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ON THE QUENCHING BEHAVIOR OF THE MEMS WITH FRINGING FIELD

By

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Abstract. The singular parabolic problem $u_t - \triangle u = \lambda \frac{1+\delta|\nabla u|^2}{(1-u)^2}$ on a bounded domain Ω of $\mathbb{R}^n$ with Dirichlet boundary condition models the microelectromechanical systems (MEMS) device with fringing field. In this paper, we focus on the quenching behavior of the solution to this equation. We first show that there exists a critical value $\lambda_\delta^* > 0$ such that if $0 < \lambda < \lambda_\delta^*$, all solutions exist globally, while for $\lambda > \lambda_\delta^*$, all the solutions will quench in finite time. The estimate of the quenching time in terms of large voltage λ is investigated. Furthermore, the quenching set is a compact subset of Ω, provided Ω is a convex bounded domain in $\mathbb{R}^n$. In particular, if the domain Ω is radially symmetric, then the origin is the only quenching point. We not only derive the one-side estimate of the quenching rate, but also further study the refined asymptotic behavior of the finite quenching solution.

1. Introduction. Micro- and nanoelectromechanical systems (MEMS and NEMS) are indubitably the hottest topic in engineering nowadays. These devices have been playing important roles in the development of many commercial systems, such as accelerometers, optical switches, microgrippers, micro force gauges, transducers, micropumps, etc. Yet much research remains to be done. A deeper understanding of basic phenomena will advance the design in MEMS and NEMS.

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The simplified physical model of MEMS is the idealized electrostatic device. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and that lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when V is increased beyond a certain critical value V^*, which is known as *pull-in voltage*, the steady-state of the elastic membrane is lost and proceeds to quench or touch down at finite time.

In designing almost any MEMS or NEMS device based on the interaction of electrostatic forces with elastic structures, the designers will always confront the "pull-in" instability. This instability refers to the pheonomena of quenching or touch down as we described previously when the applied voltage is beyond certain critical value V^*. It is easy to see that this instability severely restricts the stable range of operation of many devices [22]. Hence much reseach has been done in understanding and controlling the instability. Most investigations of MEMS and NEMS have followed Nathanson's lead [20] and used some sort of small aspect ratio approximation to simplify the mathematical model. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [22].

The instability of the simplified mathematical model (cf. [14]) has also been observed and analyzed in [14], [6], [13], etc. This model is described by a partial differential equation:

$$\begin{cases} u_t - \triangle u = \dfrac{\lambda}{(1-u)^2} & \text{for } (x,t) \in \Omega_T \\ u(x,t) = 0 & x \in \partial\Omega_T \\ u(x,0) = 0 & x \in \Omega, \end{cases} \tag{1.1}$$

where $\circ_T = \circ \times [0,T)$, T is the maximal time of existence of the solution. The study of (1.1) starts from its stationary equation. It is shown in [5] that there exists a pull-in voltage $\lambda^* := \lambda^*(\Omega) > 0$ such that:

a. If $0 \leq \lambda < \lambda^*$, there exists at least one solution to the stationary equation of (1.1).
b. If $\lambda > \lambda^*$, there is no solution to the stationary equation of (1.1).

Concerning the evolutionary equation (1.1), [6] dealt with the issues of global convergence as well as finite and infinite time quenching of (1.1). It asserts that for the same λ^* above, the following hold:

(1) If $\lambda \leq \lambda^*$, then there exists a unique solution $u(x,t)$ to (1.1) which globally converges pointwisely as $t \to +\infty$ to its unique minimal steady-state.
(2) If $\lambda > \lambda^*$, then a unique solution $u(x,t)$ to (1.1) must quench in finite time.

More refined analysis of the quenching behavior of (1.1) is in [6], [13] and the references therein.

As pointed out in [23], (1.1) is only a leading-order outer approximation of an asymptotic theory based on expansion in the small aspect ratio. The fringing term $\delta|\nabla u|^2$ is the first-order correction. The model (1.1) is modified as:

$$\begin{cases} u_t - \triangle u = \lambda \dfrac{1+\delta|\nabla u|^2}{(1-u)^2}, & (x,t) \in \Omega_T \\ u(x,t) = 0, & (x,t) \in \partial\Omega_T \\ u(x,0) = 0, & x \in \Omega. \end{cases} \tag{$F_{\lambda,\delta}$}$$

In this paper, we aim to understand how the fringing term affects the behavior of the solution to $(F_{\lambda,\delta})$, including the pull-in voltage, quenching time, quenching behavior, etc.

The stationary equation of $(F_{\lambda,\delta})$,

$$\begin{cases} -\triangle u = \lambda \dfrac{1+\delta|\nabla u|^2}{(1-u)^2}, & x \in \Omega \subset \mathbb{R}^n \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$SF_{\lambda,\delta}$}$$

has been studied in [26]. The authors show that for fixed $\delta > 0$, there exists a pull-in voltage $\lambda_\delta^* > 0$ such that for $\lambda > \lambda_\delta^*$ there are no solutions to $(SF_{\lambda,\delta})$; for $0 < \lambda < \lambda_\delta^*$ there are at least two solutions; and when $\lambda = \lambda_\delta^*$ there exists a unique solution. Furthermore, for $\lambda < \lambda_\delta^*$ the equation $(SF_{\lambda,\delta})$ has a minimal solution u_λ and $\lambda \mapsto u_\lambda$ is increasing for $\lambda \in (0, \lambda_\delta^*)$.

The instability of $(F_{\lambda,\delta})$ is stated in the following theorem.

Theorem 1.1 (Theorem 2.3, [25]). *For fixed $\delta > 0$, suppose λ_δ^* is the pull-in voltage in [26]; then the following hold:*

(1) *If $\lambda \le \lambda_\delta^*$, then there exists a unique global solution $u(x,t)$ of $(F_{\lambda,\delta})$ which converges as $t \to \infty$ monotonically and pointwisely to its unique minimal steady-state.*
(2) *If $\lambda > \lambda_\delta^*$, then the unique solution $u(x,t)$ for $(F_{\lambda,\delta})$ must quench in finite time.*

In the literature, we say the solution u quenches if it reaches $u = 1$. Although the proof of this theorem has been briefly sketched in [25] with the right-hand side of $(F_{\lambda,\delta})$ being, rather than $\frac{1+\delta|\nabla u|^2}{(1-u)^2}$, an even more general nonlinearity $g(u)(1+\delta|\nabla u|^2)$, where $g : [0,1) \to \mathbb{R}_+$ satisfying

g is a C^2, positive, nondecreasing and convex function such that $\lim_{u\to 1^-} g(u) = +\infty$,

$$\int_0^1 g(s)ds = +\infty.$$

We believe the argument there is not rigorous, since when passing to the limit, it is not clear why $\lim_{t\to+\infty} \nabla k(x,t) = m(x)$ and $\lim_{t\to+\infty} \triangle k(x,t) = \triangle m(x)$. Instead, in this paper we adapt the argument in [1] to give a detailed proof.

The pull-in voltage λ_δ^* has been estimated in [25]:

$$\frac{4}{27} \frac{||\xi||_\infty}{||\xi||_\infty^2 + \delta||\triangle\xi||_\infty^2} \le \lambda_\delta^* \le \lambda^*. \tag{1.2}$$

We show in this paper that

$$\lim_{\delta\to\infty} \lambda_\delta^* = 0.$$

This improves the upper bound in (1.2) dramatically for $\delta \gg 1$.

From Theorem 1.1, we know that the solution quenches in finite time when $\lambda \geq \lambda_\delta^*$, denoted $T = T(\lambda, \delta) < \infty$. The precise definition of quenching time T is

$$T = \sup\{t > 0 : \, ||u(\cdot, \tau)||_\infty < 1, \, \forall \tau \in [0, t]\}.$$

It has been shown in [25] that $T = \mathcal{O}\left((\lambda - \lambda_\delta^*)^{-\frac{1}{2}}\right)$, provided that $\lambda > \lambda_\delta^*$ is sufficiently close to λ_δ^* and $\delta \ll 1$. For $\lambda \gg \lambda_\delta^*$, we show the following result:

THEOREM 1.2. The quenching time $T = T(\lambda, \delta)$ for the solution u of $(F_{\lambda,\delta})$ verifies

$$\limsup_{\lambda \to \infty} \lambda T = \frac{1}{3}.$$

This result is valid for $(F_{\lambda,\delta})$ with or without fringing term. However, it is known that the quenching time for $(F_{\lambda,\delta})$ without fringing term satisfies

$$\lim_{\lambda \to \infty} \lambda T = \frac{1}{3}.$$

The numerical results in section 6.1 suggest that $\lim_{\lambda \to \infty} \lambda T = 0$. Actually, with the similar argument in [27], we show that

$$T \leq \frac{||\phi||_1}{3\lambda ||\phi||_1 + ||\triangle \phi||_1},$$

where $\phi \geq 0$ is any C^2 function in Ω and $\phi = 0$ on $\partial\Omega$, and $|| \cdot ||_1$ is the L^1 norm. This implies that

$$T \lesssim \frac{1}{\lambda},$$

if $\lambda \gg 1$. The notation $a \lesssim b$ means there exists some constant $C > 0$ such that $a \leq Cb$. This is a finer decaying rate than $\mathcal{O}(\lambda^{-\frac{1}{2}})$, which was obtained in [25]. Besides the quenching time, we are also interested in the quenching set. The mathematical definition of quenching set is

$$\Sigma = \{x \in \bar{\Omega} : \, \exists \, (x_n, t_n) \in \Omega_T, \; s.t. \; x_n \to x, \; t_n \to T, \; u(x_n, t_n) \to 1\}.$$

We assume $\Omega \subset \mathbb{R}^n$ is a convex bounded domain. By the moving plane argument, we assert that the quenching set is a compact subset of Ω. And if $\Omega = B_R$, the ball centered at the origin with the radius R, then the quenching solution is radially symmetric (cf. [8]) and the only quenching point is the origin.

THEOREM 1.3. Suppose $\Omega = B_R$. If $\lambda > \lambda_\delta^*$, then the solution quenches only at $r = 0$. That is, the origin is the unique quenching point.

To understand the quenching behavior of the finite time quenching solution to $(F_{\lambda,\delta})$, we begin with the one-side quenching estimate, which has been derived in [18] for only the one dimensional case.

LEMMA 1.4 (One-side quenching estimate). If $\Omega \subset \mathbb{R}^n$ is a convex bounded domain, and $u(x,t)$ is a quenching solution of $(F_{\lambda,\delta})$ in finite time, then there exists a bounded positive constant $M > 0$ such that

$$M(T - t)^{\frac{1}{3}} \leq 1 - u(x, t),$$

for all $\Omega \times (0, T]$. Moreover, $u_t \to +\infty$ as u touches down.

Actually, we show in this paper that under certain conditions (namely (1.6)), the solution quenches in finite time T with the rate

$$1 - u(x,t) \sim (3\lambda(T-t))^{\frac{1}{3}},$$

as $t \to T^-$, provided $\Omega \in \mathbb{R}$ or $\Omega \in \mathbb{R}^n$, $n \geq 2$, is a radially symmetric domain.

This result comes from the similarity variables, which were first suggested in [9]-[11]. Let us make the similarity transformation at some point $a \in \Omega_\eta$ as in [9] and [13]:

$$y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad u(x,t) = 1-(T-t)^{\frac{1}{3}} w_a(y,s), \tag{1.3}$$

where $\Omega_\eta = \{x \in \Omega : dist(x, \partial\Omega) > \eta\}$, for some $\eta \ll 1$. First, the point a can be identified as a nonquenching point if $w_a(y,s) \to \infty$, as $s \to +\infty$ uniformly in $|y| \leq C$, for any constant $C > 0$. This is called the *nondegeneracy* phenomena in [11]. This property is not difficult to derive. It follows immediately from the comparison principle and the nondegeneracy of (1.1) obtained in [13].

The basis of the method, the similarity variables in [13], is the scaling property of (1.1): the fact that if u solves (1.1) near $(0,0)$, then so do the rescaled functions

$$1 - u_\gamma(x,t) = \gamma^{-\frac{2}{3}}\left[1 - u(\gamma x, \gamma^2 t)\right], \tag{1.4}$$

for each $\gamma > 0$. If $(0,0)$ is a quenching point, then the asymptotics of the quenching are encoded in the behavior of u_γ as $\gamma \to 0$. Unfortunately, compared with (1.1), $(F_{\lambda,\delta})$ doesn't possess this nice property. That is, it is not rescale-invariant. This is where the difficulty in analysis arises and the condition (1.6) comes from. Essentially, we characterize the asymptotic behavior near a singularity, assuming a certain upper bound on the rate of the gradient's blow-up. The condition (1.6) in some degree forces the solution of $(F_{\lambda,\delta})$ to converge to the *self-similar* solution of (1.1) as $t \to T^-$. We call u the self-similar solution to (1.1) if u defines $\mathbb{R}^n \times (0, +\infty)$ and $u_\gamma = u$ for every γ (see (1.4)).

Hence, the study of the asymptotic behavior of u near the singularity is equivalent to understanding the behavior of $w_a(y,s)$, as $s \to +\infty$, which satisfies the equation:

$$\frac{\partial w_a}{\partial s} = \triangle w_a - \frac{y}{2}\cdot\nabla w_a + \frac{1}{3}w_a - \frac{\lambda}{w_a^2} - \lambda\delta e^{\frac{s}{3}}\frac{|\nabla w_a|^2}{w_a^2}. \tag{1.5}$$

THEOREM 1.5. Suppose w_a is the solution to (1.5) quenching at $x = a$ in finite time T. Assume further that

$$\int_{s_0}^{\infty} s e^{\frac{s}{3}} \int_{B_s} \rho|\nabla w_a|^2 dy ds < \infty, \tag{1.6}$$

for some $s_0 \gg 1$, where $\rho(y) = e^{-\frac{|y|^2}{4}}$, B_s is defined in (5.10). Then $w_a(y,s) \to w_{a\,\infty}(y)$, as $s \to \infty$ uniformly on $|y| \leq C$, where $C > 0$ is any bounded constant, and $w_{a\,\infty}(y)$ is a bounded positive solution of

$$\triangle w - \frac{1}{2}y\cdot\nabla w + \frac{1}{3}w - \frac{\lambda}{w^2} = 0 \tag{1.7}$$

in $\mathbb{R}^n$. Moreover, if $\Omega \in \mathbb{R}$ or $\Omega \in \mathbb{R}^n$, $n \geq 2$, is a convex bounded domain, then we have

$$\lim_{t\to T^-}(1-u(x,t))(T-t)^{-\frac{1}{3}} = (3\lambda)^{\frac{1}{3}}$$

uniformly on $|x-a| \le C\sqrt{T-t}$ for any bounded constant C.

