega \equiv 1 \text{ on } B_2.$$

Then for any v_h in S_h there exists χ in S_h with support in D_h^4 and with $\chi \equiv v_h$ on D_h^1. Further,

$$\|\omega^2 v_h - \chi\|_{H^1(D_h^4)} \leqslant CLh\left\{d^{-2}\|v_h\|_{L_2(D_h^4\setminus B_1)} + d^{-1}\|v_h\|_{H^1(D_h^4\setminus B_1)}\right\}.$$

Again the above is easily extended to more general domains.

For a discussion of superapproximation, see Nitsche and Schatz [16] and also Bramble, Nitsche, and Schatz [4]. The proofs there are easily adjusted to include, e.g., isoparametric modifications. Often, χ can simply be taken as a local interpolant of $\omega^2 v_h$.

4. Local $\mathring{H}^1$-Estimates. This section is devoted to proving Theorem 4.1 below. It is assumed that $\mathfrak{R}_h \subseteq \mathfrak{R}$.

The result and proof are similar to those in [16], but care needs to be exercised to account for the discrepancy between $\mathfrak{R}_h$ and $\mathfrak{R}$, and to trace constants depending on sizes of domains. Therefore we feel that a self-contained proof is in order.

Let $B = B(y, d)$ and $B' = B(y, 2d)$ be closed concentric balls centered at y and of radii d and $2d$, respectively. Set

$$D_h = B \cap \mathfrak{R}_h, \qquad D_h' = B' \cap \mathfrak{R}_h.$$

For a domain Ω, let

$$S_h^{\not{\partial}}(\Omega) = \{\chi \in S_h: \operatorname{supp} \chi \subseteq \Omega \cap \mathfrak{R}_h\}.$$

THEOREM 4.1. *Assume that $\mathfrak{R}_h \subseteq \mathfrak{R}$ and that the assumptions of Section 3 hold. There exist constants C and $c > 0$, independent of y, d and h, such that for $d \geqslant ch$ the following holds: If $v \in \mathring{H}^1(\mathfrak{R})$ and $v_h \in S_h$ with*

$$\int \nabla(v - v_h) \cdot \nabla\chi = 0 \quad \textit{for } \chi \in S_h^{\not{\partial}}(D_h'), \tag{4.1}$$

then

$$\|v - v_h\|_{\dot{H}^1(D_h)} \leqslant C\left(\|v\|_{\dot{H}^1(D_h')} + d^{-1}\|v\|_{L_2(D_h')} + d^{-1}\|v - v_h\|_{L_2(D_h')}\right). \tag{4.2}$$

Remark 4.1. Writing $v - v_h = (v - \chi) - (v_h - \chi)$ for any $\chi \in S_h$, the first two terms on the right of (4.2) can be replaced by

$$\inf_{\chi \in S_h} \left(\|v - \chi\|_{\dot{H}^1(D_h')} + d^{-1}\|v - \chi\|_{L_2(D_h')} \right).$$

Proof. We shall need a few auxiliary domains "between" D_h and D_h'; for this let $B^k = B(y, (1 + 1/k) \cdot d)$, $k = 1, 2, \ldots,$ and $D_h^k = B^k \cap \mathfrak{R}_h$, $k = 2, 3, 4$. Then $D_h \subseteq D_h^4 \subseteq D_h^3 \subseteq D_h^2 \subseteq D_h'$.

Consider first functions $v_h \in S_h$ which are "discrete harmonic" in D_h^2, i.e., such that

$$\int_{\mathfrak{R}_h} \nabla v_h \cdot \nabla \chi = 0 \quad \text{for } \chi \in \mathring{S}_h^{\not p}(D_h^2). \tag{4.3}$$

We shall show then that for $d \geqslant ch$, c large enough,

$$\|v_h\|_{\dot{H}^1(D_h)} \leqslant C d^{-1} \|v_h\|_{L_2(D_h^2)}. \tag{4.4}$$

We introduce an infinitely differentiable cutoff function ω, $0 \leqslant \omega \leqslant 1$, such that

$$\omega \equiv 1 \quad \text{on } B, \qquad \operatorname{supp} \omega \subseteq B^5,$$

and with

$$\|\omega\|_{W_\infty^k(R^N)} \leqslant C_k d^{-k}, \qquad k = 1, 2, \ldots. \tag{4.5}$$

Such a function is easily constructed by change of variables in one valid for $d = 1$. Now

$$\|v_h\|_{\dot{H}^1(D_h)} \leqslant \|\omega v_h\|_{\dot{H}^1(\mathfrak{R}_h)}. \tag{4.6}$$

Here

$$\begin{aligned}
\|\omega v_h\|^2_{\dot{H}^1(\mathfrak{R}_h)} &= \int_{\mathfrak{R}_h} \nabla(\omega v_h) \cdot \nabla(\omega v_h) \\
&= \int_{\mathfrak{R}_h} \nabla \omega \cdot v_h \nabla(\omega v_h) + \int_{\mathfrak{R}_h} \nabla v_h \cdot \omega \nabla(\omega v_h) \\
&= \int_{\mathfrak{R}_h} \nabla \omega \cdot v_h \nabla(\omega v_h) + \int_{\mathfrak{R}_h} \nabla v_h \cdot \nabla(\omega^2 v_h) - \int_{\mathfrak{R}_h} \nabla v_h \cdot (\nabla \omega) \omega v_h.
\end{aligned}$$

The last term on the right equals

$$-\int_{\mathfrak{R}_h} \nabla(\omega v_h) \cdot (\nabla \omega) v_h + \int_{\mathfrak{R}_h} |\nabla \omega|^2 v_h^2$$

and hence, cancelling terms and using the discrete harmonicity of v_h, (4.3),

$$\|\omega v_h\|^2_{\dot{H}^1(\mathfrak{R}_h)} = \int_{\mathfrak{R}_h} |\nabla \omega|^2 v_h^2 + \int_{\mathfrak{R}_h} \nabla v_h \cdot \nabla(\omega^2 v_h - \chi) \quad \text{for any } \chi \in \mathring{S}_h^{\not p}(D_h^2).$$

For the rest of the proof we drop the h's in the notation for D_h, D_h^k, and D_h'.

We next use Schwarz' inequality, the properties of ω, and, for choosing χ, the superapproximation hypothesis A.6. Note that, since ω is supported in B^5, only the behavior of v_h on D^4 need influence χ, provided d is sufficiently large relative to h. We obtain

$$\begin{aligned}
\|\omega v_h\|^2_{\dot{H}^1(\mathfrak{R}_h)} &\leqslant C d^{-2} \|v_h\|^2_{L_2(D^4)} \\
&\quad + C \|v_h\|_{\dot{H}^1(D^4)} \left\{ h d^{-2} \|v_h\|_{L_2(D^4)} + h d^{-1} \|v_h\|_{\dot{H}^1(D^4)} \right\}.
\end{aligned}$$

Via (4.6) we arrive at

$$\begin{aligned}\|v_h\|^2_{\dot H^1(D)} &\leq Cd^{-2}\|v_h\|^2_{L_2(D^4)} + Ch\|v_h\|_{\dot H^1(D^4)}d^{-2}\|v_h\|_{L_2(D^4)} \\ &\quad + Chd^{-1}\|v_h\|^2_{\dot H^1(D^4)} \\ &\leq Cd^{-2}\|v_h\|^2_{L_2(D^4)} + Chd^{-1}\|v_h\|^2_{\dot H^1(D^4)}.\end{aligned}$$

In the last step we used the fact that $hd^{-1} \leq C$.

