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On the radio number of toroidal grids

Laxman Saha   Pratima Panigrahi

Department of Mathematics
IIT Kharagpur
Kharagpur 721302
India

laxman.iitkgp@gmail.com   pratima@maths.iitkgp.ernet.in

Abstract

A radio coloring of a simple connected graph G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geqslant \text{diam}(G) + 1 - d(u, v)$ for each pair of distinct vertices u and v of G, where $\text{diam}(G)$ is the diameter of G and $d(u, v)$ is the distance between u and v in G. The *span* of a radio coloring f, $\text{span}(f)$, is the number $\max_{u\in V(G)} f(u)$. The *radio number* of G, $\text{rn}(G)$, is defined as $\min_f\{\text{span}(f) : f \text{ is a radio coloring of } G\}$. In this paper, we determine the radio number of the toroidal grid $T_{m,n}$ (the cartesian product of cycle C_m with the cycle C_n), when at least one of m and n is an even integer. Furthermore, a lower bound is given for the same when both m and n are odd integers.

1 Introduction

Radio coloring of a graph is a variation of the channel assignment problem introduced by Hale [3]. The channel assignment problem is the problem of assigning channels (non-negative integers) to the stations in an optimal way so that the interference is avoided. The interference is closely related to the location of the stations. When the distance between two stations is small, the difference in their assigned channels must be relatively large, whereas two stations at a larger distance may be assigned channels with a small difference. That is, if the channels assigned to the stations u and v are $f(u)$ and $f(v)$, respectively, then

$$|f(u) - f(v)| \geqslant l_{uv}$$

where l_{uv} depends on the distance between u and v. Chartrand et al. [2] have introduced the radio coloring of a simple connected graph by taking $l_{uv} = \text{diam}(G) + 1 - d(u, v)$, where $\text{diam}(G)$ is the diameter of G and $d(u, v)$ is the distance between u and v in G. Therefore, a *radio coloring* f of a simple connected graph G is an assignment of non-negative integers to the vertices of G such that for every two distinct vertices u and v of G,

$$|f(u) - f(v)| \geqslant \mathrm{diam}(G) + 1 - d(u, v).$$

The *span* of a radio coloring f, denoted by $\mathrm{span}(f)$, is defined as $\max_{v \in V(G)} f(v)$, and the *radio number* of G, denoted by $\mathrm{rn}(G)$, is defined as

$$\min_f \{\mathrm{span}(f) : f \text{ is a radio coloring of } G\}.$$

A radio coloring f of G is called minimal if $\mathrm{span}(f) = \mathrm{rn}(G)$. We observe that for a minimal radio coloring f, $\min_{v \in V(G)} f(v) = 0$, otherwise the span of f can be reduced further by subtracting the integer $\min_{v \in V(G)} f(v)$ from all the labels assigned to the vertices of the graph.

Finding the radio number of a graph is of great interest for its widespread applications to the channel assignment problem. So far, the radio number is known for a very limited number of families of graphs. For paths and cycles, the radio numbers were studied by Chartrand et al. [1, 2, 12] and the exact values remained open until solved by Liu and Zhu [8]. Liu et al. [6, 7] have studied the radio number of square of paths and cycles. For hypercubes the radio number have been determined by Khennoufa [5]. Ortiz et al. [11] have studied the radio number of generalized prism graphs and have computed the exact value of the radio number for some particular cases. Recently, the radio numbers of complete m-ary trees were determined by Li et al. [9].

The Cartesian product of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (u', v') are adjacent in $G \square H$ if and only if $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. For positive integers m and n, the Toroidal grids $T_{m,n}$ are the cartesian product $C_m \square C_n$. Morris et al. [10] have determined the radio number of $T_{m,n}$ when $m = n$.

In this paper, we determine the exact value of $\mathrm{rn}(T_{m,n})$ when at least one of m and n is an even integer, and give a lower bound of the same for the remaining values of m and n.

2 Lower Bound for $\mathrm{rn}(T_{m,n})$

Throughout the paper we take the vertex set of the cycle graph C_n of order n to be $\{0, 1, \ldots, n-1\}$. In this context the vertex set of $T_{m,n}$ may be represented as $V(T_{m,n}) = \{(i, j) : 0 \leqslant i \leqslant m-1 \text{ and } 0 \leqslant j \leqslant n-1\}$.

Remark 2.1 *For a cycle C_n of length n,*

(*i*) $d_{C_n}(i, j) = \min\{\ |i - j|,\ n - |i - j|\ \}$,

(*ii*) *for any vertex r of C_n,* $d_{C_n}(i, j) = d_{C_n}\left((i + r) \pmod n,\ (j + r) \pmod n\right)$,

(*iii*) $\mathit{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$, (*iv*) *for any three vertices* u, v *and* w *of* C_n, $d(u,v) + d(v,w) + d(w,u) \leqslant n$.

The following proposition can be found in ([4], Proposition 5.1).

Proposition 2.1 *If* (g,h) *and* (g',h') *are vertices of* $G\square H$, *then*

$$d_{G\square H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h').$$

Corollary 2.1 *Let* (i,j) *and* (r,s) *be any two vertices of* $T_{m,n}$. *Then*
$d((i,j),(r,s)) = \min\{|i-r|, m-|i-r|\} + \min\{|j-s|, n-|j-s|\}$.

Lemma 2.1 *For simple connected graphs* G *and* H, $\operatorname{diam}(G\square H) = \operatorname{diam}(G) + \operatorname{diam}(H)$.

Proof: For any two vertices u, v of $G\square H$, Proposition 2.1 gives $d(u,v) \leqslant \operatorname{diam}(G) + \operatorname{diam}(H)$. Now, if we choose $u = (g,h)$ and $v = (g',h')$ with $d_G(g,g') = \operatorname{diam}(G)$ and $d_H(h,h') = \operatorname{diam}(H)$, then Proposition 2.1 gives $d(u,v) = d_G(g,g') + d_H(h,h') = \operatorname{diam}(G) + \operatorname{diam}(H)$. Therefore $\operatorname{diam}(G\square H) = \operatorname{diam}(G) + \operatorname{diam}(H)$. $\square$

Corollary 2.2 *For the toroidal grids* $T_{m,n}$, $\operatorname{diam}(T_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$.

Lemma 2.2 *For any three vertices* u, v *and* w *of* $T_{m,n}$, $d(u,v) + d(v,w) + d(w,u) \leqslant m + n$.

Proof: Let $u = (x_1, y_1), v = (x_2, y_2)$ and $w = (x_3, y_3)$ be any three vertices of $T_{m,n}$, where $x_i \in V(C_m)$ and $y_i \in V(C_n)$ for $i = 1, 2, 3$. From Proposition 2.1, we have

$$\begin{aligned} d(u,v) &= d_{C_m}(x_1,x_2) + d_{C_n}(y_1,y_2) && (1)\\ d(v,w) &= d_{C_m}(x_2,x_3) + d_{C_n}(y_2,y_3) && (2)\\ d(w,u) &= d_{C_m}(x_3,x_1) + d_{C_n}(y_3,y_1). && (3) \end{aligned}$$

Adding (1)–(3), we get

$$\begin{aligned} d(u,v) + d(v,w) + d(w,u) &= d_{C_m}(x_1,x_2) + d_{C_m}(x_2,x_3) + d_{C_m}(x_3,x_1) \\ &\quad + d_{C_n}(y_1,y_2) + d_{C_n}(y_2,y_3) + d_{C_n}(y_3,y_1). \qquad (4) \end{aligned}$$

Using Remark 2.1(*iv*) and from (4) we get the desired result. $\square$

In the following theorem, we obtain a lower bound of $rn(T_{m,n})$, for $m \geqslant 3$ and $n \geqslant 3$.

