On the Raman Scattering in Liquid Helium

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General treatment is given on the final state interaction between a pair of elementary excitations produced by the Raman scattering on liquid helium. The final state interaction exerts a drastic influence on the Raman spectrum where the produced excitations have small group velocities, i.e. the excitations lie near the extremum points of the phonon-roton dispersion curve. It is shown that, in general, the Raman spectrum has no peaks at these points in spite of the infinite density of states, and correct form of the spectrum is given. The result is interpreted as a destructive interference between excitations with positive and with negative group velocities. Possible existence of a resonance, which depends on the sign of the interaction between the excitations around the extrema, is also investigated.

§ 1. Introduction

A pair of elementary excitations (phonon-rotons) is created by the Raman scattering in liquid helium. Momenta of the two excitations are of equal magnitude with opposite directions, because the momentum transfer from the electromagnetic field to liquid helium is completely negligible. The density of states of the pair becomes infinite when the excitations lie at the extremum points of the phonon-roton dispersion curve. Three such points exist corresponding to the maximum, the minimum and the plateau of the dispersion curve. Halley[1] predicted that this high density of states should induce sharp peaks to the Raman spectrum of the scattered light. Experiment by Greytak and Yan[2] has shown that there are a large peak slightly above twice the minimum energy, a broad peak almost at twice the plateau energy and no predicted peak at twice the maximum energy. They interpreted the shift of the large peak from twice the minimum energy as due to the instrumental energy resolution. However, the complete absence of the peak corresponding to the maximum has not been so well explained even from the theory by Stephen,[3] whose calculation shows that the possible peak corresponding to the maximum is smaller than the one corresponding to the minimum. They have also suggested the possible effect of the finite lifetime of an elementary excitation on the Raman spectrum.

It is the purpose of the present paper to notice that, among many possible effects, the final state interaction between the produced elementary excitations causes a drastic change to the Raman spectrum inferred from the simple argument of the state density, and to derive the correct behavior of the spectrum around the points corresponding to the extrema of the dispersion curve.

The importance of the final state interaction is easily recognized from the fact that the excitations created near the extremum points have very small group velocities. For such slow particles the Born approximation is completely invalid. In order to see the situation clearly, suppose a following simple example, i.e., the elastic scattering of a particle with dispersion relation $\omega(k)$, which has a vanishing group velocity $v = d\omega/dk$ at the finite wave number k. The scattering cross section, $\sigma = (4\pi/k^2)(2l+1)\sin^2\delta_l$, for a given partial wave l can never be infinite. It remains of the order of square of the de Broglie wave length. However, if we compute the cross section in the Born approximation by taking square of the matrix element M, multiplied by the final state density and divided by the initial group velocity, it diverges as $|M|^2/v^2$. From this example we can see that the correct transition matrix element itself should be proportional to v, and the finite cross section will come out only if the interaction is treated up to the infinite order.

Therefore, the complete analysis is required on the scattering states of the interacting pair of elementary excitations in order to discuss the correct behavior of the Raman spectrum. Such an analysis presents us an interesting scattering problem due to the non-monotonic behavior of the dispersion curve. Let us describe general features of the scatterin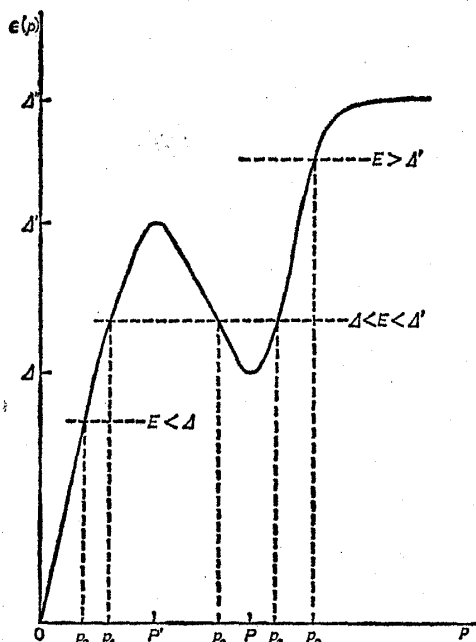g states. Figure 1 shows the energy $\epsilon(p)$ of a non-interacting pair of elementary excitations with momenta p and $-p$. It is just twice the phonon-roton dispersion curve. The minimum is at the momentum P, $\epsilon(P) = \Delta$, and the maximum at P', $\epsilon(P') = \Delta'$. Twice the plateau energy is denoted by Δ''. When the interaction is present, the scattering takes place. Because the total momentum of the pair is zero, we can classify the scattering states according to their angular momenta. Only even partial waves appear due to the Bose statistics. When the energy E of the state is in the region $E<\Delta$ or $E>\Delta'$, there is only one channel. However, when $\Delta<E<\Delta'$, there are three open channels; three waves $\sin(p_\lambda r + \delta_\lambda)$ reach the wave zone, where p_λ $(\lambda=1,2,3)$ satisfy $\epsilon(p_\lambda)=E$. Therefore, Δ or Δ' may be considered as a threshold energy for opening or closing the inelastic channels. This complicacy is analysed by introducing the eigenchannels. We notice that the group velocity of the

Fig. 1. Energy $\epsilon(p)$ of a pair of excitations with momenta p and $-p$. The threshold energies Δ, Δ' are at P, P'. Twice the plateau energy is denoted by Δ''. The solutions $\epsilon(p_\lambda)=E$ are at $p_\lambda=p_1$, p_2, p_3 when $\Delta<E<\Delta'$, and at $p_\lambda=p_0$ when $E<\Delta$ or $E>\Delta'$.

excitation with momentum p_2, $P' < p_2 < P$, is negative. Scattering problem involving such a particle of negative group velocity has never been treated so seriously. We also remark that there will be a resonant state below the threshold Δ when the interaction is attractive between the excitations around the minimum point P, because, just like a mechanism of the Cooper pair[4] in superconductor, the pair around the minimum point P forms a bound state with energy below Δ, and this bound state couples with the elastic channel. The same happens when the interaction is repulsive around the maximum point P' or around the plateau; there will be a resonance above the threshold Δ' or above Δ''.

In this paper we consider liquid helium at the absolute temperature zero in order to avoid the complications arising from other origins than the final state interaction. We do not rely on any specific model of liquid helium. However, we assume that the state which has one elementary excitation can be represented by an exact stationary eigenstate of the total Hamiltonian, which we call one-particle state. Although the exact two-particle state contains components of non-interacting three- or more-particle states, we approximate the exact two-particle state as a superposition only of the non-interacting two-particle states. The validity of the approximation is questionable at higher energies. Therefore, we do not enter into a detailed discussion on the spectrum around Δ''. We describe the treatment mainly about the threshold Δ, because, from the formal point of view, the threshold Δ' can be treated in the same way by changing the sign of the energy.