Repeat the above procedure, with appropriate notational changes, on the last term on the right to obtain

$$\begin{aligned}\|v_h\|^2_{\dot H^1(D)} &\leq Cd^{-2}\|v_h\|^2_{L_2(D^4)} + Chd^{-1}\left(d^{-2}\|v_h\|^2_{L_2(D^3)} + hd^{-1}\|v_h\|^2_{\dot H^1(D^3)}\right) \\ &\leq Cd^{-2}\|v_h\|^2_{L_2(D^3)} + Cd^{-2}h^2\|v_h\|^2_{\dot H^1(D^3)}.\end{aligned}$$

The inverse assumption A.3 is now applied to the last term to complete the proof of (4.4).

We proceed to prove (4.2). This time we employ a cutoff function, still denoted by ω, such that

$$\omega \equiv 1 \quad \text{on } B^2, \qquad \operatorname{supp} \omega \subseteq B',$$

and satisfying (4.5). Let P be the $\mathring{H}^1(\mathfrak{R}_h)$-projection to S_h. Note that since $\mathfrak{R}_h \subseteq \mathfrak{R}$, P is also the $\mathring{H}^1(\mathfrak{R})$-projection to S_h, if functions in S_h are extended by zero. Now,

$$\begin{aligned}\|v - v_h\|_{\dot H^1(D)} &= \|\omega v - v_h\|_{\dot H^1(D)} \\ &\leq \|\omega v - P(\omega v)\|_{\dot H^1(\mathfrak{R}_h)} + \|P(\omega v) - v_h\|_{\dot H^1(D)}.\end{aligned} \tag{4.7}$$

Using (4.5), we have

$$\|\omega v - P(\omega v)\|_{\dot H^1(\mathfrak{R}_h)} \leq \|\omega v\|_{\dot H^1(\mathfrak{R}_h)} \leq C\|v\|_{\dot H^1(D')} + Cd^{-1}\|v\|_{L_2(D')}. \tag{4.8}$$

Since $\omega \equiv 1$ on B^2, using (4.1) it is easily seen that $P(\omega v) - v_h \in S_h$ is discrete harmonic on D^2, (4.3). Therefore, from (4.4),

$$\begin{aligned}\|P(\omega v) - v_h\|_{\dot H^1(D)} &\leq Cd^{-1}\|P(\omega v) - v_h\|_{L_2(D^2)} \\ &\leq Cd^{-1}\|P(\omega v) - \omega v\|_{L_2(D^2)} + Cd^{-1}\|v - v_h\|_{L_2(D^2)}.\end{aligned} \tag{4.9}$$

By (4.7)–(4.9) we find that

$$\begin{aligned}\|v - v_h\|_{\dot H^1(D)} &\leq \|v\|_{\dot H^1(D')} + Cd^{-1}\|v\|_{L_2(D')} \\ &\quad + Cd^{-1}\|v - v_h\|_{L_2(D')} + Cd^{-1}\|P(\omega v) - \omega v\|_{L_2(D')}.\end{aligned} \tag{4.10}$$

To handle the last term on the right, we utilize a duality argument over the domain $\mathfrak{R}$, which has H^2-regularity for the Dirichlet problem. Thus,

$$\|P(\omega v) - \omega v\|_{L_2(D')} = \sup_{\substack{\varphi \in C_0^\infty(D') \\ \|\varphi\|_{L_2} = 1}} \int (P(\omega v) - \omega v)\varphi. \tag{4.11}$$

For each fixed φ, let ψ be the solution of the problem

$$-\Delta\psi = \varphi \quad \text{in } \mathfrak{R}, \qquad \psi = 0 \quad \text{on } \partial\mathfrak{R}.$$

Since $P(\omega v) = 0$ on $\partial\mathfrak{R}_h$, we have, from Green's formula,

$$\int (P(\omega v) - \omega v)\varphi = \int_{\mathfrak{R}_h} \nabla(P(\omega v) - \omega v)\cdot\nabla\psi - \int_{\partial\mathfrak{R}_h} \omega v\,\frac{\partial\psi}{\partial n} \equiv I_1 + I_2. \tag{4.12}$$

Here, by the properties of the projection P, by the low-order approximation assumption A.5, and by elliptic regularity,

$$\begin{aligned} I_1 &= -\int_{\mathfrak{R}_h} \nabla(\omega v)\nabla(\psi - P\psi) \leqslant C\|\omega v\|_{\dot{H}^1(\mathfrak{R}_h)} h\|\psi\|_{H^2(\mathfrak{R})} \\ &\leqslant Ch\left\{\|v\|_{\dot{H}^1(D')} + d^{-1}\|v\|_{L_2(D')}\right\}. \end{aligned} \tag{4.13}$$

For the term I_2 we note that it only enters if $B' \cap \partial\mathfrak{R}_h$ is not empty. We have

$$|I_2| \leqslant |\omega v|_{L_2(\partial\mathfrak{R}_h)} |\nabla\psi|_{L_2(\partial\mathfrak{R}_h)}. \tag{4.14}$$

Since $\partial\mathfrak{R}_h$ is uniformly Lipschitz, one knows (or easily deduces) that

$$\begin{aligned} |\omega v|_{L_2(\partial\mathfrak{R}_h)} &\leqslant C\left(\|\omega v\|_{L_2(\mathfrak{R}_h)}\|\omega v\|_{H^1(\mathfrak{R}_h)}\right)^{1/2} \\ &\leqslant C\left(d^{-1}\|v\|^2_{L_2(D')} + \|v\|_{L_2(D')}\|v\|_{\dot{H}^1(D')}\right)^{1/2} \\ &\leqslant C\left(d^{-1/2}\|v\|_{L_2(D')} + d^{1/2}\|v\|_{\dot{H}^1(D')}\right). \end{aligned} \tag{4.15}$$

Further,

$$|\nabla\psi|_{L_2(\partial\mathfrak{R}_h)} \leqslant C\left(\|\psi\|_{\dot{H}^1(\mathfrak{R}_h)}\|\psi\|_{H^2(\mathfrak{R}_h)}\right)^{1/2}.$$

Here, by elliptic regularity, $\|\psi\|_{H^2(\mathfrak{R}_h)} \leqslant C$. Also,

$$\|\psi\|^2_{\dot{H}^1(\mathfrak{R})} = \int_{D'} \psi\varphi \leqslant \|\psi\|_{L_2(D')}.$$

Since $B(y, 2d) \cap \partial\mathfrak{R}_h$ is not empty, ψ vanishes at some points on the boundary $\partial\mathfrak{R}$ that are within a distance $O(\delta) \ll d$ of D'. Considering the domain $B(y, 4d) \cap \mathfrak{R} \supset D'$, ψ vanishes on a part of its boundary which contains a fixed fraction of its total surface measure, and hence, by Poincaré's inequality,

$$\|\psi\|_{L_2(D')} \leqslant Cd\|\psi\|_{\dot{H}^1(\mathfrak{R})},$$

where it is not hard to see that the constant may be taken uniformly in d and y. Therefore, $\|\psi\|_{\dot{H}^1(\mathfrak{R})} \leqslant Cd$, and hence, $|\nabla\psi|_{L_2(\partial\mathfrak{R}_h)} \leqslant Cd^{1/2}$. Combining this with (4.14), (4.15),

$$|I_2| \leqslant C\left(\|v\|_{L_2(D')} + d\|v\|_{\dot{H}^1(D')}\right).$$

So, by (4.10)–(4.13), since $hd^{-1} \leqslant C$,

$$\|v - v_h\|_{\dot{H}^1(D)} \leqslant C\|v\|_{\dot{H}^1(D')} + Cd^{-1}\|v\|_{L_2(D')} + Cd^{-1}\|v - v_h\|_{L_2(D')}.$$

This completes the proof of Theorem 4.1.