Theorem 2.1 *For any two positive integers* $m \geqslant 3$ *and* $n \geqslant 3$,

$$\operatorname{rn}(T_{m,n}) \geqslant \begin{cases} \left\lceil \frac{m+n+6}{4} \right\rceil \left(\frac{mn-2}{2}\right) + 1, & \textit{if both of } m \textit{ and } n \textit{ are even integers;} \\ \left\lceil \frac{m+n+3}{4} \right\rceil \left(\frac{mn-2}{2}\right) + 1, & \textit{if one of } m \textit{ and } n \textit{ is an odd integer;} \\ \left\lceil \frac{m+n}{4} \right\rceil \left(\frac{mn-1}{2}\right), & \textit{if both of } m \textit{ and } n \textit{ are odd inetegers.} \end{cases}$$

Proof: Let D be the diameter of $T_{m,n}$. Let f be a radio coloring of $T_{m,n}$ and $x_1, x_2, \ldots, x_{mn}$ be an ordering of the vertices of $T_{m,n}$ such that $f(x_{j+1}) > f(x_j)$, for $1 \leqslant j \leqslant mn-1$. Then $f(x_1) = 0$ and the span of f is $f(x_n)$. Since $f(x_{j+1}) > f(x_j)$ for $1 \leqslant j \leqslant mn-1$ and f is a radio coloring of $C_m \square C_n$,

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geqslant D + 1 - d(x_{i+1}, x_i), & (5)\\ f(x_{i+2}) - f(x_{i+1}) &\geqslant D + 1 - d(x_{i+2}, x_{i+1}), & (6)\\ f(x_{i+2}) - f(x_i) &\geqslant D + 1 - d(x_{i+2}, x_i). & (7) \end{aligned}$$

Adding (5) – (7), we get

$$2(f(x_{i+2}) - f(x_i)) \geqslant 3(D+1) - (d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+1}) + d(x_{i+2}, x_i)). \quad (8)$$

Using the result of Lemma 2.2, the inequality (8) gives

$$2(f(x_{i+2}) - f(x_i)) \geqslant 3(D+1) - m - n, \text{ for all } i \text{ with } 1 \leqslant i \leqslant mn-2. \quad (9)$$

Since $f(x_{i+2}) - f(x_i)$ is an integer, the inequality (9) gives

$$f(x_{i+2}) - f(x_i) \geqslant \left\lceil \frac{3(D+1) - m - n}{2} \right\rceil, 1 \leqslant i \leqslant mn-2.$$

Let us denote $\left\lceil \frac{3(D+1)-m-n}{2} \right\rceil$ by M. Since $D = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$, M can be written as

$$M = \begin{cases} \left\lceil \frac{m+n+6}{4} \right\rceil, & \text{if both of } m \text{ and } n \text{ are even integers;}\\ \left\lceil \frac{m+n+3}{4} \right\rceil, & \text{if one of } m \text{ and } n \text{ is an odd integer;}\\ \left\lceil \frac{m+n}{4} \right\rceil, & \text{if both of } m \text{ and } n \text{ are odd inetegers.} \end{cases}$$

Now we consider the following two cases according to mn an even or an odd integer.

Case I: Here we take mn an even integer. Since $f(x_{i+2}) - f(x_i) \geqslant M$ for all $i \in \{1, 3, \ldots, mn-3\}$, by summing them up, we get $f(x_{mn-1}) - f(x_1) \geqslant \frac{M(mn-2)}{2}$. Using $f(x_{mn}) \geqslant f(x_{mn-1}) + 1$ and $f(x_1) = 0$, we finally get $f(x_{mn}) \geqslant \frac{M(mn-2)}{2} + 1$.

Case II: In this case we take mn an odd integer. Since $f(x_{i+2}) - f(x_i) \geqslant M$ for all $i \in \{1, 3, \ldots, mn-2\}$, by summing them up we get $f(x_{mn}) - f(x_1) \geqslant \frac{M(mn-1)}{2}$. Now using $f(x_1) = 0$, we get $f(x_{mn}) \geqslant \frac{M(mn-1)}{2}$. □

3 Radio Number of $T_{m,n}$

In this section, first we give an upper bound of $\text{rn}(T_{m,n})$ when at least one of m and n is an even integer and then show that this bound is the exact value of $\text{rn}(T_{m,n})$. Recall that any vertex of $T_{m,n}$ is written as a pair (i, j), where $0 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant n-1$. Throughout the paper, computations in the first and second components in (r, s) are taken modulo m and modulo n, respectively.

To give an upper bound of $\text{rn}(T_{m,n})$, we need to arrange the vertices of $T_{m,n}$ in a suitable manner and for this we will give some lemmas and a remark.

Lemma 3.1 *Let n and r be positive integers with $n > r$. If $\gcd(n, r) = p$, then all the elements of $S = \{ir \pmod n : i = 0, 1, \ldots, \frac{n}{p} - 1\}$ are distinct.*

Proof: Since $\gcd(n, r) = p$, there exist two positive integers q and s such that $n = pq$ and $r = ps$ with $\gcd(q, s) = 1$. If possible, let there exist two distinct integers $i, j \in \{0, 1, \ldots, \frac{n}{p} - 1\}$ for which $ir \pmod n = jr \pmod n$. Without loss of generality we take $i > j$. Then $(i - j)r \equiv 0 \pmod n$, which implies $i - j \equiv 0 \pmod r$. This is not true because $1 < i - j < r$. Therefore all the element of S are distinct. □

Lemma 3.2 *Let n and r be positive integers with $n > r$. If $\gcd(n, r) = p$ and b is a non-negative integer, then all the elements of $S_b = \{(ir + b) \pmod n : i = 0, 1, \ldots, \frac{n}{p} - 1\}$ are distinct.*

Proof: The proof of this lemma follows immediately from Lemma 3.1. □

Remark 3.1 *For Toroidal grids $T_{m,n}$, $d((i, j), (r, s)) = d((i+a, j+b), (r+a, s+b))$.*

In the following few lemmas we arrange the vertices of $T_{m,n}$ in a suitable manner to give a minimal radio coloring.

Lemma 3.3 *Let m and n be even integers with $m + n \equiv 0 \pmod 4$. Then there exists a partition of $V(T_{m,n})$ into two partite sets $U_1 = \{u_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$ and $U_2 = \{u'_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$ with the following properties*

(a) $d(u_i, u'_i) = \mathrm{diam}(T_{m,n})$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 1$;

(b) $d(u_i, u_{i+1}) \geqslant \frac{m+n}{4} - 1$ *and* $d(u'_i, u'_{i+1}) \geqslant \frac{m+n}{4} - 1$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$;

(c) $d(u'_i, u_{i+1}) \geqslant \frac{m+n}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$;

(d) $d(u_i, u'_{i+1}) \geqslant \frac{m+n}{4} - 1$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$.