From Theorem 1.5, one can hardly tell the effects of the fringing term $\delta|\nabla u|^2$ on the asymptotic behavior near the singularity. Therefore, it seems to be necessary to find the refined asymptotic expansion near the singularity. As the first attempt in this direction, we derive a formal expansion as in [15] and [14]. Let us consider $\Omega \subset \mathbb{R}^n$ to be a radially symmetrical domain. Then, for $r \ll 1$ and $T-t \ll 1$, we have

$$u \sim 1 - [3\lambda(T-t)]^{\frac{1}{3}} \left(1 - \frac{3^{\frac{1}{3}} n}{8\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}} \frac{r^2}{(T-t)^{\frac{2}{3}}} + \cdots \right)^{\frac{1}{3}}. \tag{1.8}$$

This expansion is quite different from the one for [14]:

$$u \sim -1 + [3\lambda(T-t)]^{\frac{1}{3}} \left(1 - \frac{3n}{4|\log{(T-t)}|} + \frac{3r^2}{8(T-t)|\log{(T-t)}|} + \cdots \right)^{\frac{1}{3}}.$$

We believe the difference is due to the fringing term, which can be clearly seen from the method of dominant balance; see detailed analysis in section 5.6.

Finally, as supplements, we numerically compute the pull-in voltages of $(F_{\lambda,\delta})$ with various δ and the quenching times of $(F_{\lambda,\delta})$ with various δ and $\lambda > \lambda_\delta^*$ using *bvp4c* in Matlab. Furthermore, we solve $(F_{\lambda,\delta})$ numerically using an appropriate finite difference scheme. The numerical simulations validate the results obtained in the previous sections.

2. Global existence or quenching in finite time. Motivated by [26], we make the following transformation:

$$v(x,t) := \zeta_{\lambda,\delta}(u(x,t)) = \int_0^{u(x,t)} e^{\frac{\lambda\delta}{1-s}} ds; \tag{2.1}$$

then $v(x,t)$ satisfies

$$\begin{cases} v_t - \triangle v = \lambda\rho_{\lambda,\delta}(v), & (x,t) \in \Omega_T \\ v(x,t) = 0, & (x,t) \in \partial\Omega_T \\ v(x,0) = 0, & x \in \Omega, \end{cases} \tag{$V_{\lambda,\delta}$}$$

where $\rho_{\lambda,\delta} = \xi_{\lambda,\delta} \circ \zeta_{\lambda,\delta}^{-1}(v)$, $\xi_{\lambda,\delta}(u) := \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}$. Since $\xi_{\lambda,\delta}$ and $\zeta_{\lambda,\delta}$ are increasing in $[0,1)$ and $\lim_{u\to 1^-} \zeta_{\lambda,\delta}(u) = \infty$, $\rho_{\lambda,\delta}$ is also increasing in $\mathbb{R}_+$. It is also not difficult to check that $\rho_{\lambda,\delta}(v)$ satisfies the following properties:

(1) $\rho_{\lambda,\delta}(v)$, $\rho'_{\lambda,\delta}(v)$ and $\rho''_{\lambda,\delta}(v) > 0$, for $v \in \mathbb{R}_+$. In fact, through direct computations we get

$$\rho'_{\lambda,\delta}(v) = \frac{1}{(1-u)^3}\left(2 + \frac{\lambda\delta}{1-u}\right); \quad \rho''_{\lambda,\delta}(v) = \frac{2}{(1-u)^4} e^{-\frac{\lambda\delta}{1-u}} \left(3 + \frac{2\lambda\delta}{1-u}\right). \tag{2.2}$$

(2) $\int_{v_0}^\infty \frac{ds}{\rho_{\lambda,\delta}(s)} < \infty$, for any $v_0 \in (0,\infty)$, since

$$\int_{v_0}^\infty \frac{ds}{\rho_{\lambda,\delta}(s)} = \int_{\zeta_{\lambda,\delta}^{-1}(v_0)}^1 \frac{\zeta'_{\lambda,\delta}(u)du}{\frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}} = \int_{\zeta_{\lambda,\delta}^{-1}(v_0)}^1 (1-u)^2 du = \frac{1}{3}\left(1 - \zeta_{\lambda,\delta}^{-1}(v_0)\right)^3 < \infty.$$

LEMMA 2.1 (Uniqueness). Suppose $u_1(x,t)$ and $u_2(x,t)$ are solutions of $(F_{\lambda,\delta})$ on $\Omega_T := \Omega \times [0,T]$ such that $||u_i||_{L^\infty(\Omega_T)} < 1$ for $i = 1,2$; then $u_1 = u_2$.

Proof. Let us denote by $v_i(x,t)$ the solutions of $(V_{\lambda,\delta})$, i.e. $v_i = \zeta_{\lambda,\delta}(u_i)$, $i = 1,2$. Then $\hat{v} = v_1 - v_2$ satisfies

$$\hat{v}_t - \triangle \hat{v} = \lambda[\rho_{\lambda,\delta}(v_1) - \rho_{\lambda,\delta}(v_2)] = \lambda \frac{\rho_{\lambda,\delta}(v_1) - \rho_{\lambda,\delta}(v_2)}{\hat{v}} \hat{v} := \lambda f \hat{v}. \tag{2.3}$$

The condition $||u_i||_{L^\infty(\Omega_T)} < 1$ is equivalent to $||v_i||_{L^\infty(\Omega_T)} < \infty$, $i = 1,2$. This implies that $||\rho_{\lambda,\delta}(v_i)||_{L^\infty(\Omega_T)} < \infty$, $i = 1,2$. Therefore, $||f||_{L^\infty(\Omega_T)} < \infty$.

We now fix $T_1 \in (0,T)$ and consider the solution ϕ of the problem

$$\begin{cases} \phi_t + \triangle \phi + \lambda f \phi = 0, & (x,t) \in \Omega_{T_1} \\ \phi(x,T_1) = \theta(x) \in C_0(\Omega), & x \in \Omega \\ \phi(x,t) = 0, & (x,t) \in \partial\Omega_{T_1}. \end{cases} \tag{2.4}$$

The standard linear theory gives the unique and bounded solution (cf. Theorem 8.1, [16]).

Multiplying ϕ to (2.3) and integrating in Ω_{T_1} on both sides yields by integration by parts that

$$\int_\Omega \hat{v}(x,T_1)\theta(x)dx = 0,$$

for arbitrary $T_1 \in (0,T)$ and $\theta(x) \in C_0(\Omega)$. This implies that $\hat{v} \equiv 0$. □

2.1. *Global existence.*

THEOREM 2.2 (Global existence). For every $\lambda \le \lambda_\delta^*$, there exists a unique global solution $u(x,t)$ of $(F_{\lambda,\delta})$, which monotonically converges as $t \to \infty$ to the minimal solution u_λ of $(SF_{\lambda,\delta})$.

Proof. This is standard and follows from the maximum principle combined with the existence of the regular minimal steady-state solutions for $\lambda \in (0, \lambda_\delta^*)$. Indeed, for any $0 < \lambda \le \lambda_\delta^*$, from Theorem 1 and Theorem 5, [26], there exists a unique minimal solution $u_\lambda(x)$ of $(SF_{\lambda,\delta})$. It is clear that 0 and u_λ are sub- and supersolutions to $(F_{\lambda,\delta})$, respectively. This implies that there exists a unique global solution $u(x,t)$ of $(F_{\lambda,\delta})$ such that $1 > u_\lambda(x) \ge u(x,t) \ge 0$ in $\Omega \times (0,\infty)$. Let us denote $v_\lambda = \zeta_{\lambda,\delta}(u_\lambda) < \infty$. Then, $0 \le v(x,t) \le v_\lambda < \infty$, where $v = \zeta_{\lambda,\delta}(u)$.

By differentiating $(V_{\lambda,\delta})$ in time and setting $w = v_t$, we get for any fixed $t_0 > 0$:

$$\begin{cases} w_t - \triangle w = \left[\dfrac{\lambda\delta}{(1-u)^4} + \dfrac{2}{(1-u)^3}\right] w, & (x,t) \in \Omega_{t_0} \\ \qquad\qquad w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t_0) \\ \qquad\qquad w(x,0) \ge 0, & x \in \Omega. \end{cases}$$

Here $\left[\frac{\lambda\delta}{(1-u)^4} + \frac{2}{(1-u)^3}\right]$ is a locally bounded nonnegative function, and by the strong maximum principle, we get that $v_t = w > 0$ for Ω_{t_0} or $w = 0$. The second case cannot happen; otherwise $u(x,t) = u_\lambda(x)$ for any $t > 0$. It follows that $w = v_t > 0$ for all $\Omega \times (0,\infty)$. Moreover, since $v(x,t)$ is bounded, the monotonicity in time implies that the unique solution $v(x,t)$ converges to some steady-state, denoted as $v_{ss}(x)$, as $t \to \infty$, i.e. $u(x,t) \to u_{ss}(x)$, as $t \to \infty$. Hence, $1 > u_\lambda(x) \ge u_{ss}(x) > 0$ in Ω.

Next, we claim that $u_{ss}(x)$ is a solution of $(SF_{\lambda,\delta})$. Let us consider $v_1(x)$ satisfying

$$\begin{cases} -\triangle v_1 = \lambda\rho_{\lambda,\delta}(v_{ss}), & x \in \Omega \\ v_1(x) = 0, & x \in \partial\Omega. \end{cases}$$

Let $\bar{v}(x,t) = v(x,t) - v_1(x)$; then $\bar{v}$ satisfies $\bar{v}(x,0) = -v_1(x)$, $\bar{v}|_{\partial\Omega\times(0,\infty)} = 0$ and

$$\bar{v}_t - \triangle\bar{v} = \lambda\left[\frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2} - \frac{e^{\frac{\lambda\delta}{1-u_{ss}}}}{(1-u_{ss})^2}\right], \tag{2.5}$$

in Ω. The right-hand side of (2.5) tends to zero in $L^2(\Omega)$, as $t \to \infty$, which follows from

$$\left|\frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2} - \frac{e^{\frac{\lambda\delta}{1-u_{ss}}}}{(1-u_{ss})^2}\right| \overset{(2.2)}{\leq} e^{\frac{\lambda\delta}{1-u_\lambda}}\left[\frac{\lambda\delta}{(1-u_\lambda)^4} + \frac{2}{(1-u_\lambda)^3}\right]|u - u_{ss}|$$

and Hölder's inequality. A standard eigenfunction expansion implies that $w(x,t)$ converges to zero in $L^2(\Omega)$ as $t \to \infty$. That is, $v(x,t) \to v_1(x)$, as $t \to \infty$. Combined with the fact that $v(x,t) \to v_{ss}(x)$ pointwisely as $t \to \infty$, we deduce that $v_1(x) = v_{ss}(x)$ in $L^2(\Omega)$, which implies $v_{ss}(x)$ is also a solution to the stationary equation $(V_{\lambda,\delta})$ and the corresponding $u_{ss}(x)$ is also a solution to $(SF_{\lambda,\delta})$. The minimal property of u_λ yields that $u_\lambda \equiv u_s$ in Ω, from which follows that for every $x \in \Omega$, we have $u(x,t) \uparrow u_\lambda(x)$, as $t \to \infty$. □

2.2. *Finite-time quenching.*

THEOREM 2.3 (Finite-time quenching). For every $\lambda > \lambda_\delta^*$, there exists a finite time $T = T(\lambda,\delta)$ at which the unique solution $u(x,t)$ of $(F_{\lambda,\delta})$ quenches.

Proof. By contradiction, let $\lambda > \lambda_\delta^*$ and suppose there exists a solution $u(x,t)$ of $(F_{\lambda,\delta})$ in $\Omega \times (0,\infty)$.

Claim: Given any $\varepsilon \in (0, \lambda - \lambda_\delta^*)$, $(F_{\lambda-\varepsilon,\delta})$ has a global solution u_ε, which is uniformly bounded in $\Omega \times (0,\infty)$ by some constant $C_\varepsilon < 1$.