5. The Main Result. This section contains the main result of the paper.

THEOREM 5.1. *Let the assumptions of Section* 3 *hold. There exists a constant C such that if u in* $\mathcal{C}^0(\mathfrak{R})$ *and* u_h *in* S_h, $u_h = Pu$, *satisfy* (1.1), *then*

$$\|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant C\left(\ln\frac{1}{h}\right)^{\bar{r}} \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\mathfrak{R}_h)}, \tag{5.1}$$

where $\bar{r} = 1$ *for* $r = 2$, $\bar{r} = 0$ *for* $r \geqslant 3$.

The rest of the section is devoted to a proof of Theorem 5.1. We first note, for simplicity in writing, that it suffices to establish the estimate

$$\|u - u_h\|_{L_\infty(\mathfrak{R}_h)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{L_\infty(\mathfrak{R}_h)}; \tag{5.1$'$}$$

for then (5.1) would follow upon writing $u - u_h = (u - \chi) - (u_h - \chi)$ for $\chi \in S_h$. We may also assume in the proof that $u \in \mathcal{C}^1(\mathfrak{R})$.

For further simplicity in writing, we shall often employ the convention that, in norms and integrals over the mesh-domain $\mathfrak{R}_h$, the domain is surpressed in the notation. Thus, $\|u\|_{L_\infty} = \|u\|_{L_\infty(\mathfrak{R}_h)}$. We remind the reader that $\mathfrak{R}_h \subseteq \mathfrak{R}$ is assumed.

Let x_0 be a point in $\mathfrak{R}_h$ where

$$|(u - u_h)(x_0)| = \|u - u_h\|_{L_\infty}. \tag{5.2}$$

We shall first show that we may assume that $\operatorname{dist}(x_0, \partial\mathfrak{R}_h) \geqslant c'h$ for some $c' > 0$; cf. Remark 5.1 below.

Lemma 5.1. *There exists a constant $c' > 0$ such that if* $\operatorname{dist}(x_0, \partial\mathfrak{R}_h) \leqslant c'h$, *then*

$$\|u - u_h\|_{L_\infty} \leqslant 2\|u\|_{L_\infty}. \tag{5.3}$$

Proof. Set $\delta_0 = \operatorname{dist}(x_0, \partial\mathfrak{R}_h)$. Since $u_h = 0$ on $\partial\mathfrak{R}_h$, we have, by the mean value theorem,

$$\|u - u_h\|_{L_\infty} \leqslant |u(x_0)| + |u_h(x_0)| \leqslant \|u\|_{L_\infty} + \delta_0\|\nabla u_h\|_{L_\infty}.$$

Using the inverse property A.3,

$$\begin{aligned}\|u - u_h\|_{L_\infty} &\leqslant \|u\|_{L_\infty} + c\delta_0 h^{-1}\|u_h\|_{L_\infty}\\ &\leqslant \left(1 + c\delta_0 h^{-1}\right)\|u\|_{L_\infty} + c\delta_0 h^{-1}\|u - u_h\|_{L_\infty}.\end{aligned}$$

If $c\delta_0 h^{-1} \leqslant 1/3$, we obtain (5.3). This proves the lemma.

Thus, in the remainder of this section we assume that $\operatorname{dist}(x_0, \partial\mathfrak{R}_h) \geqslant c'h$, $c' > 0$. We need some more notation. Let τ be a finite element in the partition that has x_0 in it, and let τ' be the part of τ with $\operatorname{dist}(\tau', \partial\mathfrak{R}_h) \geqslant c'h$. Then $x_0 \in \tau'$ is assumed. Assume also that c' is so small that the employment of the inverse property A.3 over τ' is justified.

The notation just introduced will be fixed for the rest of the section.

We have, by A.3,

$$\begin{aligned}|(u - u_h)(x_0)| &\leqslant \|u\|_{L_\infty} + |u_h(x_0)| \leqslant \|u\|_{L_\infty} + Ch^{-N/2}\|u_h\|_{L_2(\tau')}\\ &\leqslant \|u\|_{L_\infty} + Ch^{-N/2}\|u\|_{L_2(\tau')} + Ch^{-N/2}\|u - u_h\|_{L_2(\tau')}\\ &\leqslant C\|u\|_{L_\infty} + Ch^{-N/2}\|u - u_h\|_{L_2(\tau')}.\end{aligned} \tag{5.4}$$

We proceed to estimate the last term on the right. We first use a duality argument:

$$\|u - u_h\|_{L_2(\tau')} = \sup_{\substack{\varphi \in \mathcal{C}_0^\infty(\tau')\\ \|\varphi\|_{L_2} = 1}} \int_{\tau'} (u - u_h)\varphi. \tag{5.5}$$

For each fixed φ, let v be the solution of the Dirichlet problem

$$-\Delta v = \varphi \quad \text{in } \mathfrak{R}, \qquad v = 0 \quad \text{on } \partial\mathfrak{R}. \tag{5.6}$$

Such a v can be considered, loosely, as a scaled smooth "Green's function" with singularity at x_0. By Green's formula, and letting $v_h \in S_h$ be the $\mathring{H}^1$-projection of v,

$$\begin{aligned}\int_{\tau'} (u - u_h)\varphi &= -\int_{\partial \mathfrak{R}_h} u \frac{\partial v}{\partial n} + \int_{\mathfrak{R}_h} \nabla(u - u_h) \cdot \nabla v \\ &= -\int_{\partial \mathfrak{R}_h} u \frac{\partial v}{\partial n} + \int_{\mathfrak{R}_h} \nabla u \cdot \nabla(v - v_h) \equiv I_1 + I_2.\end{aligned} \tag{5.7}$$

To estimate I_1, we have

$$|I_1| \leqslant \|u\|_{L_\infty} \int_{\partial \mathfrak{R}_h} |\nabla v|,$$

and we appeal then to the following result.