Proof: To prove the lemma we consider the following cases.

Case I: Here we take $\frac{m}{2}$ and $\frac{n}{2}$ are both even integers. For $j \in \{0, 1, \ldots, n - 1\}$, let us define $D_j = \{(i, j) : 0 \leqslant i \leqslant m - 1\}$. Then the sets $D_0, D_1, \ldots, D_{n-1}$ are pairwise disjoint and $V(T_{m,n}) = \bigcup_{j=0}^{n-1} D_j$. Let

$$\tau : \{0, 1, \ldots, mn - 1\} \to \{(u, v) : 0 \leqslant u \leqslant m - 1 \text{ and } 0 \leqslant v \leqslant n - 1\}$$

be a function, defined by

$$\begin{aligned} \tau(4i) &= \left(\frac{i(m+2)}{2}, 0\right), \quad 0 \leqslant i \leqslant m - 1; \\ \tau(4i + 1) &= \tau(4i) + \left(\frac{m}{2}, \frac{n}{2}\right), \quad 0 \leqslant i \leqslant m - 1; \end{aligned}$$

$$
\begin{aligned}
&= \left(i\left(\frac{m}{2}+1\right)+\frac{m}{2},\frac{n}{2}\right), \quad 0\leqslant i\leqslant m-1;\\
\tau(4i+2) &= \left(\frac{i(m+2)}{2}+\frac{3m}{4},\frac{3n}{4}\right), \quad 0\leqslant i\leqslant m-1;\\
\tau(4i+3) &= \tau(4i+2)+\left(\frac{m}{2},\frac{n}{2}\right), \quad 0\leqslant i\leqslant m-1;\\
&= \left(\frac{i(m+2)}{2}+\frac{m}{4},\frac{n}{4}\right), \quad 0\leqslant i\leqslant m-1;\\
\tau(4mj+l) &= \tau(l)+(0,j), \quad 0\leqslant l\leqslant 4m-1 \text{ and } 1\leqslant j\leqslant \frac{n}{4}-1.
\end{aligned}
$$

We show that τ is a bijection. Since $\gcd(\frac{m}{2}+1,m)=1$ when $\frac{m}{2}$ is an even integer, using Lemma 3.1 we get that all the elements of $\{\tau(4i)\colon i=0,1,\dots,m-1\}$ are distinct. By using Lemma 3.2 we get for every $j\in\{1,2,3\}$ all the elements of $\{\tau(4i+j)\colon i=0,1,\dots,m-1\}$ are distinct. Note that $\{\tau(4i+j)\colon i=0,1,\dots,m-1\}=D_{\frac{jn}{4}}$, for $j\in\{0,1,2,3\}$. Thus $\{\tau(4i+j)\colon i=0,1,\dots,m-1\}=D_0\cup D_{\frac{n}{4}}\cup D_{\frac{n}{2}}\cup D_{\frac{3n}{4}}$, where $j\in\{0,1,2,3\}$. So $\tau(r)\neq\tau(s)$, for two distinct integers $r,s\in\{0,1,\dots,4m-1\}$. Next we observe that for $j\in\{1,2,\dots,\frac{n}{4}-1\}$, $\{\tau(4mj+l)\colon 0\leqslant i\leqslant 4m-1\}=\{\tau(i)+(0,j)\colon 0\leqslant i\leqslant 4m-1\}=D_j\cup D_{\frac{n}{4}+j}\cup D_{\frac{n}{2}+j}\cup D_{\frac{3n}{4}+j}$. Therefore τ is a bijection. Suppose $u_i=\tau(2i)$, $u_i'=\tau(2i+1)$ for $0\leqslant i\leqslant\frac{mn}{2}-1$ and $U_1=\{u_i\colon 0\leqslant i\leqslant\frac{mn}{2}-1\}$, $U_2=\{u_i'\colon 0\leqslant i\leqslant\frac{mn}{2}-1\}$. Then $U_1\cup U_2$ is a partition of $V(T_{m,n})$. Next we show that conditions $(a),(b),(c)$ and (d) hold for this partition. From the definition of τ it is clear that $\tau(2i+1)=\tau(2i)+\left(\frac{m}{2},\frac{n}{2}\right)$, for all $i\in\{0,1,\dots,\frac{mn}{2}-1\}$, thus $d(u_i,u_i')=d(\tau(2i),\tau(2i+1))=\frac{m}{2}+\frac{n}{2}$ (diameter of $T_{m,n}$), for all $i\in\{0,1,\dots,\frac{mn}{2}-1\}$. Therefore condition (a) is satisfied. Now, for $i\in\{0,1,\dots,2m-2\}$,

$$
\begin{aligned}
d(u_i,u_{i+1}) &= d(\tau(2i),\tau(2i+2))\\
&= \begin{cases} \min\{\frac{3m}{4},m-\frac{3m}{4}\}+\min\{\frac{3n}{4},n-\frac{3n}{4}\}, & \text{if } i \text{ is even;}\\ \min\{\frac{m}{4}-1,m-\frac{m}{4}+1\}+\min\{\frac{n}{4},n-\frac{n}{4}\}, & \text{if } i \text{ is odd;}\end{cases}\\
&= \begin{cases} \frac{m}{4}+\frac{n}{4}, & \text{if } i \text{ is even;}\\ \frac{m}{4}+\frac{n}{4}-1, & \text{if } i \text{ is odd;}\end{cases}\\
&\geqslant \frac{m}{4}+\frac{n}{4}-1.
\end{aligned}
$$

Similarly, we may show that $d(u_i',u_{i+1}')\geqslant\frac{m}{4}+\frac{n}{4}-1$, $d(u_i',u_{i+1})\geqslant\frac{m}{4}+\frac{n}{4}$ and $d(u_i,u_{i+1}')\geqslant\frac{m}{4}+\frac{n}{4}$ for all $i\in\{0,1,\dots,2m-2\}$. Here $\tau(4m-2)=\left(\frac{m}{4}-1,\frac{3n}{4}\right)$, $\tau(4m-1)=\left(\frac{3m}{4}-1,\frac{n}{4}\right)$, $\tau(4m)=(0,1)$ and $\tau(4m+1)=\left(\frac{m}{2},\frac{n}{2}+1\right)$; thus

$$
\begin{aligned}
d(u_{2m-1},u_{2m}) &= d(\tau(4m-2),\tau(4m))\\
&= \frac{m}{4}+\frac{n}{4};\\
d(u_{2m-1}',u_{2m}) &= d(\tau(4m-1),\tau(4m))\\
&= \frac{m}{4}+\frac{n}{4};\\
d(u_{2m-1},u_{2m}') &= d(\tau(4m-2),\tau(4m+1))\\
&= \frac{m}{4}+\frac{n}{4}.
\end{aligned}
$$

Therefore conditions $(b), (c)$ and (d) hold for all $i \in \{0, 1, \ldots, 2m-1\}$. Since $\tau(4mj + l) = \tau(l) + (0, j)$ for $0 \leqslant l \leqslant 4m-1$ and $1 \leqslant j \leqslant \frac{n}{4} - 1$, using Remark 2.1 we check that conditions $(b), (c)$ and (d) are also hold for all $i \in \{2m, 2m+1 \ldots, \frac{mn}{2} - 1\}$. Thus the lemma holds in this case.