In the next section we give necessary formulas for the description of the scattering states of two particles. Special consideration is paid for the particles with negative group velocities. We formulate in § 3 the coupling with the electromagnetic field. We show that the Raman matrix element is characterized by two functions, form factors, which couple with the S- and D-wave scattering states respectively. Although it has been the main concern of the previous papers,[1],[3] we do not go into detail for the calculation of the form factors, because it requires some specific model for the excitations. In order to get an explicit expression for the Raman spectrum, we consider in § 4 a simple example, in which the final state interaction is taken as separable. This example shows that the Raman spectrum has sharp minima rather than peaks at the points where the density of states is infinite. General treatment is given in § 5 on the behavior of the Raman spectrum near the threshold. We notice that, although the spectrum found in § 5 shows similar behavior to the well-known elastic scattering cross section near the inelastic threshold, the mechanism involved is completely different; in our case the state density is proportional to $1/\sqrt{E-\Delta}$, while in the scattering problem the inelastic state density is proportional to $\sqrt{E-\Delta}$. Possible resonance is generally analysed in § 6. In order to see the physical origin of the unexpected behavior of the Raman spectrum, we investigate in § 7 the wave function between the interacting pair. We show that large cancellation occurs in the wave func-

tion due to a specific interference between waves of positive group velocity and of negative group velocity. Some aspects on the higher excitations in liquid helium are discussed in § 8. Proofs of some mathematical lemmas are given in the Appendix.

§ 2. Description of two-particle scattering state

In this section we consider the scattering states of a pair of elementary excitations in liquid helium with total momentum zero.

Suppose liquid helium at the absolute temperature zero. It is in the ground state Ψ_0. An elementary excitation is known to obey the energy-momentum relation $\omega(p)$, the experimental phonon-roton dispersion relation. (We use a unit system in which $\hbar = 1$.) We assume that the state which has one elementary excitation can be represented by a stationary eigenstate of the total Hamiltonian. We denote this one-particle state by $B_p^+ \Psi_0$. We can require that the operators B_p, B_p^+ should satisfy the boson commutation relations, $[B_p, B_{p'}^+] = \delta_{pp'}$, $[B_p, B_{p'}] = [B_p^+, B_{p'}^+] = 0$. We expand the total Hamiltonian $\mathscr{H}$ in powers of B_p, B_p^+,

$$\mathscr{H} = \mathscr{E}_0 + \sum_p \omega(p) B_p^+ B_p$$

$$+ \frac{1}{2V} \sum_{Q, p, p'} W_Q(p, p') B_{Q/2+p}^+ B_{Q/2-p}^+ B_{Q/2-p'} B_{Q/2+p'} + \cdots. \qquad (2\cdot1)$$

Such terms as B^+, B^+B^+, $B^+B^+B^+$, B^+B^+B, $B^+B^+B^+B^+$, $B^+B^+B^+B$, and their Hermitian conjugates do not appear because the ground state and the one-particle states are the exact eigenstates of the total Hamiltonian. Actually, the total Hamiltonian also contains terms which describe other degrees of freedom than phonon-rotons. We assume that such terms remain constant for the discussion of the scattering states of the two elementary excitations. Evidently, the interaction $W_Q(p, p')$ satisfies the symmetry conditions,

$$W_Q(p, p') = W_Q(p', p), \qquad (2\cdot2)$$

$$W_Q(p, p') = W_Q(p, -p') = W_Q(-p, p') = W_Q(-p, -p'). \qquad (2\cdot3)$$

Let us consider the two-particle states of total momentum zero. The state $B_p^+ B_{-p}^+ \Psi_0$ is not the eigenstate of the total Hamiltonian. We approximate the exact two-particle eigenstate Ψ_λ as a superposition of the non-interacting two-particle states,

$$\Psi_\lambda = \sum_p f_\lambda(p) B_p^+ B_{-p}^+ \Psi_0. \qquad (2\cdot4)$$

The amplitude $f_\lambda(p)$ is symmetric,

$$f_\lambda(p) = f_\lambda(-p), \qquad (2\cdot5)$$

and satisfies the normalization condition

$$\langle \Psi_\lambda, \Psi_\mu \rangle = 2 \sum_p f_\lambda{}^*(p) f_\mu(p) = \delta_{\lambda\mu}. \qquad (2\cdot6)$$

The variational principle, $\delta \langle \Psi_\lambda, \mathcal{H}\Psi_\lambda \rangle / \langle \Psi_\lambda, \Psi_\lambda \rangle$, determines the best amplitude. From (2·1) and (2·4) we have

$$\{E - \epsilon(p)\} f_\lambda(p) = \frac{1}{V} \sum_{p'} W_0(p, p') f_\lambda(p'), \qquad (2\cdot7)$$

where E is the excitation energy from the ground state to Ψ_λ, and $\epsilon(p)$ the non-interacting energy of the pair, $\epsilon(p) = 2\omega(p)$. We take the outgoing wave solutions $f_\lambda{}^{(+)}(p) = ((2\pi)^3/V) \psi^{(+)}(p, p_\lambda)$ in the limit of the normalization volume $V \to \infty$,

$$\psi^{(+)}(p, p_\lambda) = \tfrac{1}{2} \{\delta(p - p_\lambda) + \delta(p + p_\lambda)\}$$

$$+ \frac{1}{E - \epsilon(p) + i\eta} \int W_0(p, p') \psi^{(+)}(p', p_\lambda) dp'/(2\pi)^3, \qquad (2\cdot8)$$

where η is a small positive constant and p_λ the incident momentum, $\epsilon(p_\lambda) = E$. Because we are describing the scattering states of identical particles, p_λ is restricted on half the momentum space, $(p_\lambda)_z > 0$.

Now we decompose the scattering states in partial waves,

$$p p_\lambda \psi^{(+)}(p, p_\lambda) = \sum_l u^{(l)}(p, p_\lambda) \sum_m Y_{lm}(\hat{p}) Y_{lm}^*(\hat{p}_\lambda) \qquad (2\cdot9)$$

and

$$p p' W_0(p, p')/(2\pi)^3 = \sum_l W^{(l)}(p, p') \sum_m Y_{lm}(\hat{p}) Y_{lm}^*(\hat{p}'). \qquad (2\cdot10)$$

Only even partial waves, $l = 0, 2, 4, \cdots$, appear due to the symmetry conditions (2·5) and (2·3). The equation of the partial wave $u^{(l)}(p, p_\lambda)$ is

$$u^{(l)}(p, p_\lambda) = \delta(p - p_\lambda) + \frac{1}{E - \epsilon(p) + i\eta} T^{(l)}(p, p_\lambda, E), \qquad (2\cdot11)$$

where the scattering amplitude $T^{(l)}(p, p', E)$ satisfies the integral equation

$$T^{(l)}(p, p', E) = W^{(l)}(p, p') + \int \frac{W^{(l)}(p, p'') T^{(l)}(p'', p', E)}{E - \epsilon(p'') + i\eta} dp''. \qquad (2\cdot12)$$

The interesting feature of negative group velocity manifests itself in the wave function,

$$\psi_\lambda{}^{(+)}(r) = \int \exp(ip \cdot r) \psi^{(+)}(p, p_\lambda) dp. \qquad (2\cdot13)$$

It is also decomposed into partial waves,

$$r \psi_\lambda{}^{(+)}(r) = 4\pi \sum_l u_\lambda{}^{(l)}(r) \sum_m Y_{lm}(\hat{r}) Y_{lm}^*(\hat{p}_\lambda). \qquad (2\cdot14)$$