Lemma 5.2. *For v as in* (5.6) *with $\varphi \in \mathcal{C}_0^\infty(\tau')$ of unit L_2-norm,*

$$\int_{\partial \mathfrak{R}_h} |\nabla v| \leqslant Ch^{N/2}, \tag{5.8}$$

$$\int_{\mathfrak{R} \setminus \mathfrak{R}_h} |\nabla v| \leqslant C\delta h^{N/2}. \tag{5.9}$$

Admitting this lemma for a moment, we have

$$|I_1| \leqslant Ch^{N/2} \|u\|_{L_\infty}. \tag{5.10}$$

To estimate I_2, use Green's formula over each element,

$$I_2 = -\sum_i \int_{\tau_i^h} u \Delta(v - v_h) + \sum_i \int_{\partial \tau_i^h} u \frac{\partial}{\partial n}(v - v_h).$$

Then, from A.2,

$$|I_2| \leqslant C\|u\|_{L_\infty} \big(\|\nabla(v - v_h)\|_{W_1^{1,h}} + h^{-1} \|\nabla(v - v_h)\|_{L_1} \big).$$

We now record the crucial

Lemma 5.3. *For v as in* (5.6) *with $\varphi \in \mathcal{C}_0^\infty(\tau')$ of unit L_2-norm, and v_h its $\mathring{H}^1$-projection,*

$$\|\nabla(v - v_h)\|_{W_1^{1,h}(\mathfrak{R}_h)} + h^{-1} \|\nabla(v - v_h)\|_{L_1(\mathfrak{R}_h)} \leqslant Ch^{N/2} \left(\ln \frac{1}{h} \right)^{\bar{r}}. \tag{5.11}$$

The proof of this will be given later in this section. Using the lemma,

$$|I_2| \leqslant Ch^{N/2} \left(\ln \frac{1}{h} \right)^{\bar{r}} \|u\|_{L_\infty}.$$

Combining the above estimate with (5.10) into (5.7) and (5.5),

$$\|u - u_h\|_{L_2(\tau')} \leqslant Ch^{N/2} \left(\ln \frac{1}{h} \right)^{\bar{r}} \|u\|_{L_\infty},$$

so that by (5.4) the desired result (5.1)′ obtains.

It remains now to prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Let us first consider

$$\int_{\partial\mathfrak{R}} |\nabla v| = \int_{\partial\mathfrak{R}} \left|\frac{\partial v}{\partial n}\right|,$$

which equals

$$\sup_{\substack{|\eta|_{L_\infty(\partial\mathfrak{R})}=1 \\ \eta\in\mathcal{C}^\infty(\partial\mathfrak{R})}} \int_{\partial\mathfrak{R}} \frac{\partial v}{\partial n}\eta.$$

If w denotes the harmonic extension of η into $\mathfrak{R}$, then, since $v = 0$ on $\partial\mathfrak{R}$, Green's second formula gives

$$-\int_{\partial\mathfrak{R}} \frac{\partial v}{\partial n}\eta = -\int_{\mathfrak{R}} (\Delta v)w = \int_{\tau'} \varphi w \leqslant Ch^{N/2}\|\varphi\|_{L_2}\|w\|_{L_\infty(\mathfrak{R})} \leqslant Ch^{N/2},$$

where we used the maximum principle in the last step. Hence,

$$\int_{\partial\mathfrak{R}} |\nabla v| \leqslant Ch^{N/2}. \tag{5.12}$$

We need to show the same estimate with $\partial\mathfrak{R}$ replaced by $\partial\mathfrak{R}_h$. To do so, let us work on a coordinate patch, where, after a smooth transformation,

$$\begin{aligned} x &= (x', x_N), \qquad x' \in \Omega' \subset\subset R^{N-1}, \\ \partial\mathfrak{R} &= \{x\colon x_N = 0, x' \in \Omega'\}, \\ \partial\mathfrak{R}_h &= \{x\colon x_N = b(x'), x' \in \Omega'\}, \end{aligned}$$

with A.1, $0 \leqslant b(x') \leqslant C\delta \leqslant Ch^2$, and where $b(x')$ is sectionally smooth and uniformly Lipschitz. Note that hence $(1 + |\nabla b|^2)^{1/2}$ is uniformly bounded below and above so that we may freely go from integrals over Ω' to surface integrals over the corresponding part of $\partial\mathfrak{R}_h$, and vice versa. With Dv a generic first derivative,

$$Dv(x', b(x')) = Dv(x', 0) + \int_0^{b(x')} \frac{\partial}{\partial x_N} Dv(x', z)\, dz.$$

Here, $v(x) = \int_{\tau'} G^x(y)\varphi(y)\, dy$, so that, by the properties of the Green's function, (2.2), (2.3), and since $\operatorname{dist}(\tau', \partial\mathfrak{R}_h) \geqslant c'h$ and $|z| \leqslant Ch^2$,

$$\left|\frac{\partial}{\partial x_N} Dv(x', z)\right| \leqslant \int_{\tau'} \frac{C}{|y - (x', z)|^N} |\varphi(y)|\, dy \leqslant \frac{Ch^{N/2}}{|x' - x_0'|^N + h^N},$$

with $x_0 = (x_0', x_{0,N})$.

Remark 5.1. To ensure the above estimate is the reason for our assumption that $\operatorname{dist}(\tau', \partial\mathfrak{R}_h) \geqslant c'h$ and the ensuing additional work in Lemma 5.1.

Hence, using (5.12) and an elementary calculation,

$$\begin{aligned} &\int_{\Omega'} |Dv(x', b(x'))|\, dx' \\ &\qquad \leqslant \int_{\Omega'} |Dv(x', 0)|\, dx' + Ch^{N/2}\int_0^{C\delta} dz \int_{\Omega'} \frac{dx'}{|x' - x_0'|^N + h^N} \\ &\qquad \leqslant Ch^{N/2} + Ch^{N/2-1}\delta \leqslant Ch^{N/2}. \end{aligned}$$

This proves (5.8).

For (5.9), in the transformed coordinates we have the estimate (5.8) over any level piece $\{x = (x', x_N),\ x' \in \Omega',\ x_N = k,\ k \leq C\delta\}$. An integration in the x_N direction then gives (5.9).

This completes the proof of Lemma 5.2.

We are now left with proving Lemma 5.3; this will occupy us for the rest of this section.

Proof of Lemma 5.3. Set $e = v - v_h$. We shall first show that

$$\|\nabla e\|_{L_1} \leq Ch^{N/2+1}\left(\ln \frac{1}{h}\right)^{\bar{r}}. \tag{5.13}$$

It will be seen later that this is the hard step in proving (5.11). Recall our notational convention that a nondisplayed domain equals $\mathfrak{R}_h$.

We need some auxiliary notation. For this, recall our fixed notation x_0 and τ', cf. (5.2) and the discussion immediately before (5.4). Set

$$A_j = \{x: 2^{-j} \leq |x - x_0| \leq 2^{-j+1}\}, \qquad j \text{ integer}, \tag{5.14}$$

$$\Omega_j = A_j \cap \mathfrak{R}_h. \tag{5.15}$$

Assume for simplicity that $\mathfrak{R}_h = \overline{\bigcup_{j=0}^{\infty} \Omega_j}$. Next let $C_* \geq 1$ be a quantity to be chosen later (sufficiently large but independent of h) and let $J = J(C_*, h)$ be the integer such that

$$2^{-J} \geq C_* h > 2^{-J-1}. \tag{5.16}$$

Further introduce

$$B_* = \{x: |x - x_0| \leq 2^{-J}\}, \qquad \Omega_* = B_* \cap \mathfrak{R}_h. \tag{5.17}$$

For C_* large enough, Ω_* contains τ' which contains x_0. Also set

$$d_j = 2^{-j}, \tag{5.18}$$

and

$$\begin{cases} A_j' = A_{j-1} \cup A_j \cup A_{j+1}, \quad A_j'' = A_{j-1}' \cup A_j' \cup A_{j+1}', \ldots, \\ \qquad\qquad\qquad\qquad\qquad\qquad A_j^{v} = A_{j-1}^{iv} \cup A_j^{iv} \cup A_{j+1}^{iv}; \\ \Omega_j' = A_j' \cap \mathfrak{R}_h \ (= \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}), \ldots, \Omega_j^{v} = A_j^{v} \cap \mathfrak{R}_h. \end{cases} \tag{5.19}$$

Note that

$$\mathfrak{R}_h = \left(\bigcup_{j=0}^{J} \Omega_j\right) \cup \Omega_*; \tag{5.20}$$

assume also that C_* is large enough so that with a positive constant c,

$$\operatorname{dist}(\tau', A_j^{v}) \geq c d_j, \qquad j = 0, \ldots, J + 1. \tag{5.21}$$

A sketch of the situation might be helpful, Figure 1. (In the sketch we place x_0 quite close to $\partial\mathfrak{R}_h$, this being the harder case. Note also that the sketch is not to scale.)