Example 3.1 Let $m = 8$ and $n = 4$. Then $U_1 = \{u_i: 0 \leqslant i \leqslant 15\}$=$\{\tau(2i): 0 \leqslant i \leqslant 15\}$=$\{$(0,0), (6,3), (5,0), (3,3), (2,0), (0,3), (7,0), (5,3), (4,0), (2,3), (1,0), (7,3), (6,0), (4,3), (3,0), (1,3)$\}$ and $U_2 = \{u'_i: 0 \leqslant i \leqslant 15\}$=$\{\tau(2i+1): 0 \leqslant i \leqslant 15\}$=$\{$(4,2), (2,1), (1,2), (7,1), (6,2), (4,1), (3,2), (1,1), (0,2), (6,1), (5,2), (3,1), (2,2), (0,1), (7,2), (5,1)$\}$.

Case II: Here we consider both $\frac{m}{2}$ and $\frac{n}{2}$ are odd integers i.e., $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$. First we take $m \equiv 6 \pmod 8$. By division algorithm an integer $r \in \{0, 1, \ldots, mn-1\}$ may be written as $r = nj + 2i$ or $r = nj + 2i + 1$ according as r is an even or odd integer, where $0 \leqslant i \leqslant \frac{n}{2} - 1$ and $0 \leqslant j \leqslant m-1$. Let a mapping $\tau: \{0, 1, \ldots, mn-1\} \to \{(u, v): 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$ be defined as

$$\begin{aligned}
\tau(nj+2i) &= \begin{cases} \left(\frac{j(m-2)}{4}, \frac{i(n+2)}{4}\right), & \text{if } i \equiv 0 \pmod 2; \\ \left(\frac{(j+1)(m-2)}{4}, \frac{i(n+2)}{4}\right), & \text{if } i \equiv 1 \pmod 2; \end{cases} \\
\tau(nj+2i+1) &= \tau(nj+2i) + \left(\frac{m}{2}, \frac{n}{2}\right) \\
&= \begin{cases} \left(\frac{j(m-2)}{4} + \frac{m}{2}, \frac{i(n+2)}{4} + \frac{n}{2}\right), & \text{if } i \equiv 0 \pmod 2; \\ \left(\frac{(j+1)(m-2)}{4} + \frac{m}{2}, \frac{i(n+2)}{4} + \frac{n}{2}\right), & \text{if } i \equiv 1 \pmod 2; \end{cases}
\end{aligned}$$

where $0 \leqslant i \leqslant \frac{n}{2} - 1$ and $0 \leqslant j \leqslant m-1$. We show that τ is a bijection. Let $\tau(r) = \tau(s)$ for two distinct integers r and s in the domain of τ. By division algorithm we may write $r = nj_1 + i_1$ and $r = nj_2 + i_2$, where $0 \leqslant j_1, j_2 \leqslant m-1$ and $0 \leqslant i_1, i_2 \leqslant n-1$. If i_1 and i_2 are both even or odd integers, then the second components of $\tau(r)$ and $\tau(s)$ give $\frac{(i_1-i_2)(n+2)}{8} \equiv 0 \pmod n$, because $\tau(r) = \tau(s)$. Since $\gcd\left(\frac{n+2}{4}, n\right) = 1$ or 2 according as $n \equiv 2 \pmod 8$ or $n \equiv 6 \pmod 8$, we have $\left(\frac{i_1-i_2}{2}\right) \equiv 0 \pmod{\frac{n}{2}}$ and this gives $i_1 = i_2$. Then first components of $\tau(r)$ and $\tau(s)$ give $\left(\frac{(j_1-j_2)(m-2)}{4}\right) \equiv 0 \pmod m$. Since $\gcd\left(\frac{m-2}{4}, m\right) = 1$ when $m \equiv 6 \pmod 8$, we have $j_1 - j_2 \equiv 0 \pmod m$ and this gives $j_1 = j_2$. Thus we get a contradiction to the fact $r \neq s$. Now we take one of i_1 and i_2 as an even integer. Without loss of generality we take i_1 an even integer. Since $\tau(r) = \tau(s)$, second components of $\tau(r)$ and $\tau(s)$ give $\left(\frac{n}{2} + \frac{(i_2-i_1-1)(n+2)}{8}\right) \equiv 0 \pmod n$. Subtracting n we get $\left(\frac{n}{2} - \frac{(i_2-i_1-1)(n+2)}{8}\right) \equiv 0 \pmod n$. But the set $\left\{\frac{i(n+2)}{4} \pmod n: 0 \leqslant i \leqslant \frac{n}{2} - 1\right\}$ does not contain $\frac{n}{2}$. Thus $\left(\frac{n}{2} + \frac{(i_2-i_1-1)(n+2)}{8}\right) \not\equiv 0 \pmod n$. Therefore $\tau(r) \neq \tau(s)$ if either i_1 or i_2 is an even integer. Hence τ is a bijection. Suppose $u_i = \tau(2i)$, $u'_i = \tau(2i+1)$ for $0 \leqslant i \leqslant \frac{mn}{2} - 1$ and $U_1 = \{u_i: 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$, $U_2 = \{u'_i: 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$. Then $U_1 \cup U_2$ is a partition of $V(T_{m,n})$. We check that conditions $(a), (b), (c)$ and (d) hold for this partition. Hence the lemma is true for $m \equiv 6 \pmod 8$ and $n \equiv 2 \pmod 4$. Since $T_{m,n}$ is isomorphic to $T_{n,m}$, the lemma is also true for $m \equiv 2 \pmod 4$ and $n \equiv 6 \pmod 8$. Thus the only remaining case is $m \equiv 2 \pmod 8$ and $n \equiv 2 \pmod 8$. For these values of m and n we define τ as

$$\begin{aligned}\tau(nj+2i) &= \begin{cases} \left(\frac{j(m+2)}{4}, \frac{i(n+2)}{4}\right), & \text{if } i \equiv 0 \pmod 2; \\ \left(\frac{j(m+2)+m-2}{4}, \frac{i(n+2)}{4}\right), & \text{if } i \equiv 1 \pmod 2; \end{cases} \\ \tau(nj+2i+1) &= \tau(nj+2i) + \left(\frac{m}{2}, \frac{n}{2}\right).\end{aligned}$$

By similar way as in the above we check that τ is a bijection. Suppose $u_i = \tau(2i)$ and $u'_i = \tau(2i+1)$, for $0 \leqslant i \leqslant \frac{mn}{2} - 1$. If we take $U_1 = \{u_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$, $U_2 = \{u'_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$, then $U_1 \cup U_2$ is a partition of $V(T_{m,n})$ and conditions $(a), (b), (c)$ and (d) also hold.