Then, $u_\lambda{}^{(l)}(r)$ is connected with $u^{(l)}(p, p_\lambda)$ by

$$u_\lambda{}^{(l)}(r) = i^l \int p r j_l(pr) u^{(l)}(p, p_\lambda) dp, \qquad (2\cdot15)$$

where $j_l(pr)$ is a spherical Bessel function, and it behaves at large r as $i^l prj_l(pr) \sim \sin pr$ for even l. From Fig. 1 the equation $\epsilon(p_\lambda) = E$ has three solutions $p_\lambda = p_1, p_2, p_3$ when $\Delta < E < \Delta'$, while it has only one solution $p_\lambda = p_0$ when $E < \Delta$ or $E > \Delta'$. As is shown in the Appendix A, the asymptotic form of $u_\lambda^{(l)}(r)$ at large r is

$$u_\lambda^{(l)}(r) \sim \sin p_\lambda r - \pi \sum_{\mu=1}^{3} \frac{\sigma_\mu}{v_\mu} \exp(i\sigma_\mu p_\mu r) T_{\mu\lambda} \qquad (2\cdot16)$$

for $\Delta < E < \Delta'$, and

$$u_{\lambda=0}^{(l)}(r) \sim \sin p_0 r - \frac{\pi}{v_0} T_{00} \exp(ip_0 r) \qquad (2\cdot17)$$

for $E < \Delta$ or $E > \Delta'$. In these equations v_λ is the magnitude of the group velocity, $v_\lambda = |d\epsilon/dp|_{p=p_\lambda}$, σ_λ is the sign of the group velocity, $(d\epsilon/dp)_{p=p_\lambda} = \sigma_\lambda v_\lambda$, and $T_{\mu\lambda}$ is the T-matrix element on the energy shell, $T_{\mu\lambda} = T(p_\mu(E), p_\lambda(E), E)$. We notice that the outgoing spherical wave is represented by e^{ipr} or $-e^{-ipr}$ according as its group velocity is positive or negative.

We consider the eigenchannels for $\Delta < E < \Delta'$. Introducing $\phi_\lambda^{(-)}(r)$ and $\phi_\lambda^{(+)}(r)$, the incoming and outgoing spherical waves normalized by unit flux, by

$$\phi_\lambda^{(\pm)}(r) = \frac{\sigma_\lambda}{\sqrt{v_\lambda}} \exp(\pm i\sigma_\lambda p_\lambda r), \qquad (2\cdot18)$$

we write the asymptotic form $(2\cdot16)$ as (dropping l when no confusion arises)

$$-2iu_\lambda(r) \sim \phi_\lambda^{(-)}(r) - \sum_\mu S_{\mu\lambda} \phi_\mu^{(+)}(r), \qquad (2\cdot19)$$

where $S_{\mu\lambda}$ is the element of the unitary S-matrix,

$$S_{\mu\lambda} = \delta_{\mu\lambda} - 2\pi i \frac{T_{\mu\lambda}}{\sqrt{v_\mu v_\lambda}}. \qquad (2\cdot20)$$

It is diagonalized by an orthogonal matrix $(f_{\lambda\alpha})$ as

$$\sum_{\mu,\lambda} S_{\mu\lambda} f_{\mu\alpha} f_{\lambda\alpha} = \exp(2i\delta_\alpha). \qquad (\alpha = 1, 2, 3) \qquad (2\cdot21)$$

Also we have a relation from $(2\cdot20)$

$$\sum_{\mu,\lambda} \frac{T_{\mu\lambda}}{\sqrt{v_\mu v_\lambda}} f_{\mu\alpha} f_{\lambda\alpha} = -\frac{1}{\pi} \exp(i\delta_\alpha) \sin \delta_\alpha. \qquad (2\cdot22)$$

From $(2\cdot19)$ and $(2\cdot21)$ the eigenchannel wave function $U_\alpha(r)$ has the asymptotic form

$$U_\alpha(r) \equiv \sum_\lambda u_\lambda(r) \frac{f_{\lambda\alpha}}{\sqrt{v_\lambda}} \sim \exp(i\delta_\alpha) \sum_\lambda \frac{f_{\lambda\alpha}}{\sqrt{v_\lambda}} \sin(p_\lambda r + \sigma_\lambda \delta_\alpha). \qquad (2\cdot23)$$

Notice the sign factor at the eigenphase δ_α.

Sometimes it is convenient to use the reaction matrix K. We define $K(p,$

$p', E)$ by the integral equation

$$K(p, p', E) = W(p, p') + \mathscr{P} \int \frac{W(p, p'')K(p'', p', E)}{E - \epsilon(p'')} dp''. \qquad (2 \cdot 24)$$

The symbol $\mathscr{P}$ denotes the principal value integral. From $(2 \cdot 12)$ T is connected with K by the Heitler equation (see Appendix B),

$$T(p, p', E) = K(p, p', E) - i\pi \sum_{\mu} K(p, p_{\mu}, E) \frac{1}{v_{\mu}} T(p_{\mu}, p', E) \qquad (2 \cdot 25)$$

for $\varDelta < E < \varDelta'$. Denoting $K_{\mu\lambda} = K(p_{\mu}(E), p_{\lambda}(E), E)$, we have from $(2 \cdot 22)$ and $(2 \cdot 25)$

$$\sum_{\mu, \lambda} \frac{K_{\mu\lambda}}{\sqrt{v_{\mu} v_{\lambda}}} f_{\mu\alpha} f_{\lambda\alpha} = -\frac{1}{\pi} \tan \delta_{\alpha}. \qquad (2 \cdot 26)$$

From $(2 \cdot 11)$ and $(2 \cdot 25)$ the eigenchannel wave function in momentum space $U(p, \alpha)$ can be expressed by

$$U(p, \alpha) = \sum_{\lambda=1}^{3} u(p, p_{\lambda}) \frac{f_{\lambda\alpha}}{\sqrt{v_{\lambda}}}$$

$$= \exp(i\delta_{\alpha}) \cos \delta_{\alpha} \sum_{\lambda=1}^{3} \left\{ \delta(p - p_{\lambda}) + \mathscr{P} \frac{K(p, p_{\lambda}, E)}{E - \epsilon(p)} \right\} \frac{f_{\lambda\alpha}}{\sqrt{v_{\lambda}}}. \qquad (2 \cdot 27)$$

Corresponding equations for one-channel case, $E < \varDelta$ or $E > \varDelta'$, are easily written down. In particular, we have

$$T_{00} = -\frac{v_0}{\pi} \exp(i\delta_0) \sin \delta_0, \qquad K_{00} = -\frac{v_0}{\pi} \tan \delta_0 \qquad (2 \cdot 28)$$

and

$$U(p, \alpha = 0) = u(p, p_0) \frac{1}{\sqrt{v_0}} = \frac{\exp(i\delta_0)}{\sqrt{v_0}} \cos \delta_0 \left\{ \delta(p - p_0) + \mathscr{P} \frac{K(p, p_0, E)}{E - \epsilon(p)} \right\}. \qquad (2 \cdot 29)$$

We remark that $T(p, p', E)$ defined by $(2 \cdot 12)$ is an analytic function with respect to the complex variable E. The analytic continuation of $T_{11} = T(p_1(E), p_1(E), E > \varDelta)$ beyond the threshold below $\varDelta$ coincides with $T_{00} = T(p_0(E), p_0(E), E < \varDelta)$. The same is true for the S-matrix; $S_{11}(E > \varDelta)$ analytically continues to $S_{00}(E < \varDelta)$. On the contrary the reaction matrix $K(p, p', E)$ defined with the principal value integral is in general not an analytic function of E. In our case, however, Appendices C and D show that $K(p, p', E)$ is a regular function of $E - \varDelta$ above the threshold, but its analytic continuation below $\varDelta$ is not the reaction matrix below $\varDelta$. Equation $(2 \cdot 24)$ defines different analytic functions above and below $\varDelta$. Their connection is discussed in § 6.