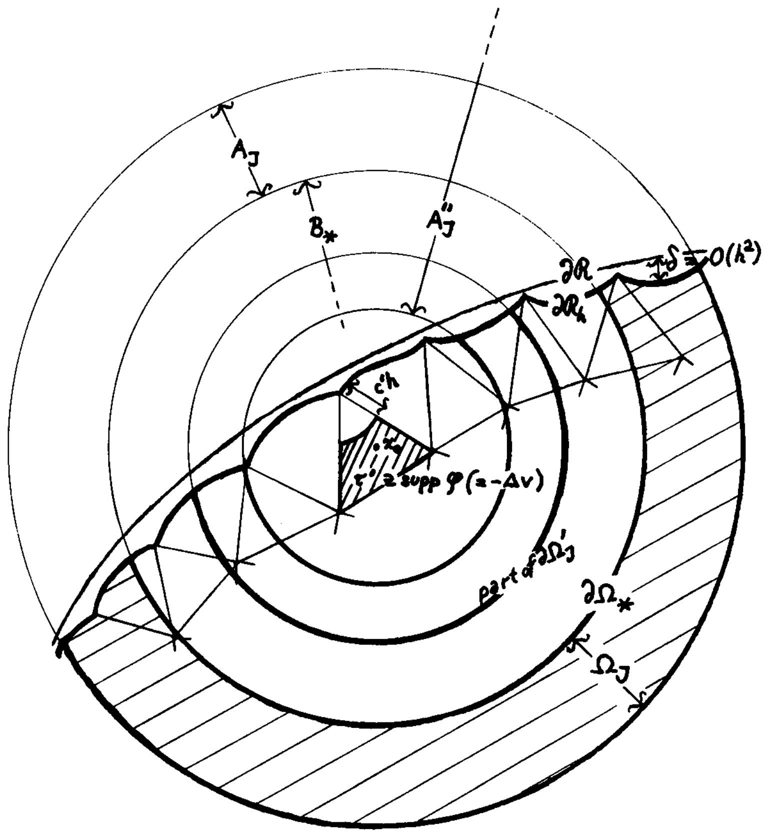

FIGURE 1

We have now

$$\|\nabla e\|_{L_1} = \|\nabla e\|_{L_1(\Omega_*)} + \sum_0^J \|\nabla e\|_{L_1(\Omega_j)}. \tag{5.22}$$

Here, by the low-order approximation property A.5 and by elliptic regularity for (5.6),

$$\begin{aligned}\|\nabla e\|_{L_1(\Omega_*)} &\leqslant CC_*^{N/2}h^{N/2}\|e\|_{\dot H^1(\mathcal{R}_h)} \\ &\leqslant CC_*^{N/2}h^{N/2} \inf_{\chi\in S_h}\|v-\chi\|_{\dot H^1(\mathcal{R}_h)} \\ &\leqslant CC_*^{N/2}h^{N/2+1}\|v\|_{H^2(\mathcal{R})} \leqslant CC_*^{N/2}h^{N/2+1}.\end{aligned} \tag{5.23}$$

Next,

$$\|\nabla e\|_{L_1(\Omega_j)} \leqslant 2^N d_j^{N/2}\|e\|_{\dot H^1(\Omega_j)},$$

so that, with

$$S = \sum_0^J d_j^{N/2}\|e\|_{\dot H^1(\Omega_j)}, \tag{5.24}$$

we have, by (5.22), (5.23),

$$\|\nabla e\|_{L_1} \leqslant CC_*^{N/2}h^{N/2+1} + 2^{N/2}S. \tag{5.25}$$

Remark 5.2. Note that for the function v, which is harmonic away from the region Ω_*, one has

$$c d_j^{N/2}\|v\|_{H^1(\Omega_j)} \leqslant \|v\|_{W_1^1(\Omega_j')} \leqslant C d_j^{N/2}\|v\|_{H^1(\Omega_j')},$$

with positive constants c and C. A similar estimate can be derived for the "discrete harmonic" function v_h. Therefore, the bound in (5.25) appears sharp. Note further that the right-hand side of (5.25) can be bounded by a weighted $\mathring{H}^1$-norm, viz.,

$$C\left(\ln \frac{1}{h}\right)^{1/2}\left(\int_{\mathfrak{R}_h}\left(\operatorname{dist}(x, \tau') + C_* h\right)^N |\nabla e(x)|^2\, dx\right)^{1/2},$$

cf. [14], [15], [17].

To estimate each term in S we use the local $\mathring{H}^1$-estimates of Theorem 4.1. Since A_j can be covered by a bounded number of balls of radius $d_j/4$, Theorem 4.1 applies with $D_h = \Omega_j$, $D_h' = \Omega_j'$, and $d = d_j$. Heeding Remark 4.1, we thus obtain

$$(5.26)\quad \begin{aligned} d_j^{N/2}\|e\|_{\dot{H}^1(\Omega_j)} &\leqslant d_j^{N/2} C \inf_{\chi \in S_h}\left(\|v-\chi\|_{\dot{H}^1(\Omega_j')} + d_j^{-1}\|v-\chi\|_{L_2(\Omega_j')}\right) \\ &\qquad + C d_j^{N/2-1}\|e\|_{L_2(\Omega_j')} \\ &\leqslant C d_j^{N} \inf_{\chi \in S_h}\left(\|v-\chi\|_{W_\infty^1(\Omega_j')} + d_j^{-1}\|v-\chi\|_{L_\infty(\Omega_j')}\right) \\ &\qquad + C d_j^{N/2-1}\|e\|_{L_2(\Omega_j')}. \end{aligned}$$

By the local approximation property A.4, and since $hd_j^{-1} \leqslant C$,

$$(5.27)\quad \begin{aligned} &\inf_{\chi \in S_h}\left(\|v-\chi\|_{W_\infty^1(\Omega_j')} + d_j^{-1}\|v-\chi\|_{L_\infty(\Omega_j')}\right) \\ &\qquad \leqslant C h^{r-1}\|v\|_{W_\infty^r(A_j''\cap\mathfrak{R})} + C h^{-1}\delta \sum_{m=1}^{M} d_j^{m-1}\|v\|_{W_\infty^m(A_j''\cap\mathfrak{R})}. \end{aligned}$$