Example 3.2 Let $m = 14$ and $n = 10$. Then $U_1 = \{u_i : 0 \leqslant i \leqslant 69\}$=$\{\tau(2i) : 0 \leqslant i \leqslant 69\}$=$\{(j,0), (3+j,3), (j,6), (3+j,9), (j,2) : 0 \leqslant j \leqslant 13\}$ and $U_2 = \{u'_i : 0 \leqslant i \leqslant 69\}$=$\{\tau(2i+1) : 0 \leqslant i \leqslant 69\}$=$\{(7+j,5),\ (10+j,8),\ (7+j,1),\ (10+j,4),\ (7+j,7) : 0 \leqslant i \leqslant 69\}$. Here the first component x of a pair (x,y) is taken modulo 14. □

Lemma 3.4 *Let m and n be two even integers with $m+n \equiv 2 \pmod 4$. Then there exists a partition of $V(T_{m,n})$ into two partite sets $U_1 = \{u_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$ and $U_2 = \{u'_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$ with the following properties*

(a) $d(u_i, u'_i) = \operatorname{diam}(T_{m,n})$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 1$;

(b) $d(u_i, u_{i+1}) \geqslant \frac{m+n-2}{4}$ *and* $d(u'_i, u'_{i+1}) \geqslant \frac{m+n-2}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$;

(c) $d(u'_i, u_{i+1}) \geqslant \frac{m+n+2}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$;

(d) $d(u_i, u'_{i+1}) \geqslant \frac{m+n-2}{4} - 1$, *for* $i = 0, 1, \ldots, \frac{mn}{2} - 2$.

Proof: Since both m and n are even integers with $m+n \equiv 2 \pmod 4$, either $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$ or $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$. Since $T_{m,n}$ is isomorphic to $T_{n,m}$, without loss of generality we may assume $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$. For $j \in \{0, 1, \ldots, n-1\}$, let us define $D_j = \{(i,j) : 0 \leqslant i \leqslant m-1\}$. Then the sets $D_0, D_1, \ldots, D_{n-1}$ are pairwise disjoint and $V(T_{m,n}) = \bigcup_{j=0}^{n-1} D_j$.
Let a function $\tau : \{0, 1, \ldots, mn-1\} \to \{(u,v) : 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$ be defined as : for $0 \leqslant i \leqslant \frac{m}{2} - 1$

$$\begin{aligned}\tau(4i) &= \left(\frac{i(m-2)}{2}, 0\right); \\ \tau(4i+1) &= \left(\frac{i(m-2)}{2} + \frac{m}{2}, \frac{n}{2}\right);\end{aligned}$$

$$\begin{aligned}
\tau(4i+2) &= \left(\frac{i(m-2)}{2}+\frac{m-2}{4}, \frac{n}{4}\right);\\
\tau(4i+3) &= \left(\frac{i(m-2)}{2}+\frac{3m-2}{4}, \frac{3n}{4}\right);\\
\tau(2m+4i) &= \left(\frac{i(m+2)}{2}, \frac{n}{2}\right);\\
\tau(2m+4i+1) &= \left(\frac{i(m+2)}{2}+\frac{m}{2}, 0\right);\\
\tau(2m+4i+2) &= \left(\frac{(i+1)(m+2)}{2}+\frac{m-2}{4}, \frac{3n}{4}\right);\\
\tau(2m+4i+3) &= \left(\frac{i(m+2)}{2}+\frac{m+2}{4}, \frac{n}{4}\right);
\end{aligned}$$

and for $1 \leqslant j \leqslant \frac{n}{4}-1$ and $0 \leqslant l \leqslant 4m-1$

$$\tau(4mj+l) \;=\; \tau(l)+(j,j).$$

We show that τ is a bijection. Since $\gcd(\frac{m-2}{2}, m)=2$, using Lemma 3.1, we check that all the elements of $\{\tau(4i)\colon i=0,1,\dots,\frac{m}{2}-1\}$ are distinct. Also by using Lemma 3.2, it is clear that for every $j \in \{1,2,3\}$ all the elements of $\{\tau(4i+j)\colon i=0,1,\dots,\frac{m}{2}-1\}$ are distinct. Note that $\{\tau(4i+j)\colon i=0,1,\dots,\frac{m}{2}-1\} \subset D_{\frac{jn}{4}}$, $j \in \{0,1,2,3\}$. Thus $\tau(r) \neq \tau(s)$, for two distinct integers $r,s \in \{0,1,\dots,2m-1\}$. Similarly, we check that $\tau(2m+i) \neq \tau(2m+j)$, for two distinct integers $i,j \in \{0,1,\dots,2m-1\}$. Note that for $i,j \in \{0,1,\dots,2m-1\}$ with $i \neq j$, if $\tau(i)=\tau(2m+j)$ then at least one of the following holds

(i) $\frac{i(m-2)}{2} \pmod m = \left(\frac{j(m+2)}{2}+\frac{m}{2}\right) \pmod m$,

(ii) $\left(\frac{i(m-2)}{2}+\frac{m}{2}\right) \pmod m = \frac{j(m+2)}{2} \pmod m$,

(iii) $\left(\frac{i(m-2)}{2}+\frac{m-2}{4}\right) \pmod m = \left(\frac{j(m+2)}{2}+\frac{m+2}{4}\right) \pmod m$,

(iv) $\left(\frac{i(m-2)}{2}+\frac{m-2}{4}+\frac{m}{2}\right) \pmod m = \left(\frac{(j+1)(m+2)}{2}+\frac{m-2}{4}\right) \pmod m$.

Since $\frac{i(m-2)}{2} \pmod m$ is an even integer and $\left(\frac{j(m+2)}{2}+\frac{m}{2}\right) \pmod m$ is an odd integer, (i) is not true. For similar reason (ii) does not hold. Again $\left(\frac{i(m-2)}{2}+\frac{m-2}{4}\right) \pmod m$ is an even or odd integer according as $m \equiv 2 \pmod 8$ or $m \equiv 6 \pmod 8$ and $\left(\frac{j(m+2)}{2}+\frac{m+2}{4}\right) \pmod m$ is an even or odd integer according as $m \equiv 6 \pmod 8$ or $m \equiv 2 \pmod 8$. Thus (iii) does not hold. Due to the similar reason (iv) also does not hold. Therefore $\tau(r) \neq \tau(s)$, for two distinct integers $r,s \in \{0,1,\dots,4m-1\}$. It is easy to see that $\{\tau(i)\colon i=0,1,\dots,4m-1\} = D_0 \cup D_{\frac{n}{4}} \cup D_{\frac{n}{2}} \cup D_{\frac{3n}{4}}$. Next observe that for $j \in \{1,2,\dots,\frac{n}{4}-1\}$, $\{\tau(4mj+l)\colon 0 \leqslant i \leqslant$

$4m-1\} = \{\tau(i)+(j,j)\colon 0 \leqslant i \leqslant 4m-1\} = D_j \cup D_{\frac{n}{4}+j} \cup D_{\frac{n}{2}+j} \cup D_{\frac{3n}{4}+j}$. Therefore τ is a bijection. Suppose $u_i = \tau(2i)$, $u'_i = \tau(2i+1)$ for $0 \leqslant i \leqslant \frac{mn}{2}-1$ and $U_1 = \{u_i\colon 0 \leqslant i \leqslant \frac{mn}{2}-1\}$, $U_2 = \{u'_i\colon 0 \leqslant i \leqslant \frac{mn}{2}-1\}$. Then $U_1 \cup U_2$ is a partition of $V(T_{m,n})$. By similar way as in Lemma 3.3 we check that conditions (a), (b), (c) and (d) are satisfied. □