§ 3. Coupling with the electromagnetic field

We consider a process; liquid helium at the ground state is excited by incident light with frequency ν, wave number k, polarization e, and the incident light is scattered to the light with ν', k', e'.

The Raman matrix element which leads to the non-interacting two-particle state $B_p{}^+ B_{-p}^+ \Psi_0$ is known to be

$$M^{(0)}(e, e', p) = \langle B_p{}^+ B_{-p}^+ \Psi_0, \mathcal{O}\Psi_0 \rangle , \tag{3·1}$$

where

$$\mathcal{O} = \frac{1}{2c^2 V \sqrt{\nu\nu'}} \left\{ J \cdot e' \frac{1}{\mathcal{E}_0 + \nu - \mathcal{H}} J \cdot e + J \cdot e \frac{1}{\mathcal{E}_0 - \nu' - \mathcal{H}} J \cdot e' \right\}, \tag{3·2}$$

J being the total current operator of all the charged particles in the liquid, $J = \sum_i e_i p_i / m_i$. Because $M^{(0)}(e, e', p)$ is a scalar, it is expressed by two form factors $\Gamma_0(p)$, $\Gamma_2(p)$ as

$$p M^{(0)}(e, e', p) = \frac{(2\pi)^3}{V} \left(\frac{e}{mc}\right)^2 \left\{ \frac{e \cdot e'}{\sqrt{4\pi}} \Gamma_0(p) + \sqrt{\frac{5}{16\pi}} (3\hat{p} \cdot e \hat{p} \cdot e' - e \cdot e') \Gamma_2(p) \right\}. \tag{3·3}$$

When the final state interaction is present, the correct matrix element[5] is

$$M(e, e', p_\lambda) = \langle \Psi_\lambda{}^{(-)}, \mathcal{O}\Psi_0 \rangle , \tag{3·4}$$

where $\Psi_\lambda{}^{(-)}$ is the incoming wave solution. Noting $\langle \Psi_\lambda{}^{(-)}| = \sum_p f_\lambda{}^{(-)*}(p) \langle B_p{}^+ B_{-p}^+ \Psi_0|$ and $f_\lambda{}^{(-)*}(p) = f_\lambda{}^{(+)}(p)$, we have from (2·9) and (3·3)

$$p_\lambda M(e, e', p_\lambda) = \frac{(2\pi)^3}{V} \left(\frac{e}{mc}\right)^2 \left\{ \frac{e \cdot e'}{\sqrt{4\pi}} M_0(p_\lambda) + \sqrt{\frac{5}{16\pi}} (3\hat{p}_\lambda \cdot e \hat{p}_\lambda \cdot e' - e \cdot e') M_2(p_\lambda) \right\}, \tag{3·5}$$

where l-wave matrix elements are

$$M_l(p_\lambda) = \int \Gamma_l(p) u^{(l)}(p, p_\lambda) dp . \tag{3·6}$$

The transition probability is calculated by the general rule,

$$\mathcal{W} = 2\pi \sum_{p_\lambda, k'} |M(e, e', p_\lambda)|^2 \delta(\nu - \nu' - \epsilon(p_\lambda)) . \tag{3·7}$$

The energy transfer $\nu - \nu'$ is the excitation energy E, $E = \nu - \nu'$. Inserting (3·5) into (3·7) and performing p_λ-summation over the half momentum space, we have

$$d\mathcal{W} = \left(\frac{e}{mc}\right)^4 \pi k'^2 dk' d\Omega_{\hat{k}'} \left\{ (e \cdot e')^2 \mathcal{S}^{(0)}(E) + \frac{1}{4}(3 + (e \cdot e')^2) \mathcal{S}^{(2)}(E) \right\}, \tag{3·8}$$

where the l-wave spectral function $\mathcal{S}^{(l)}(E)$ is

$$\mathcal{S}^{(l)}(E) = \sum_\lambda \frac{1}{v_\lambda} |M_l(p_\lambda)|^2 . \tag{3·9}$$

We introduce the eigenchannel matrix elements $L_\alpha{}^{(l)}$ by

$$L_\alpha{}^{(l)} = \int \Gamma_l(p) U^{(l)}(p, \alpha) dp , \qquad (3\cdot10)$$

which are written from (2·27) and (2·29) as

$$L_\alpha{}^{(l)} = \exp(i\delta_\alpha) \cos \delta_\alpha \sum_\lambda D^{(l)}(p_\lambda, E) \frac{f_{\lambda\alpha}}{\sqrt{v_\lambda}} , \qquad (3\cdot11)$$

where $D^{(l)}(p, E)$ is given by

$$D^{(l)}(p, E) = \Gamma_l(p) + \mathscr{P} \int \frac{\Gamma_l(p') K^{(l)}(p', p, E)}{E - \epsilon(p')} dp' . \qquad (3\cdot12)$$

We can express $\mathscr{S}^{(l)}(E)$ by $L_\alpha{}^{(l)}$,

$$\mathscr{S}^{(l)}(E) = \sum_\alpha |L_\alpha{}^{(l)}|^2 . \qquad (3\cdot13)$$

In Eq. (3·9) and in the following we understand the λ- or α-summation to be performed for λ or $\alpha = 1, 2, 3$ when $\varDelta < E < \varDelta'$, and for λ or $\alpha = 0$ when $E < \varDelta$ or $E > \varDelta'$.

We do not go into the calculation of form factors $\Gamma_l(p)$, nor into the pair interaction $W^{(l)}(p, p')$, because it requires some specific model for phonon-rotons. Notice the Hamiltonian $\mathscr{H}$ in Eq. (3·2); it should describe the motions of all the charged particles which compose the liquid. Of course, the low energy expansion (2·1) cannot be used. In this paper we assume that $\Gamma_{(l)}(p)$ and $W^{(l)}(p, p')$ are regular functions of p or p'.

From the next section we drop l because no formal differences exist between the treatments of *S*- and *D*-waves.