Recall, (5.21), that $\operatorname{dist}(\tau', A_j'') \geqslant c d_j$, $c > 0$ may be assumed. Since φ is supported in τ', the properties of the Green's function, (2.2), (2.3), give

$$(5.28)\qquad \|v\|_{W_\infty^l(A_j''\cap\mathfrak{R})} \leqslant C d_j^{2-N-l} h^{N/2}, \qquad l = 1, \ldots, \operatorname{Max}(r, M).$$

Substituting now (5.28) into (5.27), and the result of that into (5.26), we obtain

$$(5.29)\quad \begin{aligned} d_j^{N/2}\|e\|_{\dot{H}^1(\Omega_j)} &\leqslant C d_j^{2-r} h^{N/2+r-1} + C d_j \delta h^{N/2-1} \\ &\qquad + C d_j^{N/2-1}\|e\|_{L_2(\Omega_j')}. \end{aligned}$$

Inserting this into (5.25) and summing the geometric series and, for $r = 2$, noting that the sum involves approximately $\ln(1/h)$ terms, and also remembering that $\delta \leqslant Ch^2$, we find that

$$(5.30)\quad \begin{aligned} \|\nabla e\|_{L_1(\mathfrak{R}_h)} &\leqslant C C_*^{N/2} h^{N/2+1} + 2^{N/2} S \\ &\leqslant C C_*^{N/2} h^{N/2+1} + C h^{N/2+1}\sum_0^J d_j^{2-r} h^{r-2} + C h^{N/2+1}(\delta h^{-2})\sum_0^J d_j \\ &\qquad + C\sum_0^J d_j^{N/2-1}\|e\|_{L_2(\Omega_j')} \\ &\leqslant C h^{N/2+1}\left(C_*^{N/2} + \left(\ln \frac{1}{h}\right)^{\bar{r}}\right) + C\sum_0^{J+1} d_j^{N/2-1}\|e\|_{L_2(\Omega_j)}. \end{aligned}$$

Remark 5.3. If $r = 2$, $N = 2$, we may now easily conclude the proof of (5.13). For then we estimate the last sum in (5.30) by

$$\sum_{0}^{J+1} \|e\|_{L_2(\Omega_j)} \leq C\left(\ln \frac{1}{h}\right)^{1/2} \|e\|_{L_2} \leq Ch^2\left(\ln \frac{1}{h}\right)^{1/2};$$

the last estimate here is well known by the low-order approximation hypothesis A.5 and a duality argument.

In general, our argument is more involved; to estimate $\|e\|_{L_2(\Omega_j)}$ we call on an additional local duality procedure. Write

$$\|e\|_{L_2(\Omega_j)} = \sup_{\substack{\eta \in \mathcal{C}_0^\infty(\Omega_j) \\ \|\eta\|_{L_2} = 1}} \int_{\Omega_j} e\eta. \tag{5.31}$$

For each such fixed η, let w be the solution of

$$-\Delta w = \eta \quad \text{in } \mathfrak{R}, \qquad w = 0 \quad \text{on } \partial\mathfrak{R}.$$

Then, for any χ in S_h,

$$\int_{\Omega_j} e\eta = \int_{\mathfrak{R}} \nabla e \cdot \nabla w = \int_{\mathfrak{R}} \nabla e \cdot \nabla(w - \chi). \tag{5.32}$$

We shall now construct an approximation χ to w that, roughly speaking, will be the low-order approximation of A.5 on Ω_j, and will be the high-order local approximation of A.4 outside of Ω_j. The blending of the two will be accomplished via "superapproximation", A.6. (We thank K. Eriksson for his help in this argument.)

Let ω, $0 \leq \omega \leq 1$, be a smooth function on R^N such that (cf. (5.19) for notation)

$$\omega^2 \equiv 1 \quad \text{on } A_j''', \qquad \operatorname{supp} \omega^2 \subseteq A_j^{\text{iv}}, \tag{5.33}$$

and

$$\|\omega\|_{W_\infty^k(R^N)} \leq C d_j^{-k}, \qquad k = 0, \ldots, K \quad (\text{cf. A.6}), \tag{5.34}$$

where C is independent of j. (Construct such a function on unit size domains and then scale.)

Let χ_H be the high-order local approximant to w of A.4, and let χ_L denote the low-order global approximant to w of A.5. Set $\psi = \omega^2(\chi_L - \chi_H)$, and let $\psi_S \in S_h$ be the "super"-approximaion to ψ given in A.6. Then

$$\psi_S \equiv 0 \quad \text{outside } \Omega_j^{\text{v}}, \tag{5.35}$$

and

$$\psi_S \equiv \psi \quad \text{in } \Omega_j''. \tag{5.36}$$

We now set $\chi = \chi_H + \psi_S$; then, on Ω_j'', $\chi = \chi_H + \psi = \chi_L$, and on $\mathfrak{R}_h \setminus \Omega_j^{\text{v}}$, $\chi = \chi_H$.

We use the χ just constructed in (5.32). Then,

$$
\begin{aligned}
\int_{\mathfrak{R}} \nabla e \cdot \nabla(w-\chi) &= \int_{\mathfrak{R}} \nabla e \cdot \nabla\big(\omega^2 w + (1-\omega^2)w - \chi_H - \psi_S\big) \\
&= \int_{\mathfrak{R}} \nabla e \cdot \nabla\big(\omega^2(w-\chi_L)\big) \\
&\quad + \int_{\mathfrak{R}} \nabla e \cdot \nabla\big((1-\omega^2)(w-\chi_H)\big) + \int_{\mathfrak{R}} \nabla e \cdot \nabla(\psi - \psi_S) \\
&\equiv J_1 + J_2 + J_3.
\end{aligned} \tag{5.37}
$$

We proceed to estimate the three terms above.

For J_1: By (5.33), (5.34), and A.5,

$$
\begin{aligned}
|J_1| &\leqslant C\|e\|_{\dot{H}^1(\mathfrak{R}\cap A_j^{iv})}\big(d_j^{-1}\|w-\chi_L\|_{L_2(\mathfrak{R})} + \|w-\chi_L\|_{\dot{H}^1(\mathfrak{R})}\big) \\
&\leqslant C\big(\|\nabla v\|_{L_2((\mathfrak{R}\setminus\mathfrak{R}_h)\cap A_j^{iv})} + \|e\|_{\dot{H}^1(\mathfrak{R}_h\cap A_j^{iv})}\big)h.
\end{aligned}
$$

By the Green's function representation, $v(x) = \int_{\tau'} G^x(y)\varphi(y)\,dy$ (cf. (5.6)), and by (5.21),

$$
\begin{aligned}
\|\nabla v\|_{L_2((\mathfrak{R}\setminus\mathfrak{R}_h)\cap A_j^{iv})} &\leqslant C\big(\delta d_j^{N-1}\big)^{1/2}\|\nabla v\|_{L_\infty((\mathfrak{R}\setminus\mathfrak{R}_h)\cap A_j^{iv})} \\
&\leqslant C\big(\delta d_j^{N-1}\big)^{1/2} d_j^{1-N}h^{N/2} = C\delta^{1/2}d_j^{1/2-N/2}h^{N/2}.
\end{aligned}
$$

Thus,

$$
|J_1| \leqslant Ch^{N/2+1}\delta^{1/2}d_j^{1/2-N/2} + Ch\|e\|_{\dot{H}^1(\mathfrak{R}_h\cap A_j^{v})}. \tag{5.38}
$$