Lemma 3.5 *Let m and n be two integers with $m+n \equiv 3 \pmod 4$. Then there exist a partition of $V(T_{m,n})$ into two partite sets $U_1 = \{u_i\colon 0 \leqslant i \leqslant \frac{mn}{2}-1\}$ and $U_2 = \{u'_i\colon 0 \leqslant i \leqslant \frac{mn}{2}-1\}$ with the following properties*

(a) $d(u_i, u'_i) = \operatorname{diam}(T_{m,n})$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-1$;

(b) $d(u_i, u_{i+1}) \geqslant \frac{m+n-3}{4}$ *and* $d(u'_i, u'_{i+1}) \geqslant \frac{m+n-3}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$;

(c) $d(u'_i, u_{i+1}) \geqslant \frac{m+n+1}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$;

(d) $d(u_i, u'_{i+1}) \geqslant \frac{m+n-3}{4}-1$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$.

Proof: Since $m+n \equiv 3 \pmod 4$, exactly one of m and n is an odd integer. Also $T_{m,n}$ is isomorphic to $T_{n,m}$. Thus without loss of generality we may assume that m is an odd integer and n is an even integer. We consider the following two cases.

Case I: In this case we take $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$. Let a function $\tau\colon \{0,1,\ldots,mn-1\} \to \{(u,v)\colon 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$ be defined as

$$\begin{aligned}
\tau(4i) &= \left(\frac{i(m+1)}{2}, 0\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+1) &= \left(\frac{i(m+1)}{2} + \frac{m+1}{2}, \frac{n}{2}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+2) &= \left(\frac{i(m+1)}{2} + \frac{m+1}{4}, \frac{n}{4}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+3) &= \left(\frac{(i+1)(m+1)}{2} + \frac{m+1}{4}, \frac{3n}{4}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4mj+l) &= \tau(l) + (j,j), \quad 1 \leqslant j \leqslant \frac{n}{4}-1 \text{ and } 0 \leqslant l \leqslant 4m-1.
\end{aligned}$$

Then we proceed in the similar way as in Case I of Lemma 3.3 and show that the lemma holds in this case.

Case II: Here we take $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$. Consider a mapping $\tau\colon \{0,1,\ldots,mn-1\} \to \{(u,v)\colon 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$ defined by

$$\begin{aligned}
\tau(nj+2i) &= \begin{cases} \left(\frac{j(m-1)}{4}, \frac{i(n-2)}{4}\right), & \text{if } i \equiv 0 \pmod 2;\\ \left(\frac{(j+1)(m-1)}{4}, \frac{i(n-2)}{4}\right), & \text{if } i \equiv 1 \pmod 2; \end{cases}\\
\tau(nj+2i+1) &= \tau(nj+2i) + \left(\frac{m+1}{2}, \frac{n}{2}\right)\\
&= \begin{cases} \left(\frac{j(m-1)}{4} + \frac{m+1}{2}, \frac{i(n-2)}{4} + \frac{n}{2}\right), & \text{if } i \equiv 0 \pmod 2;\\ \left(\frac{(j+1)(m-1)}{4} + \frac{m+1}{2}, \frac{i(n-2)}{4} + \frac{n}{2}\right), & \text{if } i \equiv 1 \pmod 2; \end{cases}
\end{aligned}$$

where $0 \leqslant i \leqslant \frac{n}{2}-1$ and $0 \leqslant j \leqslant m-1$. Then we proceed in the similar way as in Case I of Lemma 3.3 and show that the lemma holds in this case. $\square$

Lemma 3.6 *Let m and n be two integers with $m+n \equiv 1 \pmod 4$. Then there exists a partition of $V(T_{m,n})$ into two partite sets $U_1 = \{u_i : 0 \leqslant i \leqslant \frac{mn}{2}-1\}$ and $U_2 = \{u'_i : 0 \leqslant i \leqslant \frac{mn}{2}-1\}$ with properties*

(*a*) $d(u_i, u'_i) = \mathrm{diam}(T_{m,n})$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-1$*;*

(*b*) $d(u_i, u_{i+1}) \geqslant \frac{m+n-1}{4}$ *and* $d(u'_i, u'_{i+1}) \geqslant \frac{m+n-1}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$*;*

(*c*) $d(u'_i, u_{i+1}) \geqslant \frac{m+n+3}{4}$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$*;*

(*d*) $d(u_i, u'_{i+1}) \geqslant \frac{m+n-1}{4} - 1$, *for* $i = 0, 1, \ldots, \frac{mn}{2}-2$.

Proof: Since $m+n \equiv 1 \pmod 4$, exactly one of m and n is an odd integer. Without loss of generality we may assume that m is an odd integer and n is an even integer. We consider the following two cases.

Case I: In this case we take $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$. Let the function $\tau : \{0, 1, \ldots, mn-1\} \to \{(u, v) : 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$ be defined as

$$\begin{aligned}
\tau(4i) &= \left(\frac{i(m+1)}{2}, 0\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+1) &= \left(\frac{i(m+1)}{2} + \frac{m-1}{2}, \frac{n}{2}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+2) &= \left(\frac{i(m+1)}{2} + \frac{3m+1}{4}, \frac{3n}{4}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4i+3) &= \left(\frac{i(m+1)}{2} + \frac{m-1}{2}, \frac{n}{4}\right), \quad 0 \leqslant i \leqslant m-1;\\
\tau(4mj+l) &= \tau(l) + (j, j), \quad 1 \leqslant j \leqslant \frac{n}{4} - 1 \text{ and } 0 \leqslant l \leqslant 4m-1.
\end{aligned}$$

Then we proceed in the similar way as in Case I of Lemma 3.3 and show that the lemma holds in this case.

Example 3.3 Let $m = 5$ and $n = 8$. Then $U_1 = \{u_i : 0 \leqslant i \leqslant 19\}$=$\{\tau(2i) : 0 \leqslant i \leqslant 19\}$={(0,0), (4,6), (3,0), (2,6), (1,0), (0,6), (4,0), (3,6), (2,0), (1,6), (1,1), (0,7), (4,1), (3,7), (2,1), (1,7), (0,1), (4,7), (3,1), (2, 7)} and $U_2 = \{u'_i : 0 \leqslant i \leqslant 19\}$=$\{\tau(2i+1) : 0 \leqslant i \leqslant 19\}$={(2,4), (1,2), (0,4), (4,2), (3,4), (2,2), (1, 4), (0,2), (4,4), (3,2), (3,5), (2,3), (1,5), (0,3), (4, 5), (3,3), (2,5), (1,3), (0, 5), (4,3)}.