§ 4. Separable interaction

Before going into the general treatment of the Raman spectrum, we discuss in this section an explicitly soluble example, in which the pair interaction is taken as separable,

$$W(p, p') = gV(p) V(p') . \qquad (4\cdot1)$$

In the case of the separable interaction, Eq. (2·12) for $T(p, p', E)$ is easily solved to give

$$T(p, p', E) = V(p) \frac{1}{1/g + R(E) + iI(E)} V(p') , \qquad (4\cdot2)$$

where

$$R(E) = \mathscr{P} \int \frac{V^2(p)}{\epsilon(p) - E} dp , \qquad (4\cdot3)$$

$$I(E) = \pi \sum_\lambda \frac{V_\lambda{}^2}{v_\lambda} , \qquad (4\cdot4)$$
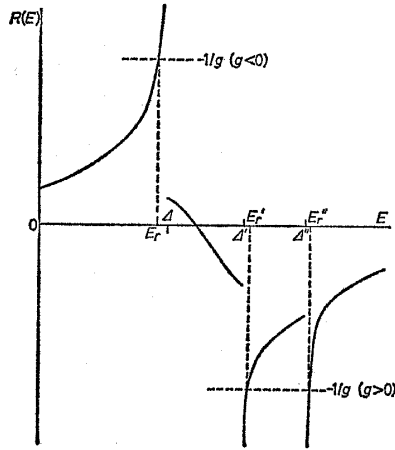
Fig. 2. Schematic graph for $R(E)$. The solutions $1/g + R(E) = 0$ are denoted by E_r when $g < 0$, and by E_r', E_r'' when $g > 0$.
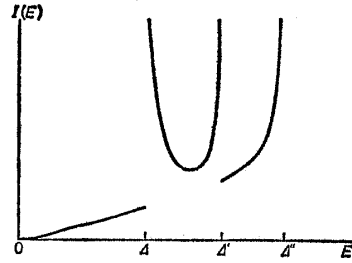

Fig. 3. Schematic graph for $I(E)$.

$V(p_\lambda)$ being simply denoted by V_λ. $R(E)$ and $I(E)$ are connected by a dispersion relation,

$$R(E) = \frac{\mathscr{P}}{\pi} \int \frac{I(\epsilon)}{\epsilon - E} d\epsilon . \qquad (4 \cdot 5)$$

Figures 2 and 3 show schematically the behaviors of $R(E)$ and $I(E)$. Notice the singular behaviors near $E = \varDelta, \varDelta'$ and $\varDelta''$. If we make a parabolic approximation to $\epsilon(p)$ near its minimum,

$$\epsilon(p) = (p - P)^2 / 2\mu + \varDelta, \qquad (4 \cdot 6)$$

$R(E)$ and $I(E)$ behave below and above the threshold $\varDelta$ as

$$R(E) = \pi V^2(P) \sqrt{\frac{2\mu}{\varDelta - E}}, \qquad E \lesssim \varDelta, \qquad (4 \cdot 7)$$

$$I(E) = \pi V^2(P) \sqrt{\frac{2\mu}{E - \varDelta}}, \qquad E \gtrsim \varDelta. \qquad (4 \cdot 8)$$

Similar behaviors are observed near $\varDelta'$ and $\varDelta''$.

We discuss the phase shifts. Introducing a unit vector,

$$f_\lambda = \sqrt{\frac{\pi}{I(E) v_\lambda}} V_\lambda, \qquad \sum_\lambda f_\lambda^2 = 1, \qquad (4 \cdot 9)$$

we have from $(4 \cdot 2)$

$$\pi \sum_{\lambda, \mu} \frac{T_{\mu\lambda}}{\sqrt{v_\mu v_\lambda}} f_\mu f_\lambda = \frac{I(E)}{1/g + R(E) + iI(E)} . \qquad (4 \cdot 10)$$

Comparing $(4 \cdot 10)$ with $(2 \cdot 22)$, we find that one eigenphase shift is given by

$$\tan \delta = -\frac{I(E)}{1/g + R(E)}, \qquad (4 \cdot 11)$$

and the other two eigenphase shifts δ_2, δ_3 for $\varDelta < E < \varDelta'$ satisfy

$$\sin \delta_2 = \sin \delta_3 = 0. \qquad (4 \cdot 12)$$
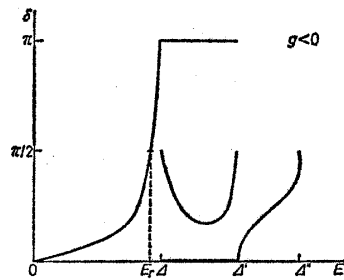
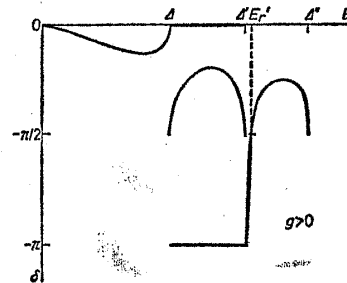Fig. 4. Schematic plot of the phase shift δ when $g<0$.

Fig. 5. Schematic plot of the phase shift δ when $g>0$.

Therefore, only one eigenphase shift determines the scattering states. This property is a special character of the separable approximation. We show the behaviors of the phase shifts in Fig. 4 when the interaction is attractive, $g<0$, and in Fig. 5 when repulsive, $g>0$. Resonance energy E_r is determined by

$$1/g + R(E_r) = 0. \tag{4.13}$$

Now we calculate the Raman spectrum. Quite a simple formula is obtained if we put

$$\Gamma(p) = cV(p). \tag{4.14}$$

Qualitative behavior of the spectrum does not change even if we use more general form for $\Gamma(p)$. From (2.11), (3.6), (3.9), (4.2) and (4.14) we have

$$\mathscr{S}(E) = \frac{1}{\pi}\left(\frac{c}{g}\right)^2 \frac{I(E)}{\{1/g + R(E)\}^2 + I^2(E)}, \tag{4.15}$$

which determines the spectrum by (3.8). Figure 6 shows qualitatively the behavior of (4.15) when $g<0$, and Fig. 7 when $g>0$. We notice the singular behavior just above the threshold $\varDelta$, where Eq. (4.15) is written by using (4.8) as

$$\mathscr{S}(E) = \frac{c^2\alpha^2\sqrt{E-\varDelta}}{E-\varDelta+g^2\beta^2}, \tag{4.16}$$
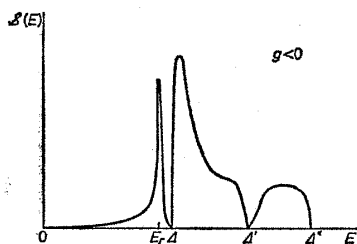

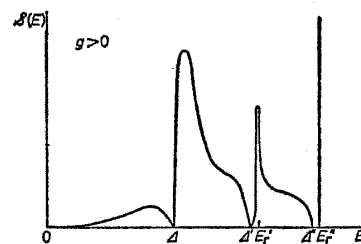Fig. 6. Schematic graph of the Raman spectrum $\mathscr{S}(E)$ when $g<0$ for the separable interaction.

Fig. 7. Schematic graph of the Raman spectrum $\mathscr{S}(E)$ when $g>0$ for the separable interaction.

α, β being some constants. Without the final state interaction $(g=0)$, it diverges as $(E-\varDelta)^{-1/2}$. However, it behaves as $(E-\varDelta)^{1/2}$ when $g\neq0$. Similar behaviors are observed at the other extremum points. We also notice the occurrence of a resonance. Its position depends on the sign of the pair interaction, g.