We follow a similar argument as in [1] or [7]. Let

$$g(u) = \frac{1}{(1-u)^2}, \quad h(u) = \int_0^u \frac{ds}{g(s)}, \quad 0 \leq u \leq 1;$$
$$\tilde{g}(u) = \frac{\lambda - \varepsilon}{\lambda(1-u)^2}, \quad \tilde{h}(u) = \int_0^u \frac{ds}{\tilde{g}(s)}, \quad 0 \leq u \leq 1;$$

and

$$\Phi_\varepsilon(u) = \tilde{h}^{-1} \circ h(u).$$

Direct computations yield that

$$\Phi_\varepsilon(u) = 1 - \left[\frac{\varepsilon}{\lambda} + \frac{\lambda-\varepsilon}{\lambda}(1-u)^3\right]^{\frac{1}{3}} \leq C_\varepsilon < 1,$$

for $0 \leq u \leq 1$, where $C_\varepsilon = 1 - \left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{3}}$. Moreover, it is easy to check that $\Phi_\varepsilon(0) = 0$, $0 \leq \Phi_\varepsilon(s) < s$, for $s \geq 0$, and $\Phi_\varepsilon(s)$ is increasing and concave with

$$\Phi_\varepsilon'(s) = \frac{\tilde{g} \circ \Phi_\varepsilon(s)}{g(s)} > 0.$$

Setting $w_\varepsilon = \Phi_\varepsilon(u)$, we have

$$\begin{aligned}
-\triangle w_\varepsilon &= -\Phi''_\varepsilon(u)|\nabla u|^2 - \Phi'_\varepsilon(u)\triangle u \geq \Phi'_\varepsilon(u)\left[\lambda\frac{(1+\delta|\nabla u|^2)}{(1-u)^2} - u_t\right]\\
&= \lambda(1+\delta|\nabla u|^2)\frac{\Phi'_\varepsilon(u)}{(1-u)^2} - (w_\varepsilon)_t = \lambda(1+\delta|\nabla u|^2)\tilde{g}(w_\varepsilon) - (w_\varepsilon)_t\\
&= \frac{(\lambda-\varepsilon)(1+\delta|\nabla u|^2)}{(1-w_\varepsilon)^2} - (w_\varepsilon)_t.
\end{aligned}$$

Notice that

$$\Phi'_\varepsilon(u) = \frac{\frac{\lambda-\varepsilon}{\lambda}(1-u)^2}{\left[\frac{\varepsilon}{\lambda} + \frac{\lambda-\varepsilon}{\lambda}(1-u)^3\right]^{\frac{2}{3}}} \leq \frac{\frac{\lambda-\varepsilon}{\lambda}(1-u)^2}{\left[\frac{\lambda-\varepsilon}{\lambda}(1-u)^3\right]^{\frac{2}{3}}} = \left(\frac{\lambda-\varepsilon}{\lambda}\right)^{\frac{1}{3}} < 1.$$

Hence,

$$|\nabla w_\varepsilon|^2 = (\Phi'_\varepsilon(u))^2|\nabla u|^2 \leq |\nabla u|^2.$$

Furthermore,

$$-\triangle w_\varepsilon \geq \frac{(\lambda-\varepsilon)(1+\delta|\nabla w_\varepsilon|^2)}{(1-w_\varepsilon)^2} - (w_\varepsilon)_t.$$

This means that $w_\varepsilon = \Phi_\varepsilon(u) \leq C_\varepsilon$ is the supersolution to $(F_{\lambda-\varepsilon,\delta})$. Since zero is a subsolution of $(F_{\lambda-\varepsilon,\delta})$, we deduce that there exists a unique global solution u_ε for $(F_{\lambda-\varepsilon,\delta})$ satisfying $0 \leq u_\varepsilon \leq w_\varepsilon \leq C_\varepsilon < 1$ uniformly in $\Omega \times (0,\infty)$.

Let $v_\epsilon = \zeta_{\lambda-\epsilon,\delta}(u_\epsilon)$. It is clear to see that v_ϵ is a global classical solution to $(V_{\lambda-\epsilon,\delta})$. And it has been checked previously that $\rho_{\lambda-\epsilon,\delta}$ is a nondecreasing, convex function, and there exists some $v_0 > 0$ such that $\rho_{\lambda-\epsilon,\delta}(v_0) > 0$ and

$$\int_{v_0}^{\infty} \frac{ds}{\rho_{\bar{\lambda},\delta}(s)} < \infty.$$

Therefore, from Theorem 1, [1], we obtain a weak solution to the stationary equation of $(V_{\lambda-\epsilon,\delta})$, where $0 < \varepsilon < \lambda - \lambda^*_\delta$. In fact, using the Sobolev embedding theorem and a boot-strap argument, any weak solution to the stationary equation of $(V_{\lambda-\epsilon,\delta})$ satisfying $\rho_{\lambda-\epsilon,\delta}(v) \in L^1(\Omega)$ is indeed smooth. This contradicts the nonexistence result in [26]. □

3. Estimates for the pull-in voltage and the finite quenching time. A lower bound of λ^*_δ is given in Theorem 2.2, [25], i.e.

$$\lambda_l := \frac{4}{27}\frac{||\xi||_\infty}{||\xi||^2_\infty + \delta||\triangle\xi||^2_\infty} \leq \lambda^*_\delta, \tag{3.1}$$

where ξ is the solution to $-\triangle\xi = 1$, $x \in \Omega$ with the Dirichlet boundary condition. And it is not difficult to see that λ^*, the pull-in voltage for (1.1), is an upper bound for λ^*_δ, due to the comparison principle. From [14], $\lambda^* \leq \frac{4}{27}\mu_0$, where $\mu_0 > 0$ is the first eigenvalue of $-\triangle\phi_0 = \mu_0\phi_0$, $x \in \Omega$ with Dirichlet boundary condition. We shall derive an upper bound for λ^*_δ to show explicit dependence of δ if $\delta \ll 1$:

PROPOSITION 3.1 (Upper bound for λ^*_δ, $\delta \ll 1$). The pull-in voltage $\lambda^*_\delta < \infty$ of $(F_{\lambda,\delta})$ has the upper bound

$$\lambda^*_\delta \leq \lambda_{u,1} := \frac{4}{27}\mu_0 \left(1 - \frac{1}{27||\xi||_\infty}\delta + \mathcal{O}(\delta^2)\right),$$

where ξ is the solution to $-\triangle\xi = 1$, $x \in \Omega$ with Dirichlet boundary condition if $\delta \ll 1$.

Proof. This argument is used in many estimates of the pull-in voltage (cf. Theorem 3.1, [21] or Theorem 2.1, [14]). Let $\mu_0 > 0$ and $\phi_0 > 0$ be the first eigenpair $-\triangle\phi_0 = \mu_0\phi_0$ in Ω with Dirichlet boundary condition. We multiply the stationary equation of $(V_{\lambda,\delta})$ by ϕ_0, integrate the resulting equation over Ω, and use Green's identity to get

$$\int_\Omega [-\mu_0 v + \lambda\rho_{\lambda,\delta}(v)]\phi_0 dx = \int_\Omega \left(-\mu_0 \int_0^u e^{\frac{\lambda\delta}{1-s}} ds + \lambda \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}\right) dx = 0.$$

Noting that $\int_0^u e^{\frac{\lambda\delta}{1-s}} ds \leq e^{\frac{\lambda\delta}{1-u}} u$, we get that if $\lambda > \frac{4}{27}\mu_0$, then

$$-\mu_0 \int_0^u e^{\frac{\lambda\delta}{1-s}} ds + \lambda \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2} \geq -\mu_0 e^{\frac{\lambda\delta}{1-u}} u + \lambda \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2} > 0, \tag{3.2}$$

for any $u \in [0,1]$. Therefore, there is no solution to the stationary equation of $(V_{\lambda,\delta})$, and so to $(SF_{\lambda,\delta})$, if $\lambda > \frac{4}{27}\mu_0$. That is, $\lambda^*_\delta \leq \frac{4}{27}\mu_0$. This is the upper bound obtained in [14]. In this way, we ignore the effect of δ completely. Let us go back to (3.2) and see that if

$$\lambda \geq \max_{u\in[0,1]} \left[\mu_0(1-u)^2 \int_0^u e^{\lambda_l\delta\left(\frac{1}{1-s} - \frac{1}{1-u}\right)} ds\right],$$

then (3.2) holds, where λ_l is in (3.1). Let us estimate the maximum in the following:

$$\int_0^u e^{\lambda_l\delta\left(\frac{1}{1-s} - \frac{1}{1-u}\right)} ds \leq \frac{1}{2}u \left[1 + e^{\lambda_l\delta\left(1 - \frac{1}{1-u}\right)}\right],$$

due to the convexity of the integrand $e^{\lambda_l\delta\left(\frac{1}{1-s} - \frac{1}{1-u}\right)}$. Therefore, if

$$\lambda \geq \max_{u\in[0,1]} \frac{1}{2}\mu_0(1-u)^2 u \left[1 + e^{\lambda_l\delta\left(1-\frac{1}{1-u}\right)}\right] = \frac{4}{27}\mu_0 - \frac{4\mu_0}{27^2||\xi||_\infty}\delta + \mathcal{O}(\delta^2),$$

then (3.2) holds, where ξ is the solution to $-\triangle\xi = 1$ for $x \in \Omega$ with Dirichlet boundary condition, provided $\delta \ll 1$. □

Next, we show the behavior of λ^*_δ as $\delta \to \infty$.

PROPOSITION 3.2 (λ^*_δ for $\delta \gg 1$). The pull-in voltage λ^*_δ of $(F_{\lambda,\delta})$ tends to 0, as $\delta \to \infty$. That is,

$$\lim_{\delta\to\infty} \lambda^*_\delta = 0.$$

Proof. As shown in Theorem 2.2 and Theorem 2.3, the pull-in voltage λ^*_δ of $(F_{\lambda,\delta})$ is the same as that of $(SF_{\lambda,\delta})$. Let us multiply $(SF_{\lambda,\delta})$ by $\phi_0 > 0$, the first eigenfunction of $-\triangle\phi_0 = \mu_0\phi_0$ in Ω with Dirichlet boundary condition, integrate over Ω, and use Green's identity to get

$$0 = \int_\Omega \left(-\mu_0 u + \frac{\lambda}{(1-u)^2}\right)\phi_0 dx + \lambda\delta \int_\Omega \frac{|\nabla u|^2}{(1-u)^2}\phi_0 dx. \tag{3.3}$$

By integration by parts, the third term in the above equation becomes

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{(1-u)^2}\phi_0 dx &= \int_\Omega \nabla u \cdot \frac{\nabla u}{(1-u)^2}\phi_0 dx = \int_\Omega \nabla u \cdot \nabla\left(\frac{1}{1-u}\right)\phi_0 dx\\ &= -\int_\Omega \triangle u \frac{1}{1-u}\phi_0 dx - \int_\Omega \nabla u \cdot \nabla \phi_0 \frac{1}{1-u} dx\\ &= \lambda \int_\Omega \frac{1+\delta|\nabla u|^2}{(1-u)^3}\phi_0 dx + \int_\Omega \nabla(\ln(1-u))\cdot\nabla\phi_0 dx\\ &= \lambda \int_\Omega \frac{1+\delta|\nabla u|^2}{(1-u)^3}\phi_0 dx + \mu_0 \int_\Omega \ln(1-u)\phi_0 dx. \end{aligned} \tag{3.4}$$

Furthermore, for $p \geq 3$, we have

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{(1-u)^p}\phi_0 dx =& \frac{1}{p-1}\int_\Omega \nabla u \nabla\left(\frac{1}{(1-u)^{p-1}}\right)\phi_0 dx\\ =& -\frac{1}{p-1}\int_\Omega \triangle u \frac{1}{(1-u)^{p-1}}\phi_0 dx - \frac{1}{p-1}\int_\Omega \nabla u \nabla\phi_0 \frac{1}{(1-u)^{p-1}} dx\\ =& \frac{\lambda}{p-1}\int_\Omega \frac{1+\delta|\nabla u|^2}{(1-u)^{p+1}}\phi_0 dx\\ &- \frac{1}{(p-1)(p-2)}\int_\Omega \nabla\phi_0\nabla\left(\frac{1}{(1-u)^{p-2}}\right)dx\\ =& \frac{\lambda}{p-1}\int_\Omega \frac{\phi_0}{(1-u)^{p+1}}dx + \frac{\lambda\delta}{p-1}\int_\Omega \frac{|\nabla u|^2}{(1-u)^{p+1}}\phi_0 dx\\ &- \frac{\mu_0}{(p-1)(p-2)}\int_\Omega \frac{\phi_0}{(1-u)^{p-2}}dx\\ &- \frac{1}{(p-1)(p-2)}\int_{\partial\Omega} \frac{\partial\phi_0}{\partial\nu}\frac{1}{(1-u)^{p-2}}dS, \end{aligned} \tag{3.5}$$

where ν is the outward unit normal vector of $\partial\Omega$. Substituting (3.4) and (3.5) into (3.3), we get

$$\begin{aligned}0 =& \int_\Omega \left\{-\mu_0 u + \frac{\lambda}{(1-u)^2} + \lambda\delta\left[\mu_0 \ln(1-u) + \frac{\lambda}{(1-u)^3}\right]\right\}\phi_0 dx\\ &+ \sum_{p=3}^{P} \frac{\lambda^{p-1}\delta^{p-1}}{(p-1)!}\left[\lambda\int_\Omega \frac{1}{(1-u)^{p+1}}\phi_0 dx - \frac{\mu_0}{p-2}\int_\Omega \frac{1}{(1-u)^{p-2}}\phi_0 dx\right.\\ &\left. - \frac{1}{p-2}\int_{\partial\Omega}\frac{\partial\phi_0}{\partial\nu}\frac{1}{(1-u)^{p-2}}dS\right]\\ &+ \frac{\lambda^P\delta^P}{(P-1)!}\int_\Omega \frac{|\nabla u|^2}{(1-u)^{P+1}}\phi_0 dx, \end{aligned} \tag{3.6}$$

for arbitrary $P > p$. By the boundary point lemma, we have $\frac{\partial\phi_0}{\partial\nu} < 0$ on $\partial\Omega$. Hence, the term $-\int_{\partial\Omega}\frac{\partial\phi_0}{\partial\nu}\frac{1}{(1-u)^{p-2}}dS$ is positive, as is the term $\int_\Omega \frac{|\nabla u|^2}{(1-u)^{P+1}}dx$. If $\delta \gg 1$, then $\mathcal{O}(\delta^{P-1})$ is the leading order term, except the last term in (3.6). The equality (3.6) cannot hold when

$$\frac{\lambda}{(1-u)^{P+1}} - \frac{\mu_0}{(P-2)(1-u)^{P-2}} > 0$$

holds for all $u \in [0,1]$. That is,

$$\lambda > \frac{\mu_0}{P-2} \max_{u\in[0,1]} (1-u)^3 = \frac{\mu_0}{P-2}, \tag{3.7}$$

where $P = P(\delta) \to \infty$ if $\delta \to \infty$. Our result follows immediately. □

PROPOSITION 3.3 (Upper bound of T). Let ϕ be any nonnegative C^2 function such that $\phi \not\equiv 0$ and $\phi = 0$ on $\partial\Omega$. Then for λ large enough, the quenching time T for the solution to $(F_{\lambda,\delta})$ satisfies

$$T \leq \frac{||\phi||_1}{3\lambda||\phi||_1 - ||\triangle\phi||_1}, \tag{3.8}$$

where $||\cdot||_1$ is the L^1 norm of $\cdot$ in Ω.