For J_2: Note that $1-\omega^2$ is supported in $\mathfrak{R}\setminus A_j'''$. Since $\mathfrak{R}\setminus A_j''' = (\mathfrak{R}_h \setminus A_j''') \cup ((\mathfrak{R}\setminus\mathfrak{R}_h)\setminus A_j''')$,

$$
\begin{aligned}
|J_2| &= \left|\int \nabla e \cdot \nabla\big((1-\omega^2)(w-\chi_H)\big)\right| \\
&\leqslant \|\nabla e\|_{L_1(\mathfrak{R}_h)} C\big\{d_j^{-1}\|w-\chi_H\|_{L_\infty(\mathfrak{R}_h\setminus A_j''')} + \|\nabla(w-\chi_H)\|_{L_\infty(\mathfrak{R}_h\setminus A_j''')}\big\} \\
&\quad + \|\nabla v\|_{L_1(\mathfrak{R}\setminus\mathfrak{R}_h)} C\big\{d_j^{-1}\|w\|_{L_\infty((\mathfrak{R}\setminus\mathfrak{R}_h)\setminus A_j''')} + \|\nabla w\|_{L_\infty((\mathfrak{R}\setminus\mathfrak{R}_h)\setminus A_j''')}\big\}.
\end{aligned} \tag{5.39}
$$

We note that for $k \neq j-3, \ldots, j+3$, $k \geqslant J+5$ say, we have by A.4 and the Green's function representation $w(x) = \int_{\Omega_j} G^x(y)\eta(y)\,dy$,

$$
\begin{aligned}
&d_j^{-1}\|w-\chi_H\|_{L_\infty(\Omega_k)} + \|\nabla(w-\chi_H)\|_{L_\infty(\Omega_k)} \\
&\qquad \leqslant Ch^{r-1}\|w\|_{W_\infty^r(\mathfrak{R}\cap A_k')} + Ch^{-1}\delta\sum_{m=1}^{M} d_k^{m-1}\|w\|_{W_\infty^m(\mathfrak{R}\cap A_k')} \\
&\qquad \leqslant Ch^{r-1}\big(\max(d_k, d_j)\big)^{2-N-r}d_j^{N/2} \\
&\qquad\quad + Ch^{-1}\delta\sum_{m=1}^{M} d_k^{m-1}\big(\max(d_k, d_j)\big)^{2-N-m}d_j^{N/2}.
\end{aligned}
$$

Since $\mathfrak{R}_h \setminus A_j'''$ is the union of such Ω_k and a small inner "core" domain, for which a similar estimate is easily derived (for C_* large enough), we find that

$$
\begin{aligned}
&d_j^{-1}\|w-\chi_H\|_{L_\infty(\mathfrak{R}_h\setminus A_j''')} + \|\nabla(w-\chi_H)\|_{L_\infty(\mathfrak{R}_h\setminus A_j''')} \\
&\qquad \leqslant Ch^{r-1}d_j^{2-N/2-r} + Ch^{-1}\delta d_j^{1-N/2}.
\end{aligned} \tag{5.40}
$$

By Lemma 5.2,

$$\|\nabla v\|_{L_1(\mathfrak{R}\setminus\mathfrak{R}_h)} \leqslant Ch^{N/2}\delta \tag{5.41}$$

and, again by the Green's function representation,

$$d_j^{-1}\|w\|_{L_\infty((\mathfrak{R}\setminus\mathfrak{R}_h)\setminus A_j''')} + \|\nabla w\|_{L_\infty((\mathfrak{R}\setminus\mathfrak{R}_h)\setminus A_j''')} \leqslant Cd_j^{1-N/2}. \tag{5.42}$$

Using (5.40), (5.41) and (5.42) in (5.39), we see that

$$|J_2| \leqslant C\|\nabla e\|_{L_1(\mathfrak{R}_h)}\left\{h^{r-1}d_j^{2-N/2-r} + h^{-1}\delta d_j^{1-N/2}\right\} + Ch^{N/2}\delta d_j^{1-N/2}. \tag{5.43}$$

For J_3: By (5.35) and (5.36) and A.6,

$$\begin{aligned}
|J_3| = \left|\int \nabla e \cdot \nabla(\psi - \psi_S)\right| &\leqslant \|e\|_{\dot H^1(\mathfrak{R}_h\cap A_j^{v})}\|\psi - \psi_S\|_{\dot H^1((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')} \\
&\leqslant C\|e\|_{\dot H^1(\mathfrak{R}_h\cap A_j^{v})} h\left\{d_j^{-2}\|\chi_L - \chi_H\|_{L_2((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')}\right. \\
&\qquad \left. + d_j^{-1}\|\chi_L - \chi_H\|_{\dot H^1((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')}\right\} \\
&\leqslant C\|e\|_{\dot H^1(\mathfrak{R}_h\cap A_j^{v})} h\left\{d_j^{-2}\|\chi_L - w\|_{L_2} + d_j^{-1}\|\chi_L - w\|_{\dot H^1}\right. \\
&\qquad + d_j^{-2}\|\chi_H - w\|_{L_2((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')} \\
&\qquad \left. + d_j^{-1}\|\chi_H - w\|_{\dot H^1((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')}\right\}.
\end{aligned}$$

Here, by A.5,

$$d_j^{-2}\|\chi_L - w\|_{L_2} + d_j^{-1}\|\chi_L - w\|_{\dot H^1} \leqslant C\|w\|_{H^2(\mathfrak{R})} \leqslant C.$$

Further, by A.4 and the Green's function representation,

$$\begin{aligned}
d_j^{-2}\|\chi_H - w\|_{L_2((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')} &\leqslant Cd_j^{-2}d_j^{N/2}\|\chi_H - w\|_{L_\infty((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')} \\
&\leqslant Cd_j^{N/2-2}\left\{h^r\|w\|_{W_\infty^r(\mathfrak{R}\setminus A_j')} + C\delta\sum_{m=1}^{M} d_j^{m-1}\|w\|_{W_\infty^m(\mathfrak{R}\setminus A_j')}\right\} \\
&\leqslant Cd_j^{N/2-2}\left\{h^r d_j^{2-N-r}d_j^{N/2} + C\delta\sum_{m=1}^{M} d_j^{m-1}d_j^{2-N-m}d_j^{N/2}\right\} \leqslant C,
\end{aligned}$$

and, similarly,

$$d_j^{-1}\|\chi_H - w\|_{\dot H^1((\mathfrak{R}_h\cap A_j^{v})\setminus A_j'')} \leqslant C.$$

Thus,

$$|J_3| \leqslant Ch\|e\|_{\dot H^1(\mathfrak{R}_h\cap A_j^{v})}. \tag{5.44}$$

Using (5.44), (5.43) and (5.38) in (5.37), and the result in (5.32) and (5.31),

$$\begin{aligned}
\|e\|_{L_2(\Omega_j)} \leqslant Ch\|e\|_{\dot H^1(\mathfrak{R}_h\cap A_j^{v})} &+ Ch^{N/2+1}\delta^{1/2}d_j^{1/2-N/2} \\
&+ C\|\nabla e\|_{L_1(\mathfrak{R}_h)}\left(h^{r-1}d_j^{2-N/2-r} + h^{-1}\delta d_j^{1-N/2}\right) \\
&+ Ch^{N/2}\delta d_j^{1-N/2}.
\end{aligned}$$