§ 5. General treatment near threshold

In this section we show that the Raman spectrum near the threshold $\varDelta$ behaves generally as

$$\mathscr{S}(E) = \begin{cases} s_0 + s_+ z + O(z^2), & E \gtrsim \varDelta , \\ s_0 + s_- \zeta + O(\zeta^2), & E \lesssim \varDelta , \end{cases} \tag{5.1}$$

where $z = \sqrt{E-\varDelta}$ and $\zeta = \sqrt{\varDelta - E}$. The constants s_0, s_+ and s_- depend on the interaction $W(p, p')$ and the form factor $\varGamma(p)$. In the following, formulas are expanded in z or ζ up to the order necessary to give s_0, s_+ and s_-. We first treat the quantities above the threshold, and then the formulas below the threshold are derived by a suitable analytic continuation.

In order to discuss the behavior near the threshold, we distinguish the singular quantities from the regular ones. As is shown in Appendices C and D, the reaction matrix $K(p, p', E)$ defined by (2.24) is a regular function with respect to the variable $z^2 = E - \varDelta$ above the threshold. All the irregularities above the threshold arise from v_2 and v_3, which are

$$v_2 = v_3 = \sqrt{\frac{2}{\mu}}\, z\, , \tag{5.2}$$

by using the parabolic formula (4.6) near the threshold. We expand the reaction matrix on the energy shell, $K(p_\mu(E), p_\lambda(E), E)$ with respect to z by using

$$p_1(E) = Q + \frac{1}{v_Q} z^2,$$

$$\left.\begin{array}{r} p_2(E) \\ p_3(E) \end{array}\right\} = P \mp \sqrt{2\mu}\, z, \tag{5.3}$$

where Q is the other momentum than P satisfying $\epsilon(Q) = \varDelta$, and $v_Q = |d\epsilon/dp|_{p=Q}$. In order to arrange the expansion neatly, we introduce the symmetrized channels by the transformation matrix,

$$(g_{\lambda a}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}. \tag{5.4}$$

The reaction matrix $\mathfrak{K}_{ab}$ in the symmetrized channels defined by

$$\mathfrak{K}_{ab} = -\pi \sum_{\mu,\lambda}' \frac{K_{\mu\lambda}}{\sqrt{v_\mu v_\lambda}} g_{\mu a} g_{\lambda b} \tag{5.5}$$

shows the following behavior for small $z \gtrless 0$;

$$(\mathfrak{K}_{ab}) = \begin{pmatrix} F & \dfrac{1}{\sqrt{z}} A & \sqrt{z} B \\[2mm] \dfrac{1}{\sqrt{z}} A & \dfrac{1}{z} G & C \\[2mm] \sqrt{z} B & C & z H \end{pmatrix}. \tag{5.6}$$

The constants in this expression are connected with K by

$$F = -\frac{\pi}{v_Q} K(Q, Q, \varDelta_+),$$

$$G = -\pi \sqrt{2\mu}\, K(P, P, \varDelta_+),$$

$$H = -\pi \sqrt{8\mu^3}\, \frac{\partial^2}{\partial p \partial p'} K(p, p', \varDelta_+)\,|_{p=p'=P},$$

$$A = -\pi \sqrt[4]{\frac{2\mu}{v_Q^2}}\, K(Q, P, \varDelta_+), \tag{5.7}$$

$$B = -\pi \sqrt[4]{\frac{8\mu^3}{v_Q^2}}\, \frac{\partial}{\partial P} K(Q, P, \varDelta_+),$$

$$C = -\pi \mu \frac{d}{dP} K(P, P, \varDelta_+).$$

These constants determine the scattering states near the threshold. This kind of approach is a generalization of the effective range theory. (See, for example, the article by Yamaguchi.[6]) All the effects near the threshold for the various interactions $W(p, p')$ are characterized by these constants. Actually the second derivative H does not affect the constants s_0, s_+, and s_-. In (5.7), $K(p, p', \varDelta_+)$ means $\lim_{E \to \varDelta + 0} K(p, p', E)$. The reaction matrix K given by (2.24) defines different analytic functions above and below the threshold.

We compute the eigen-phase shifts δ_α. From (2.26) and (5.5) the eigenvalues of $(\mathfrak{K}_{ab})$ give $\tan \delta_\alpha$,

$$\tan \delta_{\alpha=1} = \tan \delta_Q + az,$$

$$\tan \delta_{\alpha=2} = \frac{G}{z} + \frac{A^2}{G}, \tag{5.8}$$

$$\tan \delta_{\alpha=3} = \frac{\det \mathfrak{K}}{FG - A^2} z,$$

where

$$\tan \delta_Q = F - \frac{A^2}{G},$$

$$a = \frac{(BG - AC)^2}{G(FG - A^2)} - \frac{A^2}{G^3}(FG - A^2).$$

$$(5\cdot9)$$

From (5·8) the eigenphase shifts δ_α are given by

$$\delta_{\alpha=1} = \delta_Q + a\cos^2\delta_Q z,$$

$$\delta_{\alpha=2} = \frac{\pi}{2} - \frac{z}{G},$$

$$(5\cdot10)$$

$$\delta_{\alpha=3} = \frac{\det \mathfrak{R}}{FG - A^2}z.$$

The matrix $(h_{a\alpha})$,

$$h_{a\alpha} = \sum_{\lambda=1}^{3} g_{\lambda a} f_{\lambda \alpha},$$

$$(5\cdot11)$$

which transforms $\mathfrak{R}$ from the symmetrized channels to the eigenchannels, is calculated from (5·6) and (5·8),

$$(h_{a\alpha}) = \begin{pmatrix} 1 - \frac{1}{2}(t^2 + u^2)z & t\sqrt{z} & u\sqrt{z} \\ -t\sqrt{z} - v\sqrt{z^3} & 1 - t^2 z & wz \\ -u\sqrt{z} & \frac{C}{G}z & 1 \end{pmatrix}.$$

$$(5\cdot12)$$

The constants are expressed by

$$t = \frac{A}{G},$$

$$u = \frac{AC - BG}{FG - A^2},$$

$$v = -\frac{1}{2GA}(2aG + 2uBG + t^2A^2 + u^2A^2),$$

$$(5\cdot13)$$

$$w = \frac{AB - CF}{FG - A^2},$$

and there is a relation,

$$a = (u^2 - t^2)\tan \delta_Q.$$

$$(5\cdot14)$$

We can also compute the phase shift below the threshold by the analytic continuation, because S-matrix is the same analytic function of E above and below the threshold,

$$\exp\left(2i\delta_0(\zeta)\right) = S_{\lambda=0,\lambda=0}(\zeta) = S_{\lambda=1,\lambda=1}(z=i\zeta). \tag{5·15}$$

From (5·4) and (5·11) we have

$$S_{\lambda=1,\lambda=1} = \sum_\alpha h_{1\alpha}^2 \exp\left(2i\delta_\alpha\right). \tag{5·16}$$

Inserting (5·10) and (5·12) into (5·16), we have the phase shift δ_0 below the threshold,

$$\delta_0 = \delta_Q - \left(t^2 \cos^2\delta_Q + u^2 \sin^2\delta_Q\right)\zeta. \tag{5·17}$$

Of course, we can at once write down the scattering cross sections between two elementary excitations from the phase shifts, if necessary.