Case II: Here we take $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$. First we take $n \equiv 2 \pmod 8$. Let the function

$$\tau : \{0, 1, \ldots, mn-1\} \to \{(u, v) : 0 \leqslant u \leqslant m-1 \text{ and } 0 \leqslant v \leqslant n-1\}$$

be defined as

$$\tau(2mj+2i) = \begin{cases} \left(\frac{i(m+1)}{4}, \frac{j(n-2)}{4}\right), & \text{if } i \text{ is an even integer;} \\ \left(\frac{i(m+1)}{4}, \frac{(j+1)(n-2)}{4}\right), & \text{if } i \text{ is an odd integer;} \end{cases}$$

$$\begin{aligned}\tau(2mj+2i+1) &= \tau(2mj+2i) + \left(\frac{m+1}{2}, \frac{n}{2}\right) \\ &= \begin{cases} \left(\frac{i(m+1)}{4} + \frac{m+1}{2}, \frac{j(n-2)}{4} + \frac{n}{2}\right), & \text{if } i \text{ is an even integer;} \\ \left(\frac{i(m+1)}{4} + \frac{m+1}{2}, \frac{(j+1)(n-2)}{4} + \frac{n}{2}\right), & \text{if } i \text{ is an odd integer;} \end{cases}\end{aligned}$$

where $i \in \{0, 1, \ldots, m-1\}$ and $j \in \{0, 1, \ldots, \frac{n}{2} - 1\}$. We show that τ is a bijection. If possible, let $\tau(r) = \tau(s)$ for two distinct integers r and s in the domain of τ. If r and s are both even integers, then by division algorithm r and s may be written as $r = 2mj_1 + 2i_1$ and $r = 2mj_2 + 2i_2$. If i_1 and i_2 are both even integers, then $\tau(r) = \tau(s)$ gives $\frac{(i_1-i_2)(m+1)}{4} \equiv 0 \pmod m$ and $\frac{(j_1-j_2)(n-2)}{4} \equiv 0 \pmod n$. Then we get $i_1 = i_2$ and $j_1 = j_2$, because $\gcd(\frac{m+1}{4}, m) = 1$ and $\gcd(\frac{n-2}{4}, n) = 2$ when $n \equiv 2 \pmod 8$. Therefore $\tau(r) \neq \tau(s)$ for even integers i_1 and i_2. Similarly, if i_1 and i_2 are both odd integers, then $\tau(r) \neq \tau(s)$. Without loss of generality we take i_1 an even integer and i_2 an odd integer. Then the first components of $\tau(r)$ and $\tau(s)$ are distinct. Thus we get $\tau(r) \neq \tau(s)$ for two distinct even integers r and s. From the definition of τ, $\tau(l+1) = \tau(l) + \left(\frac{m+1}{2}, \frac{n}{2}\right)$ for even integer l. Thus $\tau(r) \neq \tau(s)$ for two distinct odd integers r and s. Now we take r as an even integer and s as an odd integer, i.e., $r = 2mj_1 + 2i_1$ and $r = 2mj_2 + 2i_2 + 1$. If i_1 and i_2 are both even integers, then the second components of $\tau(r)$ and $\tau(s)$ give $\left(\frac{(j_2-j_1)(n-2)}{4} + \frac{n}{2}\right) \equiv 0 \pmod n$, which is not true because $\frac{(j_2-j_1)(n-2)}{4} + \frac{n}{2}$ is an odd integer and $j_1, j_2 \in \{0, 1, \ldots, \frac{n}{2} - 1\}$. Therefore if i_1 and i_2 are both even integers, then $\tau(r) \neq \tau(s)$. Similarly, if i_1 and i_2 are both odd integers, then $\tau(r) \neq \tau(s)$. If either i_1 or i_2 is an odd integer, then the second components of $\tau(r)$ and $\tau(s)$ give $\left(\frac{(j_2-j_1+1)(n-2)}{4} + \frac{n}{2}\right) \equiv 0 \pmod n$, which is not true because $\frac{(j_2-j_1+1)(n-2)}{4} + \frac{n}{2}$ is an odd integer and $j_1, j_2 \in \{0, 1, \ldots, \frac{n}{2} - 1\}$. Thus τ is a bijection. Suppose $u_i = \tau(2i)$ and $u_i' = \tau(2i+1)$, for $0 \leqslant i \leqslant \frac{mn}{2} - 1$. If we take $U_1 = \{u_i : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$, $U_2 = \{u_i' : 0 \leqslant i \leqslant \frac{mn}{2} - 1\}$, then $U_1 \cup U_2$ is a partition of $V(T_{m,n})$ and conditions $(a), (b), (c)$ and (d) hold.

Example 3.4 Let $m = 7$ and $n = 10$. Then $U_1 = \{u_i : 0 \leqslant i \leqslant 34\} = \{\tau(2i) : 0 \leqslant i \leqslant 34\} = \{(0,0), (2,2), (4,0), (6,2), (1,0), (3,2), (5,0), (0,2), (2,4), (4,2), (6,4), (1,2), (3,4), (5,2), (0,4), (2,6), (4,4), (6,6), (1,4), (3,6), (5,4), (0,6), (2,8), (4,6), (6,8), (1,6), (3,8), (5,6), (0,8), (2,0), (4,8), (6,0), (1,8), (3,0), (5,8)\}$ and
$U_2 = \{u_i' : 0 \leqslant i \leqslant 34\} = \{\tau(2i+1) : 0 \leqslant i \leqslant 34\} = \{(4,5), (6,7), (1,5), (3,7), (5,5), (0,7), (2,5), (4,7), (6,9), (1,7), (3,9), (5,7), (0,9), (2,7), (4,9), (6,1), (1,9), (3,1), (5,9), (0,1), (2,9), (4,1), (6,3), (1,1), (3,3), (5,1), (0,3), (2,1), (4,3), (6,5), (1,3), (3,5), (5,3), (0,5), (2,3)\}$ are partite sets of $V(T_{7,10})$. ☐

Theorem 3.1 *Let $G = (V(G), E(G))$ be a simple connected graph on $2N$ vertices with diameter D. If $d(u,v) + d(v,w) + d(w,u) \leqslant 2D$ for any three vertices $u, v, w \in V(G)$ and $V(G)$ has a partition into two partite sets $U_1 = \{u_i : i = 0, 1, \ldots, N-1\}$ and $U_2 = \{u'_i : i = 0, 1, \ldots, N-1\}$ with the following properties*

(a) $d(u_i, u'_i) = D$, *for* $i = 0, 1, \ldots, N-1$;

(b) $d(u_i, u_{i+1}) \geqslant \left\lfloor \frac{D-1}{2} \right\rfloor$ *and* $d(u'_i, u'_{i+1}) \geqslant \left\lfloor \frac{D-1}{2} \right\rfloor$, *for* $i = 0, 1, \ldots, N-2$;

(c) $d(u'_i, u_{i+1}) \geqslant \left\lfloor \frac{D-1}{2} \right\rfloor + 1$, *for* $i = 0, 1, \ldots, N-2$;

(d) $d(u_i, u'_{i+1}) \geqslant \left\lfloor \frac{D-1}{2} \right\rfloor - 1$, *for* $i = 0, 1, \ldots, N-2$;

then $rn(G)$ *satisfies*

$$rn(G) \leqslant \left(\left\lfloor \frac{D}{2} \right\rfloor + 2 \right)(N-1) + 1.$$

Proof: We define a coloring f of G as

$$\begin{aligned} f(u_i) &= i\left(\left\lfloor \frac{D}{2} \right\rfloor + 2 \right), \quad 0 \leqslant i \leqslant N-1; \\ f(u'_i) &= f(u_i) + 1, \quad 0 \leqslant i \leqslant N-1. \end{aligned}$$

Suppose $i \leqslant j$. Let $u \in \{u_i, u'_i\}$ and $v \in \{u_j, u'_j\}$. To compare the values of $D + 1 - d(u,v)$ and $|f(u) - f(v)|$ for $u, v \in V(G)$, we construct Table 1. From Table 1, it is clear that $|f(u) - f(v)| \geqslant D + 1 - d(u,v)$, for all $u, v \in V(G)$. Therefore f is a radio coloring of G and the span of f is $\left(\left\lfloor \frac{D}{2} \right\rfloor + 2 \right)(N-1) + 1$.