Proof. Using $\varphi(1-u)^2$ as the test function to $(F_{\lambda,\delta})$ and integrating over Ω,

$$\begin{aligned}\left(\int_\Omega \frac{1}{3}\left[1-(1-u)^3\right]\varphi dx\right)_t &= \int_\Omega \triangle u\varphi(1-u)^2dx + \lambda\int_\Omega (1+\delta|\nabla u|^2)\varphi dx \\ &\geq -\int_\Omega \nabla u\nabla\varphi(1-u)^2dx + 2\int_\Omega |\nabla u|^2\varphi(1-u) \\ &\quad + \lambda\int_\Omega \varphi dx \\ &\geq \frac{1}{3}\int_\Omega \left[1-(1-u)^3\right]\triangle\varphi dx + \lambda\int_\Omega \varphi dx.\end{aligned}$$

Hence, for any $t < T$, integrating from 0 to t, we obtain that

$$\begin{aligned}\frac{1}{3}\int_\Omega \varphi dx &\geq \frac{1}{3}\int_\Omega \left[1-(1-u(x,t))^3\right]\varphi dx \\ &\geq \frac{1}{3}\int_0^t\int_\Omega \left[1-(1-u)^3\right]\triangle\varphi dx + \lambda t\int_\Omega \varphi dx \geq \lambda t\int_\Omega \varphi dx - \frac{1}{3}t\int_\Omega |\triangle\varphi|dx.\end{aligned}$$

By tending t to T, we are done. □

We compare the quenching time $T = T(\lambda,\delta)$ with different λ:

PROPOSITION 3.4. Suppose $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ are solutions of $(F_{\lambda,\delta})$ with $\lambda = \lambda_1$ and λ_2, respectively. The corresponding finite quenching times are T_{λ_1} and T_{λ_2}, respectively. If $\lambda_1 > \lambda_2$, then $T_{\lambda_1} < T_{\lambda_2}$.

Proof. Let $\hat{v} = v_1 - v_2$, where v_i, $i = 1,2$, are the corresponding solutions of $(V_{\lambda_i,\delta})$, $i = 1,2$, respectively. Then $\hat{v}|_{\partial\Omega}(x,t) = \hat{v}(x,0) = 0$ and

$$\hat{v}_t - \triangle\hat{v} = \lambda_1\rho_{\lambda_1,\delta}(v_1) - \lambda_2\rho_{\lambda_2,\delta}(v_2) > \lambda_2[\rho_{\lambda_2,\delta}(v_1) - \rho_{\lambda_2,\delta}(v_2)] = \lambda_2\rho'_{\lambda_2,\delta}(\theta)\hat{v},$$

with $\rho'_{\lambda_2,\delta}(\theta) \geq 0$, for some function θ. Hence, $v_1 > v_2$ in $\Omega \times (0, \min\{T_{\lambda_1}, T_{\lambda_2}\})$. Thus, $T_{\lambda_1} < T_{\lambda_2}$. □

REMARK 3.5. Fix the voltage $\lambda > \max\{\lambda^*_{\delta_1}, \lambda^*_{\delta_2}\}$. If $\delta_1 > \delta_2 > 0$, then $T_{\delta_1} < T_{\delta_2}$, where T_{δ_i} are the finite quenching times corresponding to δ_i, $i = 1, 2$. This observation follows immediately from

$$\partial_t u_1 - \triangle u_1 = \lambda \frac{1 + \delta_1 |\nabla u_1|^2}{(1 - u_1)^2} > \lambda \frac{1 + \delta_2 |\nabla u_1|^2}{(1 - u_1)^2},$$

which means that $u_1 > u_2$ in $\Omega_{\min\{T_{\delta_1}, T_{\delta_2}\}}$. Hence, $T_{\delta_1} < T_{\delta_2}$.

4. Quenching set.

In this section, we assume that Ω is a bounded convex subset of $\mathbb{R}^n$. It is followed by the moving-plane argument that the quenching set of any finite-time quenching solution to $(F_{\lambda,\delta})$ is a compact subset of Ω.

THEOREM 4.1 (Compactness of the quenching set). *Suppose $\Omega \subset \mathbb{R}^n$ is convex and $u(x,t)$ is a solution to $(F_{\lambda,\delta})$ which quenches in finite time T. Then the set of the quenching points is a compact subset of Ω.*

Proof (Adaption of moving-plane argument). It is equivalent to show that the set of the blow-up points of v in $(V_{\lambda,\delta})$ is a compact subset of Ω.

Let us denote $x = (x_1, x') \in \mathbb{R}^n$, where $x' = (x_2, x_3, \cdots, x_n) \in \mathbb{R}^{n-1}$. Take any point $y_0 \in \partial\Omega$ and assume without loss of generality that $y_0 = 0$ and that the half space $\{x_1 > 0\}$ is tangent to Ω at y_0.

Let $\Omega^+_\alpha = \Omega \cap \{x_1 > \alpha\}$, $\alpha < 0$, $|\alpha|$ small, and $\Omega^-_\alpha = \{x = (x_1, x') \in \mathbb{R}^n : (2\alpha - x_1, x') \in \Omega^+_\alpha\}$, the reflection of Ω^+_α with respect to $\{x_1 = \alpha\}$.

First, from the maximum principle, we observe that

$$v \geq 0, \tag{4.1}$$

for $(x,t) \in \Omega_T$ and $\frac{\partial v}{\partial \nu}(t_0) < 0$ on $\partial\Omega$ for some small $t_0 \in (0, T)$.

Let us consider

$$\bar{v}(x,t) = v(x_1, x', t) - v(2\alpha - x_1, x', t),$$

for $x \in \Omega^-_\alpha$; then $\bar{v}$ satisfies

$$\partial_t \bar{v} - \triangle \bar{v} = \lambda \left[\rho_{\lambda,\delta}(v(x_1, x', t)) - \rho_{\lambda,\delta}(v(2\alpha - x_1, x', t))\right] = \lambda c(x,t)\bar{v},$$

where $c(x,t)$ is a bounded function. It is clear that $\bar{v} = 0$ on $\{x_1 = \alpha\}$ and $\bar{v} = v(x_1, x', t) \geq 0$ on $\partial\Omega^-_\alpha \cap \{x_1 < \alpha\} \times (0, T]$. If α is small enough, then $\bar{v}(x, t_0) = v(x_1, x', t_0) - v(2\alpha - x_1, x', t_0) \geq 0$, for $x \in \Omega^-_\alpha$. Applying the maximum principle, we conclude that

$$\bar{v} > 0 \quad \text{in } \Omega^-_\alpha \times (t_0, T) \quad \text{and} \quad \frac{\partial \bar{v}}{\partial v_1} = -\frac{2\partial v}{\partial x_1} > 0 \quad \text{on } \{x_1 = \alpha\}.$$

Since α is arbitrary, it follows by varying α that

$$\frac{\partial v}{\partial x_1} < 0, \tag{4.2}$$

for $x \in \Omega^+_{\alpha_0}$, $t_0 < t < T$, provided that α_0 is small enough.

Let us consider

$$J = v_{x_1} + \varepsilon_1 (x_1 - \alpha_0)$$

in $\Omega_{\alpha_0}^+ \times (t_0, T)$, where $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ is a constant to be determined later. Through direct computations, we obtain that

$$\begin{aligned}\partial_t J - \triangle J =& \triangle(v_{x_1}) + \lambda \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^3} u_{x_1} \left[2 + \frac{\lambda\delta}{1-u}\right] - \triangle(v_{x_1}) \\ =& \frac{\lambda v_{x_1}}{(1-u)^3}\left[2 + \frac{\lambda\delta}{1-u}\right] \le 0,\end{aligned}$$

in $\Omega_{\alpha_0}^+ \times (t_0, T)$, where u is the corresponding solution to $(F_{\lambda,\delta})$. Therefore, J cannot obtain positive maximum in $\Omega_{\alpha_0}^+ \times (t_0, T)$. Next, $J < 0$ on $\{x_1 = \alpha_0\}$ by (4.2). From (4.1), $\frac{\partial v(x,t_0)}{\partial x_1} \le C < 0$.

If we can show $J < 0$ on $\Gamma \times (t_0, T)$, where $\Gamma = \partial\Omega_{\alpha_0}^+ \cap \partial\Omega$, then

$$J < 0, \tag{4.3}$$

in $\Omega_{\alpha_0}^+ \times (t_0, T)$. To show (4.3), we compare v with the solution z of the heat equation

$$\begin{cases} z_t - \triangle z = 0 & \text{in} \quad \Omega \times (t_0, T) \\ z(x,t) = 0 & \text{on} \quad \partial\Omega \times (t_0, T) \\ z(x,t_0) = 0 & \text{in} \quad \Omega \times (t_0, T). \end{cases} \tag{4.4}$$

Since $\lambda\rho_{\lambda,\delta}(v) \ge 0$, we have $v \ge z$. Consequently, $\frac{\partial v}{\partial \nu} < \frac{\partial z}{\partial \nu} \le -C_0 < 0$ on $\partial\Omega \times (t_0, T)$. It follows that if $x \in \Gamma$,

$$J \le -C_0 \frac{\partial x_1}{\partial \nu} + \varepsilon_1(x_1 - \alpha_0) < 0,$$

provided ε_1 is small enough. Now, the maximum principle yields that there exists $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0)$ small enough such that $J \le 0$ in $\Omega_{\alpha_0}^+ \times (t_0, T)$, i.e.

$$-v_{x_1} = |v_{x_1}| \ge \varepsilon_1(x_1 - \alpha_0),$$

if $x' = 0$, $\alpha_0 \le x_1 < 0$. Integrating with respect to x_1, we get for any $\alpha_0 < y_1 < 0$,

$$-v(y_1, 0, t) + v(\alpha_0, 0, t) \ge \frac{\varepsilon_1}{2}|y_1 - \alpha_0|^2.$$

It follows that

$$\begin{aligned}\liminf_{t\to T^-} v(0,t) =& \liminf_{t\to T^-} \lim_{y_1\to 0^-} v(y_1, 0, t) \\ \le& \liminf_{t\to T^-} \lim_{y_1\to 0^-} \left[v(\alpha_0, 0, t) - \frac{\varepsilon_1}{2}|y_1 - \alpha_0|^2\right] < \infty.\end{aligned}$$

Thus, every point in $\{x' = 0,\ \alpha_0 < x_1 < 0\}$ is not a blow-up point. The above proof shows that α_0 can be chosen independently of $y_0 \in \partial\Omega$. Hence, by varying $y_0 \in \partial\Omega$, we conclude that there is an Ω-neighborhood Ω' of $\partial\Omega$ such that each point $x \in \Omega'$ is not a blow-up point. Since the blow-up points lie in a compact subset of Ω, it is clearly a closed set. □

In addition, if $\Omega = B_R(0)$ is a ball of radius R centered at the origin, then according to [8] we conclude that any solution $u(x,t)$ is indeed radial symmetric, i.e. $u(x,t) = u(r,t)$, with $r = |x| \in [0, R]$. Furthermore, we can show that the only possible quenching point is the origin.

THEOREM 4.2. Suppose $\Omega = B_R$. If $\lambda > \lambda_\delta^*$, then the solution quenches only at $r = 0$. That is, the origin is the unique quenching point.

LEMMA 4.3. $v_r < 0$ in $\Omega_T \cap \{r > 0\}$.

Proof. Set $\bar{v} = r^{n-1} v_r$. Then $(V_{\lambda,\delta})$ becomes

$$v_t - \frac{1}{r^{n-1}} \bar{v}_r = \lambda \rho_{\lambda,\delta}(v). \tag{4.5}$$

Differentiating with respect to r, we get

$$\bar{v}_t + \frac{n-1}{r}\bar{v}_r - \bar{v}_{rr} - \frac{\lambda}{(1-u)^3}\left(2 + \frac{\lambda\delta}{1-u}\right)\bar{v} = 0.$$

Since $\bar{v} = r^{n-1} v_r < 0$ on $\partial\Omega \times (0, T)$ (by the maximum principle) and $\bar{v}(r, 0) = 0$ by $(V_{\lambda,\delta})$, we deduce by the maximum principle that $v_r < 0$ in $\Omega_T \cap \{r > 0\}$. ☐

Proof of Theorem 1.3. Let us consider as in Theorem 2.3, [4]

$$J = \bar{v} + c(r)F(v),$$

where $\bar{v} = r^{n-1}v_r$ is defined as in Lemma 4.3, F, c are positive functions to be determined and $F' \geq 0$, $F'' \geq 0$. We aim to show that $J \leq 0$ in Ω_T. Through direct computations, we have

$$\begin{aligned}
J_t + \frac{n-1}{r}J_r - J_{rr} =& \frac{\lambda}{(1-u)^3}\left[2 + \frac{\lambda\delta}{1-u}\right]\bar{v} \\
&+ cF'f + \frac{2(n-1)}{r}cF'v_r + \frac{n-1}{r}c'F - cF''v_r^2 - 2c'F'v_r - c''F \\
\leq& \left\{\frac{\lambda}{(1-u)^3}\left[2 + \frac{\lambda\delta}{1-u}\right] + \frac{2(n-1)}{r^n}cF' - \frac{2c'F'}{r^{n-1}}\right\} J \\
&+ \left\{-\frac{\lambda}{(1-u)^3}\left[2 + \frac{\lambda\delta}{1-u}\right]cF + cF'\frac{\lambda e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}\right. \\
&\left. - \frac{2(n-1)}{r^n}c^2F'F + \frac{n-1}{r}c'F + \frac{2}{r^{n-1}}cc'F'F - c''F\right\} \\
:=& AJ + B,
\end{aligned}$$

by using $\bar{v} = J - cF$ and $\bar{v} = r^{n-1}v_r$. It is easy to see that A is a bounded function for $0 < r < R$. Let us choose

$$c(r) = \varepsilon r^n \quad \text{and} \quad F(v) = \frac{e^{\frac{\lambda\delta}{1-u}}}{(1-u)^\gamma},$$

where $u = \zeta_{\lambda,\delta}^{-1}(v)$, $\gamma \geq 0$ is some constant to be determined later. Direct computations yield that

$$B = c(r)e^{\frac{\lambda\delta}{1-u}}\left\{\frac{\lambda(\gamma-2)}{(1-u)^{\gamma+3}} + 2\varepsilon\left[\frac{\lambda\delta}{(1-u)^{2\gamma+2}} + \frac{\gamma}{(1-u)^{2\gamma+1}}\right]\right\} \leq 0,$$

if $\gamma < 1$ and $\varepsilon \ll 1$. $J = 0$ at $r = 0$, due to $c(0) = 0$, and it follows that J cannot obtain positive maximum in Ω_T or on $\{t = T\}$.