We calculate the Raman spectrum. Appendix C shows that $D(p, E)$ given by (3·12) is a regular function of z^2 above the threshold. As we have done for K, we expand $D(p_\lambda(E), E)$ in powers of z, and its first few terms characterize the coupling with the electromagnetic field near the threshold. The behavior of D in the symmetrized channels are

$$\sum_\lambda D(p_\lambda, E)\frac{g_{\lambda\alpha}}{\sqrt{v_\lambda}} = \begin{cases} X_{\alpha=1} + O(z^2), \\ \dfrac{1}{\sqrt{z}}X_{\alpha=2} + O(z^{3/2}), \\ \sqrt{z}X_{\alpha=3} + O(z^{5/2}), \end{cases} \tag{5·18}$$

where the three constants X_α are given by

$$X_1 = \frac{1}{\sqrt{v_Q}}D(Q, \varDelta_+),$$

$$X_2 = \sqrt[4]{2\mu}\,D(P, \varDelta_+), \tag{5·19}$$

$$X_3 = \sqrt[4]{8\mu^3}\,\frac{\partial}{\partial P}D(P, \varDelta_+).$$

From (3·11) and (5·11) the matrix elements L_α are connected with these constants X_α by

$$L_\alpha = \exp\left(i\delta_\alpha\right)\cos\delta_\alpha\left(X_1 h_{1\alpha} + \frac{1}{\sqrt{z}}X_2 h_{2\alpha} + \sqrt{z}X_3 h_{3\alpha}\right). \tag{5·20}$$

Inserting (5·10), (5·12) into (5·20), we have

$$L_1 = \exp\left(i\delta_Q\right)\cos\delta_Q\{(X_1 - tX_2) - (l_R - il_I)z\},$$

$$L_2 = i\frac{X_2}{G}\sqrt{z}, \tag{5·21}$$

$$L_3 = (uX_1 + wX_2 + X_3)\sqrt{z},$$

where we put

$$l_R = \tfrac{1}{2}(t^2 + u^2)X_1 + vX_2 + uX_3 + a\sin\delta_Q\cos\delta_Q(X_1 - tX_2),$$

$$l_I = a \cos^2 \delta_Q (X_1 - t X_2). \tag{5.22}$$

From L_α given by (5·21) we can calculate the Raman spectrum $\mathscr{S}(E)$ by (3·13). We find that the coefficients for the Raman spectrum in (5·1) above the threshold are given by

$$s_0 = \cos^2 \delta_Q (X_1 - t X_2)^2 \tag{5.23}$$

and

$$s_+ = \frac{X_2^2}{G^2} + (u X_1 + w X_2 + X_3)^2 - 2 \cos^2 \delta_Q (X_1 - t X_2) l_R . \tag{5.24}$$

Finally, we derive the Raman spectrum below the threshold by the analytic continuation. From (2·11) and (3·6) we have

$$M(p_\lambda(E), E) = \Gamma(p_\lambda(E)) + \int \frac{\Gamma(p) T(p, p_\lambda(E), E)}{E - \epsilon(p) + i\eta} dp . \tag{5.25}$$

As a function of E, $M(p_1(E), E)$ and $M(p_0(E), E)$ are the same analytic function. Therefore, from (3·10) and (5·11) the matrix element L_0 below the threshold is given by

$$L_0 = \frac{1}{\sqrt{v_0}} M(p_0(E), E) = \sum_{\alpha=1}^{3} h_{1\alpha} L_\alpha . \tag{5.26}$$

Inserting (5·12) and (5·21) into this expression and putting $z = i\zeta$, we have the Raman spectrum below the threshold by $\mathscr{S}(E) = |L_0|^2$. We find again s_0 given by (5·23) and s_- given by

$$s_- = 2 \cos \delta_Q (X_1 - t X_2) \left\{ \{ \sin \delta_Q (u^2 X_1 + u w X_2 + u X_3) - \cos \delta_Q \left(l_I + \frac{X_2}{G} t \right) \right\} . \tag{5.27}$$

§ 6. General analysis of a possible resonance

In § 5 we have derived the phase shift just below the threshold. In order to discuss a possible resonance it is desirable to extend the formula so as to cover the wider range of energy. In this section we analyse K-matrix below the threshold, and derive a general formula which determines the resonance energy.

The K-matrix above the threshold, $K(p, p', E > \varDelta)$, is a regular function of $E - \varDelta$. We rewrite its definition (2·24) in a form

$$K(p, p', E > \varDelta) = W(p, p')$$

$$+ \int W(p, p'') \left\{ \frac{1}{E - \epsilon(p'') + i\eta} + i\pi \sum_{\lambda=1}^{3} \frac{1}{v_\lambda(E)} \delta(p'' - p_\lambda(E)) \right\}$$

$$\times K(p'', p', E) dp'', \tag{6.1}$$

where $v_\lambda(E)$ and $p_\lambda(E)$ are given by (5·2) and (5·3). We denote the analytic continuation of $K(p, p', E > \Delta)$ beyond Δ by $K_1(p, p', \zeta)$, which follows the equation

$$K_1(p, p', \zeta) = W(p, p') + \int W(p, p'') \left[\frac{\mathscr{P}}{\Delta - \zeta^2 - \epsilon(p'')} \right.$$
$$\left. + \sqrt{\frac{\mu}{2}} \frac{\pi}{\zeta} \left\{ \delta(p'' - P + i\sqrt{2\mu}\,\zeta) + \delta(p'' - P - i\sqrt{2\mu}\,\zeta) \right\} \right]$$
$$\times K_1(p'', p', \zeta) \, dp''. \tag{6·2}$$

Equation (6·2) is derived from (6·1) by putting $E = \Delta - \zeta^2$, or $z = i\zeta$. The unusual δ-function with complex argument is defined in the following way:

$$\int \{\delta(p - P + iq) + \delta(p - P - iq)\} f(p) \, dp = f(P - iq) + f(P + iq)$$
$$= 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}}{(2n)!} \frac{d^{2n}}{dP^{2n}} f(P). \tag{6·3}$$