Hence, from (5.30),

$$
\begin{aligned}
\|\nabla e\|_{L_1} &< CC_*^{N/2}h^{N/2+1} + 2^{N/2}S \leqslant Ch^{N/2+1}\left(C_*^{N/2} + \left(\ln \frac{1}{h}\right)^{\bar r}\right) \\
&\quad + C\|\nabla e\|_{L_1} \sum_0^{J+1} \left(h^{r-1}d_j^{1-r} + h^{-1}\delta\right) \\
&\quad + C\sum_0^{J+1} hd_j^{N/2-1}\|e\|_{\dot H^1(\mathfrak{R}_h \cap A_j^v)} \\
&\quad + C\sum_0^{J+1} \left(h^{N/2}\delta + h^{N/2+1}\delta^{1/2}d_j^{-1/2}\right).
\end{aligned} \tag{5.45}
$$

Here, remembering that $\delta \leqslant Ch^2$,

$$\sum_0^{J+1} \left(h^{r-1}d_j^{1-r} + h^{-1}\delta\right) \leqslant Ch^{r-1}\left(C_*h\right)^{1-r} + Ch \ln \frac{1}{h} \leqslant \frac{C}{\left(C_*\right)^{r-1}}.$$

Further, cf. (5.24) for notation,

$$
\begin{aligned}
&\sum_0^{J+1} hd_j^{N/2-1}\|e\|_{\dot H^1(\mathfrak{R}_h \cap A_j^v)} \\
&\qquad \leqslant C\sum_0^{J} hd_j^{N/2-1}\|e\|_{\dot H^1(\Omega_j)} + Ch\left(C_*h\right)^{N/2-1}\|e\|_{\dot H^1(\mathfrak{R}_h)} \\
&\qquad \leqslant C\frac{h}{d_J}S + CC_*^{N/2-1}h^{N/2+1} \leqslant \frac{C}{C_*}S + CC_*^{N/2-1}h^{N/2+1}.
\end{aligned}
$$

Also,

$$\sum_0^{J+1} \left(h^{N/2}\delta + h^{N/2+1}\delta^{1/2}d_j^{-1/2}\right) \leqslant Ch^{N/2+1}\left(h \ln \frac{1}{h} + h^{1/2}\right) \leqslant Ch^{N/2+1}.$$

Inserting the above three estimates in (5.45),

$$
\begin{aligned}
\|\nabla e\|_{L_1} &\leqslant CC_*^{N/2}h^{N/2+1} + 2^{N/2}S \\
&\leqslant Ch^{N/2+1}\left(C_*^{N/2} + \left(\ln \frac{1}{h}\right)^{\bar r}\right) + \|\nabla e\|_{L_1}\frac{C}{\left(C_*\right)^{r-1}} + S\frac{C}{C_*}.
\end{aligned}
$$

Taking now C_* large enough, we deduce in succession that

$$S \leqslant Ch^{N/2+1}\left(C_*^{N/2} + \left(\ln \frac{1}{h}\right)^{\bar r}\right) + \|\nabla e\|_{L_1}\frac{C}{C_*^{r-1}}$$

and that

$$\|\nabla e\|_{L_1} \leqslant Ch^{N/2+1}\left(C_*^{N/2} + \left(\ln \frac{1}{h}\right)^{\bar r}\right).$$

This proves the desired estimate (5.13).

It remains now to show (5.11). In the notation of (5.14)–(5.21),

$$\|\nabla e\|_{W_1^{1,h}(\mathfrak{R}_h)} = \|\nabla e\|_{W_1^{1,h}(\Omega_*)} + \sum_0^J \|\nabla e\|_{W_1^{1,h}(\Omega_j)}.$$

Here, for any $\chi_j \in S_h$, by the inverse property A.3 (where, by subtracting constants over each element, it is seen that it suffices to include the pure gradient term),

$$\begin{aligned}\|\nabla e\|_{W_1^{1,h}(\Omega_j)} &\leq \|\nabla(v - \chi_j)\|_{W_1^{1,h}(\Omega_j)} + Ch^{-1}\|\nabla(\chi_j - v_h)\|_{L_1(\Omega_j')}\\ &\leq CI(v - \chi_j, \Omega_j', 1) + Ch^{-1}\|\nabla e\|_{L_1(\Omega_j')},\end{aligned}$$

where we have used the shorter notation

$$I(g, \Omega, p) = \|\nabla g\|_{W_p^{1,h}(\Omega)} + h^{-1}\|\nabla g\|_{L_p(\Omega)}.$$

Similarly,

$$\|\nabla e\|_{W_1^{1,h}(\Omega_*)} \leq CI\big(v - \chi_*, \Omega_* \cup \Omega_J, 1\big) + Ch^{-1}\|\nabla e\|_{L_1(\Omega_* \cup \Omega_J)}.$$

Hence,

$$\begin{aligned}\|\nabla e\|_{W_1^{1,h}(\mathfrak{R}_h)} \leq CI\big(v - \chi_*, \Omega_* \cup \Omega_J, 1\big) + C\sum_0^J I(v - \chi_j, \Omega_j', 1)\\ + Ch^{-1}\|\nabla e\|_{L_1(\mathfrak{R}_h)}.\end{aligned} \tag{5.46}$$

Here, by low-order approximation A.5,

$$\begin{aligned}I\big(v - \chi_*, \Omega_* \cup \Omega_J, 1\big) &\leq \big(8C_*h\big)^{N/2} I\big(v - \chi_*, \mathfrak{R}_h, 2\big)\\ &\leq C\big(C_*h\big)^{N/2}\|v\|_{H^2(\mathfrak{R})} \leq Ch^{N/2}.\end{aligned} \tag{5.47}$$

By local approximation A.4 and the Green's function representation of Section 2, using (5.21),

$$\begin{aligned}I(v - \chi_j, \Omega_j', 1) &\leq 4^N d_j^N I(v - \chi_j, \Omega_j', \infty)\\ &\leq Cd_j^N\left(h^{r-2}\|v\|_{W_\infty^r(\mathfrak{R}\cap A_j'')} + Ch^{-2}\delta\sum_{m=1}^M d_j^{m-1}\|v\|_{W_\infty^m(\mathfrak{R}\cap A_j'')}\right)\\ &\leq Cd_j^N\big(h^{r-2}d_j^{2-N-r}h^{N/2} + Ch^{-2}\delta d_j^{1-N}h^{N/2}\big)\\ &\leq Ch^{N/2}\big(h^{r-2}d_j^{2-r} + d_j\big),\end{aligned} \tag{5.48}$$

where the last step used that $\delta \leq Ch^2$.

Inserting (5.47) and (5.48) in (5.46) and using (5.13) for the last term of (5.46),

$$\|\nabla e\|_{W_1^{1,h}(\mathfrak{R}_h)} \leq Ch^{N/2}\left(\ln\frac{1}{h}\right)^{\bar r} + h^{N/2}\sum_0^J\big(h^{r-2}d_j^{2-r} + d_j\big) \leq Ch^{N/2}\left(\ln\frac{1}{h}\right)^{\bar r}.$$

This completes the proof of Lemma 5.3.

Theorem 5.1 is now completely verified.

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