Next, we observe that J cannot obtain positive maximum on $\{r = R\}$ if $J_r \le 0$ on $\{r = R\}$ since

$$J_r(R) = \bar{v}_r + cF'v_r + c'F \le \bar{v}_r + c'F \overset{(4.5)}{=} -R^{n-1}\lambda e^{\lambda\delta} + c'(R)F(0) \le 0,$$

provided that $\varepsilon \ll 1$. Finally, by the maximum principle, there exists $0 < t_0 < T$ such that $v_r(r, t_0) < 0$ for $0 < r \le R$ and $v_{rr}(0, t_0) < 0$. Thus, $J(r, t_0) < 0$ for $0 \le r < R$, provided $\varepsilon \ll 1$.

Therefore, by the maximum principle, we conclude that $J \le 0$ in $B_R \times [t_0, T]$, for any $0 < t_0 < T$. That is,

$$-r^{n-1}e^{\frac{\lambda\delta}{1-u}}u_r = -r^{n-1}v_r \ge c(r)F(v) = \frac{\varepsilon r^n e^{\frac{\lambda\delta}{1-u}}}{(1-u)^\gamma},$$

for $0 \le \gamma < 1$. It follows that

$$\frac{d}{dr}\left[\frac{1}{\gamma+1}[1-u(r,t)]^{\gamma+1}\right] \ge \varepsilon r.$$

Integrating from 0 to r, we obtain that

$$\frac{1}{\gamma+1}[1-u(r,t)]^{\gamma+1} - \frac{1}{\gamma+1}[1-u(0,t)]^{\gamma+1} \ge \frac{1}{2}\varepsilon r^2.$$

It is known that 0 is in the set of quenching points, so

$$[1-u(r,t)]^{\gamma+1} \ge \frac{\gamma+1}{2}\varepsilon r^2. \tag{4.6}$$

If for any $0 < r < R$, $u(r, t) \to 1$, as $t \to T$, then the left-hand side tends to 0. This contradicts (4.6). Therefore, 0 is the only quenching point. $\square$

5. Quenching behavior.

5.1. *Upper bound estimate.* We first obtain a one-side quenching estimate. A similar result has been obtained in [18] for the only one-dimension case, i.e. $x \in \mathbb{R}$.

LEMMA 5.1 (One-side quenching estimate). If $\Omega \subset \mathbb{R}^n$ is a bounded convex domain and $u(x, t)$ is a quenching solution of $(F_{\lambda,\delta})$ in finite time, then there exists a bounded positive constant $M > 0$ such that

$$M(T-t)^{\frac{1}{3}} \le 1 - u(x,t),$$

for all Ω_T. Moreover, $u_t \to +\infty$ as u quenches.

Proof. Since Ω is a convex bounded domain, we show in Theorem 4.1 that the quenching set of u is a compact subset of Ω. It now suffices to discuss the point x_0 lying in the interior domain $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$, for some small $\eta > 0$; i.e. there is no quenching point in $\Omega_\eta^c := \Omega \setminus \Omega_\eta$.

For any $t_1 < T$, we recall the maximum principle gives $u_t > 0$, for all $(x, t) \in \Omega \times (0, t_1)$. Furthermore, the boundary point lemma shows that the exterior normal derivative of u_t on $\partial\Omega$ is negative for $t > 0$. This implies that for any small $0 < t_0 < T$, there exists a positive constant $C = C(t_0, \eta)$ such that $u_t(x, t_0) \ge C > 0$, for all $x \in \bar{\Omega}_\eta$. For any $0 < t_0 < t_1 < T$, we claim that

$$J^\varepsilon(x,t) = v_t - \varepsilon\rho_{\lambda,\delta}(v) \ge 0,$$

for all $(x,t) \in \Omega_\eta \times (t_0,t_1)$, where v is the corresponding solution to $(V_{\lambda,\delta})$. In fact, it is clear that there exists $C_\eta = C(t_0,t_1,\eta) > 0$ such that $v_t = e^{\frac{\lambda\delta}{1-u}} u_t \geq C_\eta$ on $\Omega_\eta \times (t_0,t_1)$. Further, we can choose $\varepsilon = \varepsilon(t_0,t_1,\eta) > 0$ small enough so that $J^\varepsilon \geq 0$ on the parabolic boundary of $\Omega_\eta \times (t_0,t_1)$, due to the local boundedness of $\rho_{\lambda,\delta}(v)$ on $\partial\Omega_\eta \times (t_0,t_1)$. Then the claim is followed by the maximum principle and the direct computations:

$$J_t^\varepsilon - \triangle J^\varepsilon = \lambda\rho'_{\lambda,\delta}(v)J^\varepsilon + \varepsilon\rho''_{\lambda,\delta}(v)|\nabla v|^2 \geq \lambda\rho'_{\lambda,\delta}(v)J^\varepsilon,$$

due to the convexity of $\rho_{\lambda,\delta}$. This yields that for any $0 < t_0 < t_1 < T$, there exists $\varepsilon = \varepsilon(t_0,t_1,\eta) > 0$ such that

$$u_t \geq \frac{\varepsilon}{(1-u)^2},$$

for all $\Omega_\eta \times (t_0,t_1)$. This inequality implies that $u_t \to \infty$ as u touches down, and there exists $M > 0$ such that

$$M(T-t)^{\frac{1}{3}} \leq 1 - u(x,t), \tag{5.1}$$

in $\Omega_\eta \times (0,T)$, due to the arbitrariness of t_0 and t_1, where $M = M(\lambda,\delta,\eta)$. Furthermore, one can obtain (5.1) for $\Omega \times (0,T]$, due to the boundedness of u on Ω_η^c. □

5.2. *Gradient estimate.* We shall study the quenching rate for the higher derivatives of u. The idea of the proof is similar to Proposition 1, [9] and Lemma 2.6, [13].

LEMMA 5.2. Suppose u is a quenching solution of $(F_{\lambda,\delta})$ in finite time T for any point $x = a \in \Omega_\eta$, for some small $\eta > 0$. Then there exists a positive constant M' such that

$$|\nabla^m u(x,t)|(T-t)^{-\frac{1}{3}+\frac{m}{2}} \leq M', \tag{5.2}$$

$m = 1,2$, holds for $Q_R = B_R \times (T-R^2,T)$, for any $R > 0$ such that $a + R \in \Omega_\eta$.

Proof. It suffices to consider the case $a = 0$ by translation. We may focus on some fixed r, such that $\frac{1}{2}R^2 < r^2 < R^2$ and denote $Q_r = B_r \times \left(T\left(1-\left(\frac{r}{R}\right)^2\right), T\right)$.

Let us first show that $|\nabla u|$ and $|\nabla^2 u|$ are uniformly bounded on a compact subset of Q_R. Indeed, since $\rho_{\lambda,\delta}(v)$ is bounded on any compact subset D of Q_R, standard L^p estimates for heat equations (see [16]) give

$$\iint_D \left(|\nabla^2 v|^p + |v_t|^p\right) dxdt < C,$$

for $1 < p < \infty$ and any cylinder D with $\bar{D} \subset Q_R$. It also holds for u, i.e.

$$\iint_D \left(|\nabla^2 u|^p + |u_t|^p\right) dxdt < C,$$

$1 < p < \infty$, where C is a generic constant and may vary from line to line. Choosing p large, by the Sobolev embedding theorem, we conclude that u is Hölder continuous on D, and so is $\rho_{\lambda,\delta}(v)$. Therefore, Schauder's estimates for the heat equation (see [16]) show that $|\nabla v|$ and $|\nabla^2 v|$ are bounded on any compact subets of D, and so are $|\nabla u|$ and $|\nabla^2 u|$. In particular, there exists M_1 such that

$$|\nabla u| + |\nabla^2 u| \leq M_1,$$

for $(x,t) \in B_r \times \left(T\left(1-\left(\frac{r}{R}\right)^2\right), T\left(1-\frac{1}{2}\left(1-\frac{r}{R}\right)^2\right)\right)$, where M_1 depends on R, n and M given in (5.1).

We next prove (5.2) for $B_r \times \left[T\left(1-\frac{1}{2}\left(1-\frac{r}{R}\right)^2\right), T\right)$. For fixed point $(x,t) \in B_r \times \left[T\left(1-\frac{1}{2}\left(1-\frac{r}{R}\right)^2\right), T\right)$, we consider

$$\bar{u}(z,\tau) = 1 - \mu^{-\frac{2}{3}}\left[1 - u\left(x+\mu z, T - \mu^2(T-\tau)\right)\right], \tag{5.3}$$

where $\mu = \left[2\left(1-\frac{t}{T}\right)\right]^{\frac{1}{2}}$, which satisfies

$$\begin{cases} \bar{u}_\tau - \triangle_z \bar{u} = \lambda \dfrac{1+\delta\mu^{-\frac{2}{3}}|\nabla_z \bar{u}|^2}{(1-\bar{u})^2}, & (z,\tau) \in \mathcal{O}_T \\ \bar{u}(z,\tau) = 1 - \mu^{-\frac{2}{3}} < 0, \quad (z,\tau) \in \partial\mathcal{O}_T \\ \bar{u}(z,0) = \bar{u}_0(z), \quad z \in \mathcal{O}, \end{cases} \tag{5.4}$$

where $\bar{u}_0(z) = 1 - \mu^{-\frac{2}{3}}[1 - u(x+\mu z, T(1-\mu^2))] < 0$ and $\triangle_z \bar{u}_0 + \lambda \frac{1+\delta\mu^{-\frac{2}{3}}|\nabla_z \bar{u}_0|^2}{(1-\bar{u}_0)^2} > 0$ on $\mathcal{O}$. For the fixed point (x,t), we define $\mathcal{O} := \{z : x + \mu z \in \Omega\}$. It is implied by (5.3) that T is also the finite quenching time of $\bar{u}$, and the domain of $\bar{u}$ includes Q_{r_0} for some $r_0 = r_0(R) > 0$. Since the quenching set of u is a compact subset of Ω, due to Theorem 4.1, so is that of $\bar{u}$. Therefore, the argument of Lemma 1.4 can be applied to (5.4), yielding that there exists a constant $M_2 > 0$ such that

$$1 - \bar{u}(z,\tau) \geq M_2(T-\tau)^{\frac{1}{3}},$$

where M_2 depends on R, λ, δ and Ω. Applying the interior L^p estimates and Schauder's estimates to $\bar{u}$ as before, there exists $M_1' = M_1'(R,\lambda,\delta,n,M_2) > 0$ such that

$$|\nabla_z \bar{u}| + |\nabla_z^2 \bar{u}| \leq M_1', \tag{5.5}$$

for $(z,\tau) \in B_r \times \left(T\left(1-\left(\frac{r}{r_0}\right)^2\right), T\left(1-\frac{1}{2}\left(1-\frac{r}{r_0}\right)^2\right)\right)$, where we assume that $\frac{1}{2}r_0^2 < r^2 < r_0^2$. Applying (5.3) and taking $(z,\tau) = \left(0,\frac{T}{2}\right)$, (5.5) gives

$$\mu^{-\frac{1}{3}+1}|\nabla_x u| + \mu^{-\frac{1}{3}+2}|\nabla_x^2 u| \leq M_1'.$$

Thus, (5.2) follows immediately from $\mu = \left[2\left(1-\frac{t}{T}\right)\right]^{\frac{1}{2}}$. □

5.3. *Lower bound estimate.* First, we note the following local lower bound estimate.

PROPOSITION 5.3. Suppose $u(x,t)$ is a quenching solution of $(F_{\lambda,\delta})$ in finite time T. Then, there exists a bounded constant $C = C(\lambda,\Omega) > 0$ such that

$$\max_{x\in\Omega} u(x,t) \geq 1 - C(T-t)^{\frac{1}{3}}, \tag{5.6}$$

for $0 < t < T$.

Proof. Let $U(t) = \max_{x\in\Omega} u(x,t)$, $0 < t < T$, and let $U(t_i) = u(x_i,t_i)$, $i = 1,2$, with $h = t_2 - t_1 > 0$. Then,

$$U(t_2) - U(t_1) \geq u(x_1,t_2) - u(x_1,t_1) = hu_t(x_1,t_1) + o(1);$$
$$U(t_2) - U(t_1) \leq u(x_2,t_2) - u(x_2,t_1) = hu_t(x_2,t_2) + o(1).$$

It follows that $U(t)$ is Lipschitz continuous. Hence, for $t_2 > t_1$, we have

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \leq u_t(x_2, t_2) + o(1).$$

On the other hand, since $\nabla u(x_2, t_2) = 0$ and $\triangle u(x_2, t_2) \leq 0$, we obtain

$$u_t(x_2, t_2) \leq \frac{\lambda}{(1 - u(x_2, t_2))^2} = \frac{\lambda}{(1 - U(t_2))^2},$$

for $0 < t_2 < T$. Consequently, at any differentiable point of $U(t)$, it follows from the above inequalities that

$$(1 - U)^2 U_t \leq \lambda, \tag{5.7}$$

for a.e. $0 < t < T$. (5.6) is obtained by integrating (5.7) from t to T. □

5.4. *Nondegeneracy of quenching solution.* For the quenching solution $u(x, t)$ of $(F_{\lambda,\delta})$ in finite time T, we now introduce the associated similarity variables

$$y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad u(x, t) = 1 - (T - t)^{\frac{1}{3}} w_a(y, s), \tag{5.8}$$

where a is any point in Ω_η, for some small $\eta > 0$. The form of w_a defined in (1.3) is motivated by Lemma 1.4 and Proposition 5.3. Then $w_a(y, s)$ is defined in

$$W_a := \{(y, s) : \ a + ye^{-\frac{s}{2}} \in \Omega, \ s > s' = -\log T\},$$

and it solves

$$\frac{\partial w_a}{\partial s} = \triangle w_a - \frac{y}{2} \cdot \nabla w_a + \frac{1}{3} w_a - \frac{\lambda}{w_a^2} - \lambda \delta e^{\frac{s}{3}} \frac{|\nabla w_a|^2}{w_a^2}. \tag{5.9}$$

Here w_a is always strictly positive in W_a. The slice of W_a at a given time $s = s_0$ will be denoted as $\Omega_a(s_0)$:

$$\Omega_a(s_0) := W_a \cap \{s = s_0\} = e^{\frac{s_0}{2}}(\Omega - a).$$

For any $a \in \Omega_\eta$, there exists $s_0 = s_0(\eta, a) > 0$ such that

$$B_s := \{y : \ |y| < s\} \subset \Omega_a(s), \tag{5.10}$$

for $s \geq s_0$.