Table 1: Comparison between the values of $D + 1 - d(u,v)$ and $|f(u) - f(v)|$.

Pair of vertices $u, v \in U_1 \cup U_2$	Value of $j - i$	$d(u,v)$	$\lvert f(u) - f(v) \rvert$
$u, v \in U_1$ or $u, v \in U_2$	1	$\geqslant \left\lfloor \frac{D-1}{2} \right\rfloor$	$D + 1 - \left\lfloor \frac{D-1}{2} \right\rfloor$
	$\geqslant 2$	$\geqslant 1$	$\geqslant D + 2$
$u \in U_1, v \in U_2$	0	D	1
	1	$\left\lfloor \frac{D-1}{2} \right\rfloor - 1$	$D + 2 - \left\lfloor \frac{D-1}{2} \right\rfloor$
	$\geqslant 2$	$\geqslant 1$	$\geqslant D + 4$
$u \in U_2, v \in U_1$	1	$\geqslant \left\lfloor \frac{D-1}{2} \right\rfloor + 1$	$D + 1 - \left(\left\lfloor \frac{D-1}{2} \right\rfloor + 1 \right)$
	$\geqslant 2$	$\geqslant 1$	$\geqslant D + 2$

□

Theorem 3.2 *Let $G = (V(G), E(G))$ be a simple connected graph on $2N$ vertices with diameter D. If $d(u,v) + d(v,w) + d(w,u) \leqslant 2D + 1$ for any three vertices $u, v, w \in V(G)$ and $V(G)$ has a partition into two partite sets $U_1 = \{u_i : i = 0, 1, \ldots, N-1\}$ and $U_2 = \{u'_i : i = 0, 1, \ldots, N-1\}$ with the properties* *(a)* $d(u_i, u_i') = D$, *for* $i = 0, 1, \ldots, N-1$*;*

(b) $d(u_i, u_{i+1}) \geqslant \lfloor \frac{D}{2} \rfloor$ *and* $d(u_i', u_{i+1}') \geqslant \lfloor \frac{D}{2} \rfloor$, *for* $i = 0, 1, \ldots, N-2$*;*

(c) $d(u_i', u_{i+1}) \geqslant \lfloor \frac{D}{2} \rfloor + 1$, *for* $i = 0, 1, \ldots, N-2$*;*

(d) $d(u_i, u_{i+1}') \geqslant \lfloor \frac{D}{2} \rfloor - 1$, *for* $i = 0, 1, \ldots, N-2$*;*

then $rn(G)$ *satisfies*

$$rn(G) \leqslant \left(\left\lfloor \frac{D-1}{2} \right\rfloor + 2 \right) (N-1) + 1.$$

Proof: We define a coloring f of G as

$$\begin{aligned} f(u_i) &= i \left(\left\lfloor \frac{D-1}{2} \right\rfloor + 2 \right), \quad 0 \leqslant i \leqslant N-1; \\ f(u_i') &= f(u_i) + 1, \quad 0 \leqslant i \leqslant N-1. \end{aligned}$$

By the similar way as in Theorem 3.1 we show that f is a radio coloring of G with span $\left(\lfloor \frac{D-1}{2} \rfloor + 2 \right) (N-1) + 1$. □

The following theorem gives an upper bound of $\mathrm{rn}(T_{m,n})$ when at least one of m and n is an even integer.

Theorem 3.3 *Let* $m \geqslant 3$ *and* $n \geqslant 3$ *be two positive integers with at least one of them is even. Then*

$$\mathrm{rn}(T_{m,n}) \leqslant \begin{cases} \left\lceil \frac{m+n+6}{4} \right\rceil \left(\frac{mn-2}{2} \right) + 1, & \textit{if both of } m \textit{ and } n \textit{ are even integers,} \\ \left\lceil \frac{m+n+3}{4} \right\rceil \left(\frac{mn-2}{2} \right) + 1, & \textit{either } m \textit{ or } n \textit{ is an odd integer.} \end{cases} \tag{10}$$

Proof: From Corollary 2.2, the diameter of $T_{m,n}$ is $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. Let $D = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. Also Lemma 2.2 gives $d(u,v) + d(v,w) + d(w,u) \leqslant m + n$ for any three vertices u, v and w of $T_{m,n}$. Thus for any three vertices u, v and w of $T_{m,n}$,

$$d(u,v) + d(v,w) + d(w,u) \leqslant \begin{cases} 2D, & \text{if both of } m \text{ and } n \text{ are even integers;} \\ 2D+1, & \text{either } m \text{ or } n \text{ is an odd integers.} \end{cases}$$

If both of m and n are even integers, then from Lemma 3.3, Lemma 3.4 and Theorem 3.1, and if either m or n is an odd integers, then from Lemma 3.6, Lemma 3.5 and Theorem 3.2, we get the result (10). □

The following theorem gives the radio number of $T_{m,n}$ when at least one of m and n is an even integer.

Theorem 3.4 *Let* $m \geqslant 3$ *and* $n \geqslant 3$ *be two positive integers with at least one of them is even. Then*

$$\mathrm{rn}(T_{m,n}) = \begin{cases} \left\lceil \frac{m+n+6}{4} \right\rceil \left(\frac{mn-2}{2} \right) + 1, & \textit{if both of } m \textit{ and } n \textit{ are even integers,} \\ \left\lceil \frac{m+n+3}{4} \right\rceil \left(\frac{mn-2}{2} \right) + 1, & \textit{either } m \textit{ or } n \textit{ is an odd integer.} \end{cases}$$

Proof: Proof follows immediately from Theorem 2.1 and Theorem 3.3. □

4 Concluding Remark

In this paper, we give a lower bound for $\mathrm{rn}(T_{m,n})$ for all values of m and n. Further we have shown that this lower bound agrees with $\mathrm{rn}(T_{m,n})$ when at least one of m and n is an even integer. For the remaining case that $mn \equiv 1 \pmod 2$, value of $\mathrm{rn}(T_{m,n})$ is not known. In fact, no better upper bound is available for this. We have obtained an upper bound of $\mathrm{rn}(T_{m,n})$ when $mn \equiv 1 \pmod 2$, but this upper bound is too far from the lower bound given in Theorem 2.1. Using the similar kind of logic as in Theorem 2.1 one can find a lower bound of $\mathrm{rn}(G)$ when G is the cartesian product of any finite number of cycles. For this graph one may also determine the radio number when the product of lengths of cycles is an even integer.

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