Equation (5.9) could also be written in divergence form:

$$\rho w_s = \nabla(\rho \cdot \nabla w) + \frac{1}{3}\rho w - \frac{\lambda \rho}{w^2} - \lambda \delta \rho e^{\frac{s}{3}} \frac{|\nabla w|^2}{w^2}, \tag{5.11}$$

with $\rho(y) = e^{-\frac{|y|^2}{4}}$.

We shall reach the nondegeneracy of the quenching behavior. The conclusion is obtained by the comparison principle [3] and results in [13].

THEOREM 5.4. Suppose u is a quenching solution of $(F_{\lambda,\delta})$ in finite time T and a is any point in Ω_η, for some $\eta > 0$. If $w_a(y, s) \to \infty$ as $s \to \infty$ uniformly for $|y| \leq C$, where C is any positive constant, then a is not a quenching point of u.

Proof. It is easy to see that w_a in (5.9) is a subsolution of

$$\frac{\partial}{\partial s}\tilde{w} = \triangle\tilde{w} - \frac{y}{2}\cdot\nabla\tilde{w} + \frac{1}{3}\tilde{w} - \frac{\lambda}{\tilde{w}^2}$$

in $B_{s_0}\times(s_0,\infty)$. From the comparison principle (cf. [3]), we get $w_a \leq \tilde{w}$ in $B_{s_0}\times(s_0,\infty)$. If $w_a(y,s)\to\infty$, as $s\to\infty$ uniformly in $|y|\leq C$, so does $\tilde{w}(y,s)$. Our conclusion follows immediately from Theorem 2.12, [13], where $f\equiv 1$ and $\tilde{w}$ is the w_a in [13]. □

REMARK 5.5. The proof of Theorem 5.4 also implies that the quenching set of the solution to $(F_{\lambda,0})$ is a subset of that of u, the solution to $(F_{\lambda,\delta})$, $\delta>0$.

5.5. *Asymptotics of quenching solution.* In this subsection, we shall omit all the subscription a of w_a, W_a and Ω_a if no confusion will arise.

In view of (1.3), one combines Lemma 1.4 and Lemma 5.2 to reach the following estimates on w, ∇w and $\triangle w$:

COROLLARY 5.6. Suppose u is a quenching solution to $(F_{\lambda,\delta})$ in finite time T. Then the rescaled solution w satisfies

$$M\leq w\leq e^{\frac{s}{3}},\quad |\nabla w|+|\triangle w|\leq M',\quad \text{in } W,$$

where M and M' are constants in Lemma 1.4 and Lemma 5.2, respectively. Moreover, it satisfies

$$M\leq w(y_1,s)\leq w(y_2,s)+M'|y_1-y_2|, \tag{5.12}$$

for any $(y_i,s)\in W$, $i=1,2$.

LEMMA 5.7. Let s_j be an increasing sequence such that $s_j\to+\infty$, and $w(y,s+s_j)$ is uniformly convergent to a limit $w_\infty(y,s)$ in compact sets. Then either $w_\infty(y,s)\equiv\infty$ or $w_\infty(y,s)<\infty$ in $\mathbb{R}^n$.

Proof. Inequality (5.12) implies that

$$w_\infty(y_1,s)\leq w_\infty(y_2,s)+M'|y_1-y_2|,$$

and the conclusion follows. □

PROPOSITION 5.8. Suppose w is the solution of (5.9) quenching at $x=a$ in finite time T. Assume further that

$$\int_{s_0}^{\infty} se^{\frac{s}{3}}\int_{B_s}\rho|\nabla w|^2 dyds<\infty, \tag{5.13}$$

for some $s_0\gg 1$, where $\rho(y)=e^{-\frac{|y|^2}{4}}$, B_s is defined in (5.10). Then $w(y,s)\to w_\infty(y)$, as $s\to\infty$ uniformly on $|y|\leq C$, where $C>0$ is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of (1.7).

Proof. Let us adapt the arguments in the proofs of Propositions 6 and 7, [9] or Lemma 3.1, [13]. Let $\{s_j\}$ be an increasing sequence tending to ∞ and $s_{j+1}-s_j\to\infty$. Let us denote $w_j(y,s)=w(y,s+s_j)$. Applying the Arzela-Ascoli theorem on $z_j(y,s)=\frac{1}{w_j(y,s)}$ with Corollary 5.6, there is a subsequence of $\{z_j\}$, still denoted as $\{z_j\}$, such that

$$z_j(y,s)\to z_\infty(y,s)$$

uniformly on compact sets of W and

$$\nabla z_j(y,m) \to \nabla z_\infty(y,m)$$

for almost all y and for each integer m. That is, $w_j(y,s) \to w_\infty(y,s)$ uniformly on the compact sets of W and $\nabla w_j(y,m) \to \nabla w_\infty(y,m)$ for almost all y and for each integer m. From Lemma 5.7, we get that either $w_\infty(y,s) \equiv \infty$ or $w_\infty(y,s) < \infty$ in $y \in \mathbb{R}^n$. The case that $w_\infty(y,s) \equiv \infty$ could be excluded by Theorem 5.4, since a is the quenching point.

Let us define the associate energy of w at time s:

$$E[w](s) = \frac{1}{2}\int_{B_s} \rho|\nabla w|^2 dy - \frac{1}{6}\int_{B_s} \rho w^2 dy - \lambda \int_{B_s} \frac{\rho}{w} dy.$$

Direct computations yield that

$$\begin{aligned}\frac{d}{ds}E[w](s) =& \int_{B_s} \rho \nabla w \cdot \nabla w_s dy - \frac{1}{3}\int_{B_s} \rho w w_s dy + \lambda \int_{B_s} \frac{\rho}{w^2} w_s dy \\ &+ \frac{1}{2}\int_{\partial B_s} \rho |\nabla w|^2 (y\cdot\nu) dS - \frac{1}{6}\int_{\partial B_s} \rho w^2 (y\cdot \nu) dS - \lambda \int_{\partial B_s} \frac{\rho}{w}(y\cdot\nu) dS \\ =& -\int_{B_s} \nabla(\rho\cdot\nabla w) w_s dy - \frac{1}{3}\int_{B_s} \rho w w_s dy + \lambda\int_{B_s} \frac{\rho}{w^2} w_s dy \\ &+ \int_{\partial B_s} \rho(\nabla w\cdot\nu) w_s dS + \frac{1}{2}\int_{\partial B_s} \rho|\nabla w|^2 (y\cdot\nu) dS \\ &- \frac{1}{6}\int_{\partial B_s} \rho w^2 (y\cdot\nu) dS - \lambda\int_{\partial B_s} \frac{\rho}{w}(y\cdot\nu) dS \\ =& -\int_{B_s} \rho |w_s|^2 dy - \lambda\delta e^{\frac{s}{3}} \int_{B_s} \rho \frac{|\nabla w|^2}{w^2} w_s dy + G(s), \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} G(s) :=& \int_{\partial B_s} \rho(\nabla w\cdot\nu) w_s dS + \frac{1}{2}\int_{\partial B_s} \rho|\nabla w|^2 (y\cdot\nu) dS \\ &- \frac{1}{6}\int_{\partial B_s} \rho w^2 (y\cdot\nu) dS - \lambda \int_{\partial B_s} \frac{\rho}{w}(y\cdot\nu) dS, \end{aligned}$$

ν is the exterior unit normal vector to $\partial\Omega$ and dS is the surface area element. The first equality in (5.14) is followed by Lemma 2.3, [17]. Let us estimate $G(s)$ as in Lemma 2.10, [13]:

$$\begin{aligned} G(s) \le& \int_{\partial B_s} \rho(\nabla w\cdot\nu) w_s dS + \frac{1}{2}\int_{\partial B_s} \rho|\nabla w|^2 (y\cdot\nu) dS \\ \le& C_1 s^n e^{-\frac{s^2}{4}} + C_2 s^{n-1} e^{-\frac{s^2}{4}} \lesssim s^n e^{-\frac{s^2}{4}}, \end{aligned} \tag{5.15}$$

since

$$|w_s| \le C(1+|y|) + \frac{w}{3} \le \tilde{C}(1+s), \tag{5.16}$$

due to Lemma 5.7 and the fact that a is the quenching point. Hence, by integrating (5.14) in time from a to b, we have that

$$\int_a^b \int_{B_s} \rho |w_s|^2 dyds \leq E[w](a) - E[w](b) + C\int_a^b se^{\frac{s}{3}} \int_{B_s} \rho |\nabla w|^2 dyds + \tilde{C}\int_a^b G(s)ds \tag{5.17}$$

for any $a < b$. Now we shall show that w_∞ is independent of s. Let $a = m + s_j$, $b = m + s_{j+1}$ and $w = w_j$ in (5.17):

$$\int_m^{m+s_{j+1}-s_j} \int_{B_{s+s_j}} \rho \left|\frac{\partial w_j}{\partial s}\right|^2 dyds \leq E[w_j](m) - E[w_{j+1}](m) + C\int_{m+s_j}^{m+s_{j+1}} se^{\frac{s}{3}} \rho |\nabla w|^2 dyds + \tilde{C}\int_{m+s_j}^{m+s_{j+1}} G(s)ds \tag{5.18}$$

for any integer m. Since $s_j + m \to \infty$ as $j \to \infty$, the third term and the last term on the right-hand side of (5.18) tend to zero, due to (5.13) and (5.15), respectively. Since $\nabla w_j(y, m)$ is bounded and independent of j, and $\nabla w_j(y, m) \to \nabla w_\infty(y, s)$ a.e. as $j \to \infty$, we have

$$\lim_{j\to\infty} E[w_j](m) = \lim_{j\to\infty} E[w_{j+1}](m) := E[w_\infty], \tag{5.19}$$

according to the dominated convergence theorem. Thus, the right-hand side of (5.18) tends to zero as $j \to \infty$. Therefore

$$\lim_{j\to\infty} \int_m^M \int_{B_{s+s_j}} \rho \left|\frac{\partial w_j}{\partial s}\right|^2 dyds = 0 \tag{5.20}$$

for each pair of m and M. Now, from (5.16) where $\tilde{C}$ is independent of j, we get $\frac{\partial w_j}{\partial s}$ converges weakly to $\frac{\partial w_\infty}{\partial s}$. Since ρ decreases expeonentially as $|y| \to \infty$ the integral in (5.20) is lower-semicontinuous, and we conclude that

$$\int_m^M \int_{\mathbb{R}^n} \left|\frac{\partial w_\infty}{\partial s}\right|^2 dyds = 0.$$

Since m and M are arbitrary, we show that w_∞ is independent of s.

Since $\left|\frac{\partial w_j}{\partial s}\right|$ and ∇w_j are locally bounded in $\mathbb{R}^n \times (s_0, \infty)$ for some $s_0 \gg 1$, by Corollary 5.6, w_∞ is locally Lipschitzian. Each w_j solves (5.9), and condition (5.13) forces $e^s|\nabla w|^2 \to 0$, as $s \to +\infty$, so w_∞ is a stationary weak solution to (1.7). Schauder's estimates (cf. [3]) yield the desired regularity of w_∞; i.e. w_∞ is actually a strong solution. □

The solution to (1.7) in one dimension has been investigated in [2]. Also, [12] studied the radially symmetric solution to this equation of dimension $n \geq 2$. Combining Proposition 5.8 and their results, we assert the following:

THEOREM 5.9. Suppose u is a solution to $(F_{\lambda,\delta})$ quenching at $x = a$ in finite time T. Assume further that condition (5.13) is satisfied. Then we have

$$\lim_{t\to T^-}(1-u(x,t))(T-t)^{-\frac{1}{3}} = (3\lambda)^{\frac{1}{3}}$$

uniformly on $|x-a| \le C\sqrt{T-t}$ for any bounded constant C.

Proof. It is shown in Theorem 2.1, [2] and Theorem 1.6, [12] that every nonconstant (radially symmetric in dimension $n \ge 2$) solution $w(y)$ to (1.7) in $\mathbb{R}^n$ must be strictly increasing for sufficiently large $|y|$, and $w(y) \to \infty$, as $|y| \to \infty$. Therefore, w_∞ has to be a constant solution, i.e. $w_\infty \equiv (3\lambda)^{\frac{1}{3}}$. □

5.6. *Local expansion near the singularity.* In this subsection, we shall construct the local expansion of the solution $u = u(x,t)$ near the quenching point and the quenching time, provided $\Omega \in \mathbb{R}^n$ is a radially symmetric domain. It has been shown in Theorem 1.3 that the origin is the only quenching point. Let us make the following nonlinear transformation as motivated by [15] and [14]:

$$\zeta = \frac{1}{3\lambda}(1-u)^3. \tag{5.21}$$

Notice that $u = 1$ maps to $\zeta = 0$. In terms of ζ, $(F_{\lambda,\delta})$ transforms to

$$\begin{cases} \zeta_t = \triangle\zeta - \dfrac{2}{3}\dfrac{|\nabla\zeta|^2}{\zeta} - \dfrac{\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}}\dfrac{|\nabla\zeta|^2}{\zeta^{\frac{4}{3}}} - 1, \quad (x,t)\in\Omega_T, \\ \zeta(x,t) = \dfrac{1}{3\lambda}, \quad (x,t)\in\partial\Omega_T, \\ \zeta(x,0) = \dfrac{1}{3\lambda}, \quad x\in\Omega. \end{cases} \tag{5.22}$$

We shall find a formal power series solution to (5.22) near $\zeta = 0$. As in [15] and [14] we look for a locally radially symmetric solution to (5.22) in the form

$$\zeta(r,t) = \zeta_0(t) + \frac{r^2}{2!}\zeta_2(t) + \frac{r^4}{4!}\zeta_4(t) + \cdots, \tag{5.23}$$

where $r = |x|$. Substituting (5.23) into (5.22) and collecting the coefficients in r, we obtain the following coupled ODEs for ζ_0 and ζ_2:

$$\zeta_0' = -1 + n\zeta_2, \quad \zeta_2' = \frac{n+2}{3}\zeta_4 - \frac{4}{3}\frac{\zeta_2^2}{\zeta_0} - \frac{2\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}}\frac{\zeta_2^2}{\zeta_0^{\frac{4}{3}}}. \tag{5.24}$$

We are interested in the solution with $\zeta_0(T) = 0$, $\zeta_0' < 0$ and $\zeta_2 < 0$ for $T - t > 0$ and $T - t \ll 1$. We shall assume that $\zeta_4 \ll \frac{\zeta_2^2}{\zeta_0^{\frac{4}{3}}}$ near the singularity. And it is clear that $\frac{\zeta_2^2}{\zeta_0} \ll \frac{\zeta_2^2}{\zeta_0^{\frac{4}{3}}}$, since $\zeta_0 \ll 1$. Hence, (5.24) reduces to

$$\zeta_0' = -1 + n\zeta_2, \quad \zeta_2' = -\frac{2\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}}\frac{\zeta_2^2}{\zeta_0^{\frac{4}{3}}}. \tag{5.25}$$

Now we solve the system (5.25) asymptotically as $t \to T^-$. We first assume that $n\zeta_2 \ll 1$ near T. This leads to $\zeta_0 \sim T - t$ and the following differential equation for ζ_2:

$$\zeta_2' \sim -\frac{2\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}}\frac{\zeta_2^2}{(T-t)^{\frac{4}{3}}}. \tag{5.26}$$

By integrating (5.26), we obtain that

$$\zeta_2 \sim \frac{3^{\frac{1}{3}}}{2\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} + A\frac{(T-t)^{\frac{1}{3}}}{\log(T-t)} + \cdots, \tag{5.27}$$

for some unknown constant A. From (5.27), we observe that the consistency condition that $n\zeta_2 \ll 1$ as $t \to T^-$ is indeed satisfied. Substituting (5.27) into (5.25) for ζ_0, we obtain for $t \to T^-$ that

$$\zeta_0' \sim -1 + n\left(\frac{3^{\frac{1}{3}}}{2\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} + A\frac{(T-t)^{\frac{1}{3}}}{\log(T-t)} + \cdots\right). \tag{5.28}$$

Using the method of dominant balance, we look for a solution to (5.28) as $t \to T^-$ in the form

$$\zeta_0 \sim (T-t) + (T-t)\left(B_0(T-t)^{\frac{1}{3}} + B_1\frac{(T-t)^{\frac{1}{3}}}{\log(T-t)} + \cdots\right),$$

for some constants B_0 and B_1. A simple calculation yields that

$$\zeta_0 \sim (T-t) + (T-t)\left[-\frac{3^{\frac{4}{3}}n}{8\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} - \frac{3}{4}nA\frac{(T-t)^{\frac{1}{3}}}{\log(T-t)} + \cdots\right], \quad \text{as } t \to T^-. \tag{5.29}$$

The local form for ζ near the quenching point is $\zeta \sim \zeta_0 + \frac{r^2}{2}\zeta_2$. Using the leading term in ζ_2 from (5.27) and the first two terms in ζ_0 from (5.29), we obtain the local form

$$\zeta \sim (T-t)\left[1 - \frac{3^{\frac{1}{3}}n}{8\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}}\frac{r^2}{(T-t)^{\frac{2}{3}}} + \cdots\right], \tag{5.30}$$

for $r \ll 1$ and $T - t \ll 1$. Finally, using the nonlinear mapping (5.21) relating u and ζ, we conclude that

$$u \sim 1 - [3\lambda(T-t)]^{\frac{1}{3}}\left(1 - \frac{3^{\frac{1}{3}}n}{8\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}}\frac{r^2}{(T-t)^{\frac{2}{3}}} + \cdots\right)^{\frac{1}{3}}. \tag{5.31}$$

6. Numerical simulations.

6.1. *Numerical experiments on pull-in voltage and quenching time.* In section 3, we investigate the pull-in voltages λ_δ^* and the finite quenching time T of $(F_{\lambda,\delta})$. We shall verify our results in section 3 by numerically computing λ_δ^* and T for some choice of domain Ω. Let us consider the following two choices of Ω:

$$\Omega: \quad \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{(slab)},$$
$$\Omega: \quad |x| \le 1, \quad x \in \mathbb{R}^2 \quad \text{(unit disk)}.$$

To obtain λ_l and $\lambda_{u,1}$ in (3.1) and Proposition 3.1, we numerically solve $-\triangle\xi = 1$ in Ω with Dirichlet boundary condition, yielding that $||\xi||_\infty \approx 0.125$ for the slab and

TABLE 1. Pull-in voltages λ_δ^* of $(F_{\lambda,\delta})$ with $\delta = 0$, 0.1 and 0.7 for both the slab and the unit disk. The lower bound λ_l in (3.1) and the upper bound $\lambda_{u,1}$ in Proposition 3.1 are also shown. *Left:* slab; *Right:* unit disk.

δ	λ_δ^*	λ_l	$\lambda_{u,1}$
0	1.440	1.1852	1.4622
0.1	1.391	0.9581	1.4578
0.7	1.196	0.4457	1.4314

the slab

δ	λ_δ^*	λ_l	$\lambda_{u,1}$
0	0.8030	0.2080	1.4622
0.1	0.7890	0.2065	0.8523
0.7	0.712	0.1979	0.8255

the unit disk

TABLE 2. The pull-in voltages λ_δ^* tend to zero, as $\delta \to \infty$ for both the slab and the unit disk.

δ	0	0.7	7	70	700	7000
λ_δ^* (slab)	1.440	1.196	0.706	0.301	0.109	0.036
λ_δ^* (unit disk)	0.8030	0.712	0.472	0.218	0.081	0.028

$||\xi||_\infty \approx 0.712$ for the unit disk in $\mathbb{R}^2$. The first eigenpairs (μ_0, ϕ_0) of the operator $-\triangle$ with Dirichlet boundary condition in Ω and with the normalization $\int_\Omega \phi_0 dx = 1$ are explicitly given below:

$$\mu_0 = \pi^2, \quad \phi_0 = \frac{\pi}{2} \sin\left[\pi\left(x + \frac{1}{2}\right)\right] \quad \text{(slab)}, \tag{6.1}$$

$$\mu_0 = z_0^2 \approx 5.783, \quad \phi_0 = \frac{z_0}{J_1(z_0)} J_0(z_0(|x|)) \quad \text{(unit disk)}, \tag{6.2}$$

where J_0 and J_1 are Bessel functions, and $z_0 \approx 2.4048$ is the first zero of $J_0(z)$.

We first compute the pull-in voltage λ_δ^* for various δ in Table 1 for both slab (left column) and unit disk (right column). We use *bvp4c* in MatLab to determine λ_δ^* (cf. [24]). It is shown that λ_δ^* decreases as δ increases. Also, $\lambda_{u,1}$ for the case $\delta = 0.7$ and Ω is the slab provide a better upper bounds than the natural bound λ_0^* (given by the comparison principle, as in [25]).

Next, we verify the result in Proposition 3.2 by numerically computing the pull-in voltage for various $\delta = 0$, 0.7, 7, 70, 700 and 7000. The pull-in voltage λ_δ^* is also located by *bvp4c* in MatLab. It is clearly verified in Table 2 that $\lambda_\delta^* \to 0$, as $\delta \to \infty$, for both the slab and the unit disk.

About the quenching time, we use the finite-difference scheme to compute the numerical solutions to the nonlinear transformed equation (5.22). The detailed schemes are provided in section 6.2 below. We numerically verify in Table 3 that

$$\lim_{\lambda \to \infty} \lambda T = \frac{1}{3} \tag{6.3}$$

for the case without the fringing term. It is also shown numerically that (6.3) no longer holds for $\delta > 0$. Proposition 3.4 has also been verified by various δ and domains in Table 3. Moreover, we observe from the results that $\lim_{\lambda \to \infty} \lambda T = 0$ and the rate of convergence is independent of the fringing term δ.

TABLE 3. The quenching time T_{slab} and T_{disk} for $\delta = 0$, 0.1, 1 and 10 with various λ have been numerically computed, where T_{slab} and T_{disk} represent the quenching time for the slab $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and the unit disk $|x| \leq 1$ in $\mathbb{R}^2$.

δ	λ	T_{slab}	λT_{slab}	T_{disk}	λT_{disk}
0	1.5	1.073664	1.610496	0.292764	0.439146
	10	0.034122	0.34122	0.033348	0.33348
	50	0.0066666	0.3333	0.006666	0.333
	100	0.003333	0.3333	0.00333	0.333
0.1	2	0.30837	0.6167	0.19011	0.38022
	20	0.016692	0.3338	0.016668	0.033336
	200	0.000816	0.1632	0.000816	0.1632
	2000	0.000048	0.0960	0.000048	0.0960
1	2	0.24009	0.4802	0.18327	0.36654
	20	0.008658	0.1732	0.008778	0.17556
	200	0.000198	0.0396	0.000198	0.0396
10	2	0.098892	0.1978	0.101538	0.203076
	20	0.001392	0.02784	0.001398	0.02796
	200	0.000066	0.0132	0.000066	0.0132

6.2. *Numerical solution to* $(F_{\lambda,\delta})$. To numerically solve $(F_{\lambda,\delta})$, as suggested in [14], the tranformed problem (5.22) is more suitable for implementation. In fact, if we use the local behavior

$$\zeta \sim (T-t) + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}} r^2,$$

we get that

$$\frac{|\nabla\zeta|^2}{\zeta^{\frac{4}{3}}} \sim \frac{3^{\frac{2}{3}}(T-t)^{-\frac{2}{3}}}{4\delta^2\lambda^{\frac{4}{3}}\left[r^{-\frac{3}{2}} + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}}(T-t)^{-\frac{2}{3}} r^{\frac{1}{2}}\right]^{\frac{4}{3}}}, \quad \frac{|\nabla\zeta|^2}{\zeta} \sim \frac{\frac{3^{\frac{2}{3}}}{4\delta^2\lambda^{\frac{4}{3}}}(T-t)^{\frac{2}{3}}}{\frac{(T-t)}{r^2} + \frac{3^{\frac{1}{3}}}{4\delta\lambda^{\frac{2}{3}}}(T-t)^{\frac{1}{3}}}.$$

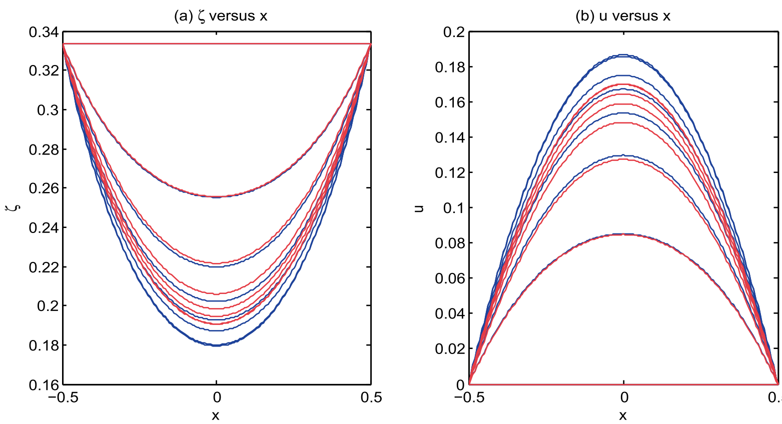
Hence, the two terms $\frac{|\nabla\zeta|^2}{\zeta^{\frac{4}{3}}}$ and $\frac{|\nabla\zeta|^2}{\zeta}$ in (5.22) are bounded in r, for any fixed t, even when t is close to T. This allows us to use a simple finite-difference scheme to compute the numerical solutions to (5.22).

Experiment 1. Let us first consider the domain slab $\left[-\frac{1}{2}, \frac{1}{2}\right]$ in one dimension with $\lambda = 1$, 1.35 or 3 and $\delta = 0$ or 0.7. This interval is discretized into $N+1$ pieces with $N = 200$; i.e. $h = \frac{1}{N+1} \approx 4.97512\times10^{-3}$ is the spatial mesh size. The time step is labelled as $dt = 6\times10^{-6}$. ζ_j^m, for $j = 1, \cdots, N+2$, is defined to be the discrete approximation to $\zeta\left(m\,dt, -\frac{1}{2} + (j-1)h\right)$. The second-order accurate in space and first-order accurate in time scheme of (5.22) are

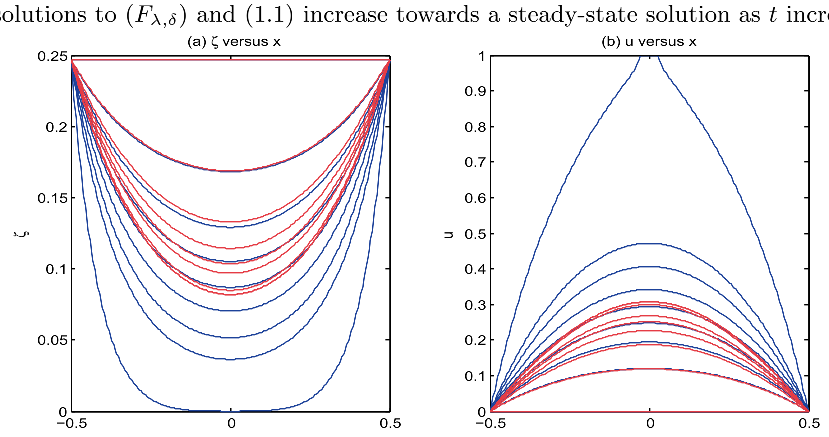
$$\zeta_j^{m+1} = \zeta_j^m + dt\left(\frac{\zeta_{j+1}^m - 2\zeta_j^m + \zeta_{j-1}^m}{h^2} - \frac{\left(\zeta_{j+1}^m - \zeta_{j-1}^m\right)^2}{6\zeta_j^m h^2} - \frac{\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}}\frac{\left(\zeta_{j+1}^m - \zeta_{j-1}^m\right)^2}{4\left(\zeta_j^m\right)^{\frac{4}{3}} h^2}\right), \quad (6.4)$$

$j = 2, \cdots, N+1$, with $\zeta_1^m = \zeta_{N+2}^m = \frac{1}{3\lambda}$ for $m > 0$ and $\zeta_j^0 = \frac{1}{3\lambda}$ for $j = 1, \cdots, N+2$. The time-step dt is chosen to satisfy $dt < \frac{h^2}{4}$ for the stability of the discrete scheme. The experimental stop time is $T_{ex} = m \times dt$, where the m is such that $\min_{j=1,\cdots,N+2}(\zeta_j^m - 0) < 10^{-10}$ for finite time quenching solution or $\max_{j=1,\cdots,N+2}(\zeta_j^{m+1} - \zeta_j^m) < 10^{-10}$ for the globally existing solution.

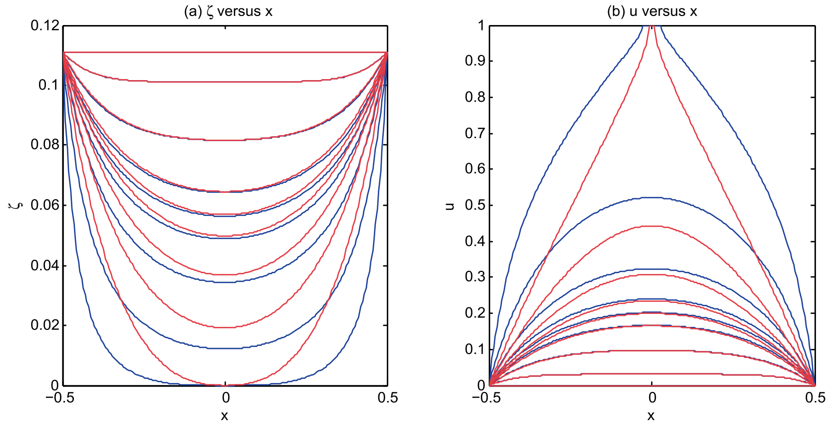
In Figure 1, we plot ζ vs. x (left) and u vs. x (right) from the discrete approximation (6.4) at a series of times. The solution to $(F_{\lambda,\delta})$ with $\delta > 0$ is drawn in blue, while that of $(F_{\lambda,0})$ (cf. (1.1)) is in red. Three different voltages are chosen: $\lambda = 1$, 1.35 and 3. It is suggested by the numerical simulation that the pull-in voltage of (1.1) should be $1.35 < \lambda^* < 3$, while that of $(F_{\lambda,0.7})$ is between 1 and 1.35. The estimate of λ^* matches well with the results in Table 1, where $\lambda_0^* = 1.440$ and $\lambda_{0.7}^* = 1.196$. As to the profiles



1.a: $\lambda = 1$. We plot at times $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 2.0$ and the experimental stop time. Both solutions to $(F_{\lambda,\delta})$ and (1.1) increase towards a steady-state solution as t increases.



1.b: $\lambda = 1.35$. We plot at times $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 3.0$ and the experimental stop time for $\delta = 0$, while at times $t = 0, 0.1, 0.2.0.3, 0.4, 0.5, 0.6, 0.66$ and the experimental stop time for $\delta = 0.7$. The solution to (1.1) still globally exists, while that of $(F_{\lambda,\delta})$ quenches in finite time.



1.c: $\lambda = 3$. We plot at times $t = 0, 0.01, 0.03, 0.05, 0.06, 0.07, 0.09, 0.12$ and the experimental stop time. Both solutions to (1.1) and $(F_{\lambda,\delta})$ quench in finite time.

FIG. 1. *Experiment 1:* For the slab domain $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with different λ. We plot ζ and u versus x at sequential times from the finite difference scheme (6.4) with $N = 200$ and $dt = 0.6 \times 10^{-5}$ and $\delta = 0$ or 0.7. *Left*: ζ versus x; *Right*: u versus x; *Blue*: solution of $(F_{\lambda,\delta})$; *Red*: that of (1.1).

of the solutions to $(F_{\lambda,\delta})$ with $\delta = 0.7$ and $\delta = 0$, the behavior is similar if they both globally exist; see Figure 1.a. The quenching profile of $(F_{\lambda,0.7})$ is much flatter than that of (1.1) if they both quench in finite time; see Figure 1.c. The quenching times T for both $\delta = 0$ and $\delta = 0.7$ in Figure 1.c. are numerically obtained to be around 0.1515 and 0.134262, respectively. This numerically verifies Remark 3.5.

Experiment 2. When we consider the unit disk in two dimensions, a second-order accurate in space and first-order accurate in time discrete approximation for (5.22), with spatial mesh size h, on $0 \le r \le 1$ and $t \ge 0$ is

$$\begin{aligned}\zeta_j^{m+1} =& \zeta_j^m + dt \left(\frac{\zeta_{j+1}^m - 2\zeta_j^m + \zeta_{j-1}^m}{h^2} + \frac{\zeta_{j+1}^m - \zeta_{j-1}^m}{2hr_j} \right. \\ & \left. - \frac{(\zeta_{j+1}^m - \zeta_{j-1}^m)^2}{6\zeta_j^m h^2} - \frac{\delta\lambda^{\frac{2}{3}}}{3^{\frac{4}{3}}} \frac{(\zeta_{j+1}^m - \zeta_{j-1}^m)^2}{4\left(\zeta_j^m\right)^{\frac{4}{3}} h^2} - 1 \right), \end{aligned} \tag{6.5}$$

where $r_j = jh$. According to [19], the discrete approximation for ζ_1 at the origin $r = 0$ is

$$\zeta_1^{m+1} = \zeta_1^m + \frac{4dt}{h^2}(\zeta_2^m - \zeta_1^m).$$

The condition at $r = 1$ is $\zeta_{N+2}^m = \frac{1}{3\lambda}$, and the initial condition is $\zeta_j^0 = \frac{1}{3\lambda}$, for $j = 1, \cdots, N+2$. The experimental stop time is $T_{ex} = m \times dt$, where the m is such that $\min_{j=1,\cdots,N+2}(\zeta_j^m - 0) < 10^{-10}$ for the finite time quenching solution or $\max_{j=1,\cdots,N+2}(\zeta_j^{m+1} - \zeta_j^m) < 10^{-10}$ for the globally existing solution.

In Figure 2, we plot ζ vs. $|x|$ (left) and u vs. $|x|$ (right) from the discrete approximation (6.5) with the voltage chosen to be $\lambda = 1$ at times $t = 0.1, 0.2, 0.3, 0.4, 0.5$ and the

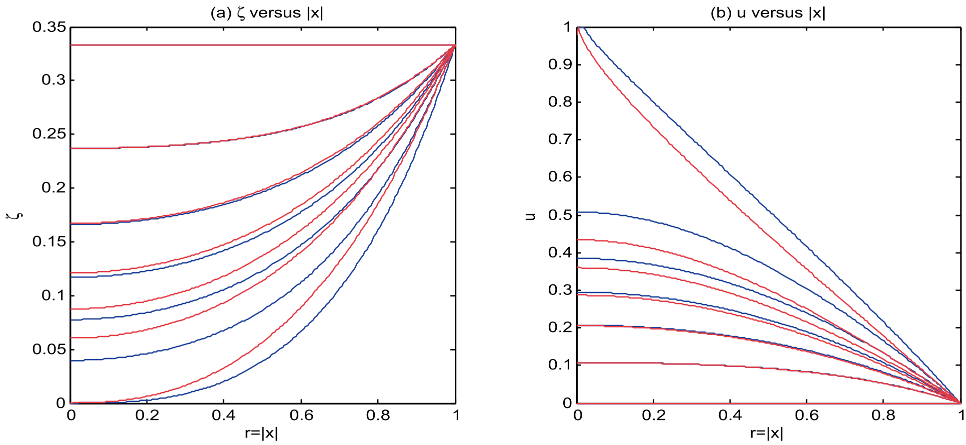


FIG. 2. *Experiment 2:* For the unit disk domain in two dimensions with $\lambda = 1$. We plot ζ and u versus x at times $t = 0.1, 0.2, 0.3, 0.4, 0.5$ and the experimental stop time from the finite difference scheme (6.4) with $N = 200$ and $dt = 0.6 \times 10^{-5}$ and $\delta = 0$ or 0.7. *Left*: ζ versus $|x|$; *Right*: u versus $|x|$; *Blue*: solution of $(F_{\lambda,\delta})$; *Red*: that of (1.1).

experimental stop time T_{ex}. The solution to $(F_{\lambda,\delta})$ with $\delta > 0$ is drawn in blue, while that of $(F_{\lambda,0})$ or (1.1) is in red. It is suggested by the numerical simulations that both the pull-in voltage λ^* of $(F_{\lambda,0})$ and $\lambda^*_{0.7}$ of $(F_{\lambda,0.7})$ are less than 1. This coincides with $\lambda^* = 0.8030$ and $\lambda^*_{0.7} = 0.712$ in Table 1 or Table 2. The quenching times T with $\delta = 0$ and 0.7 are numerically obtained to be around 0.7076 and 0.578232, respectively.

Experiment 3. Let us examine the local approximation constructed in (5.31) numerically. From Experiment 1, the numerically obtained quenching time for $(F_{3,0.7})$ in the slab domain $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is 0.134262, and from Experiment 2, the quenching time for $(F_{1,0.7})$ in the unit disk of dimension two is around 0.578232. In Figure 3, we plot ζ vs. x and $|x|$ of the discrete approximation (6.4) with $\lambda = 3$ and (6.5) with $\lambda = 1$ at time $t = 0.134004$ and $t = 0.57822$, respectively, in blue. At the same time, we plot the local approximation obtained in (5.30) in black. From Figure 3, the local approximation (5.30) matches the numerical solutions well.

7. Conclusion. In this paper, we study the equation $(F_{\lambda,\delta})$ modelling the MEMS device with the fringing term $\delta > 0$. We first show that the pull-in voltage $\lambda^*_\delta > 0$ obtained in [26] is the watershed of globally existing solutions and the finite time quenching solution of $(F_{\lambda,\delta})$. To be more precise, if $\lambda \le \lambda^*_\delta$, then the unique solution to $(F_{\lambda,\delta})$ exists globally; otherwise, the solution will quench in finite time $T < \infty$.

According to the comparision principle, a natural upper bound of λ^*_δ is λ^*, the pull-in voltage of $(F_{\lambda,0})$. In this paper, it has been slightly improved in Proposition 3.1 for $\delta \ll 1$ and numerically verified in Table 1. Moreover, we prove that $\lim_{\delta\to\infty} \lambda^*_\delta = 0$. This has been validated numerically in Table 2.

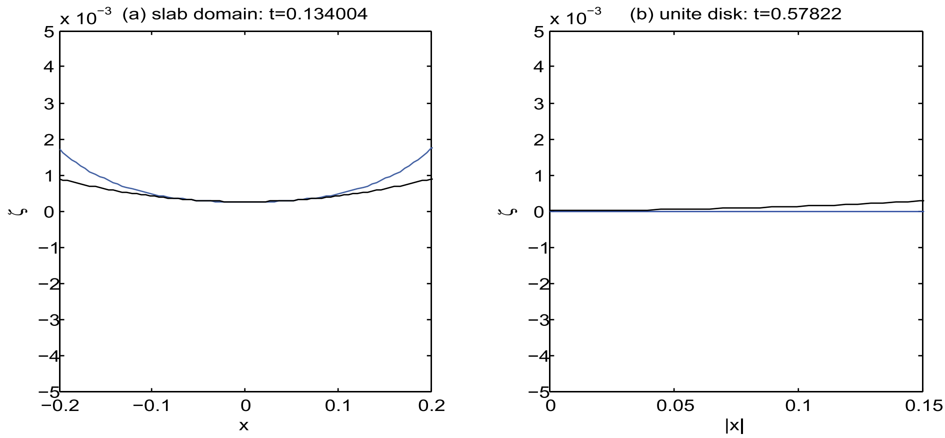


FIG. 3. *Experiment 3:* We plot ζ versus x or $|x|$ for (a) slab domain and (b) unit disk of the discrete approximations of (5.22). *Blue*: the numerical solution given by (6.4) (left column) or (6.5) (right column); *Black*: the local approximations given by (5.30).

About the quenching time T, for $\lambda > \lambda_\delta^*$, we show that it satisfies $T \lesssim \frac{1}{\lambda}$, which differs from that corresponding to $(F_{\lambda,0})$, where $\lim_{\lambda\to\infty} \lambda T = \frac{1}{3}$. We conjecture from Table 3 that $\lim_{\lambda\to\infty} \lambda T = 0$ and the rate of convergence is independent of δ.

By adapting the moving-plane argument as in [8], we show that the quenching set of $(F_{\lambda,\delta})$ is a compact set in Ω if $\Omega \subset \mathbb{R}^n$ is a bounded convex set. Furthermore, if $\Omega = B_R(0)$, the ball centered at the origin with the radius R, then the origin is the only quenching point. This is clearly seen from Figure 1 and Figure 2.

Finally, we investigate the quenching behavior of the solution to $(F_{\lambda,\delta})$ with $\lambda > \lambda_\delta^*$. It is shown in this paper that, under certain conditions, if u is the solution to $(F_{\lambda,\delta})$ quenching at $x = a$ in finite time T, then it satisfies

$$\lim_{t\to T^-} (1 - u(x,t))(T-t)^{-\frac{1}{3}} = (3\lambda)^{\frac{1}{3}}.$$

A more refined asymptotic expansion is given in (5.31). It has been verified numerically in Figure 3 that this is a good local approximation.

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