The function $K_1(p, p', \zeta)$ is not the K-matrix below the threshold; the latter obeys Eq. (2·24) at $E < \Delta$. Their connection is given by (see Appendix B)

$$K(p, p') = K_1(p, p') - \sqrt{\frac{\mu}{2}} \frac{\pi}{\zeta} \left\{ K_1(p, +) K(+, p') + K_1(p, -) K(-, p') \right\}, \tag{6·4}$$

where we simply denote $P \pm i\sqrt{2\mu}\,\zeta$ by $+$ and $-$, and we suppress the energy variable. In (6·4) and in the following, we understand $K(p, p')$ to be the K-matrix below the threshold. Putting $p' = p_0(E)$ and $p = P \pm i\sqrt{2\mu}\,\zeta$ in (6·4), we have a coupled equation for $K(+, p_0)$ and $K(-, p_0)$,

$$\left\{ \zeta + \pi \sqrt{\frac{\mu}{2}} K_1(+, +) \right\} K(+, p_0) + \pi \sqrt{\frac{\mu}{2}} K_1(+, -) K(-, p_0) = \zeta K_1(+, p_0), \tag{6·5}$$

$$\pi \sqrt{\frac{\mu}{2}} K_1(-, +) K(+, p_0) + \left\{ \zeta + \pi \sqrt{\frac{\mu}{2}} K_1(-, -) \right\} K(-, p_0) = \zeta K_1(-, p_0),$$

which is solved to give

$$K(+, p_0) = \frac{\zeta}{d(\zeta)} \left[\zeta K_1(+, p_0) + \pi \sqrt{\frac{\mu}{2}} \{ K_1(-, -) K_1(+, p_0) - K_1(+, -) K_1(-, p_0) \} \right],$$

$$K(-, p_0) = \frac{\zeta}{d(\zeta)} \left[\zeta K_1(-, p_0) + \pi \sqrt{\frac{\mu}{2}} \{ K_1(+, +) K_1(-, p_0) - K_1(-, +) K_1(+, p_0) \} \right], \tag{6·6}$$

where

$$d(\zeta) = \zeta^2 + \pi\sqrt{\frac{\mu}{2}}\zeta\{K_1(+,+) + K_1(-,-)\}$$

$$+\frac{\pi^2\mu}{2}\{K_1(+,+)K_1(-,-) - K_1(+,-)K_1(-,+)\}. \qquad (6\cdot7)$$

The formula for the phase shift δ_0 below the threshold is derived from $(6\cdot4)$ by putting $p=p'=p_0(E)$ and inserting $(6\cdot6)$,

$$-\frac{v_0}{\pi}\tan\delta_0 = K(p_0, p_0) = K_1(p_0, p_0) - \frac{n(\zeta)}{d(\zeta)}, \qquad (6\cdot8)$$

where

$$n(\zeta) = \pi\sqrt{\frac{\mu}{2}}\zeta\{K_1(p_0,+)K_1(+,p_0) + K_1(p_0,-)K_1(-,p_0)\}$$

$$+\frac{\pi^2\mu}{2}\{K_1(p_0,+)K_1(-,-)K_1(+,p_0) - K_1(p_0,+)K_1(+,-)K_1(-,p_0)$$

$$+ K_1(p_0,-)K_1(+,+)K_1(-,p_0) - K_1(p_0,-)K_1(-,+)K_1(+,p_0)\}. $$
$$(6\cdot9)$$

This formula gives the phase shift below the threshold. In particular, the resonance is determined by $d(\zeta)=0$. The resonance exists below $\varDelta$ if and only if the equation $d(\zeta)=0$ has a real positive root for ζ. Of course, it is not difficult to rederive from $(6\cdot8)$ Eq. $(5\cdot17)$ for small ζ.

§ 7.　Interference effects

As generally shown in § 5, the Raman spectrum does not diverge at the threshold, and the matrix elements L_α for eigen-channels varies as $L_1 \sim O(1)$ and $L_2 \sim L_3 \sim O(\sqrt{z})$, although the density of states is proportional to $1/z$. In order to see the physical implication of the result, we study the wave functions in this section and show that the destructive interference occurs between waves of negative group velocity $(\lambda=2)$ and of positive group velocity $(\lambda=3)$.

We first consider in the symmetrized channels the wave function $U_a(r)$, which is connected with the eigen-channel wave function $U_\alpha(r)$ by $U_a(r) = \sum_\alpha U_\alpha(r)h_{a\alpha}$. From $(2\cdot23)$ its asymptotic form for large r slightly above the threshold $\varDelta$ is

$$U_a(r) \sim \exp(i\delta_a)\sum_\lambda \sin(p_\lambda r + \sigma_\lambda\delta_a)\frac{g_{\lambda a}}{\sqrt{v_\lambda}}$$

$$= \begin{cases} \exp{(i\delta_\alpha)} \dfrac{1}{\sqrt{v_1}} \sin{(p_1 r + \delta_\alpha)}, & a = 1, \\[3mm] \exp{(i\delta_\alpha)} \dfrac{\sqrt[4]{2\mu}}{\sqrt{z}} \cos{(\sqrt{2\mu}\,zr + \delta_\alpha)} \sin Pr, & a = 2, \\[3mm] \exp{(i\delta_\alpha)} \dfrac{\sqrt[4]{2\mu}}{\sqrt{z}} \sin{(\sqrt{2\mu}\,zr + \delta_\alpha)} \cos Pr, & a = 3. \end{cases} \tag{7.1}$$

We can see the characteristic beat arising from the interference between waves of $\lambda = 2$ and $\lambda = 3$; $p_2 = P - \sqrt{2\mu}z$ and $p_3 = P + \sqrt{2\mu}z$.

Next we calculate the eigenchannel wave function $U_\alpha(r)$. From (5·10) and (5·12) we have

$$U_{\alpha=1}(r) \sim \exp{(i\delta_Q)} \Big\{ \frac{1}{\sqrt{v_Q}} \sin{(p_1 r + \delta_Q)} - \sqrt[4]{2\mu}\,t \cos{(\sqrt{2\mu}\,zr + \delta_Q)} \sin Pr$$

$$- \sqrt[4]{2\mu}\,u \sin{(\sqrt{2\mu}\,zr + \delta_Q)} \cos Pr \Big\} + O(z),$$

$$U_{\alpha=2}(r) \sim i \frac{\sqrt[4]{2\mu}}{\sqrt{z}} \sin\Big(\sqrt{2\mu}\,zr - \frac{z}{G} \Big) \sin Pr + O(z^{1/2}), \tag{7.2}$$

$$U_{\alpha=3}(r) \sim \frac{\sqrt[4]{2\mu}}{\sqrt{z}} \sin\Big(\sqrt{2\mu}\,zr + \frac{\det \mathfrak{R}}{FG - A^2} z \Big) \cos Pr + O(z^{1/2}).$$

From the above formula we can see that $U_{\alpha=2}$ and $U_{\alpha=3}$ are of the order of $1/\sqrt{z}$ at very large $r(\sqrt{2\mu}zr \gtrsim 1)$ due to flux normalization, while $U_{\alpha=1}$ is $O(1)$. However, $U_{\alpha=2}$ and $U_{\alpha=3}$ become of the order of $\sqrt{z}$ for small $z(\sqrt{2\mu}zr \lesssim 1)$. Notice that the asymptotic formulas for $U_{\alpha=2}$ and $U_{\alpha=3}$ are valid as long as $Pr \gtrsim 1$. We remark that the origin of such a destructive interference is traced back to the sign factor σ_λ in Eq. (2·16) and to the infinite density of states at the threshold.

Finally, we discuss the asymptotic form of the wave function at the resonance. Because the formalism developed in § 6 is too general, we employ the separable approximation given in § 4. At the resonance Eq. (4·2) gives

$$T(p, p', E_r) = -i \frac{V(p)\,V(p')}{I(E_r)}. \tag{7.3}$$

Inserting (7·3) into (2·11) and performing the integration in (2·15) for large r, we can easily show that

$$u_0(r) \sim i \cos p_0 r + i \frac{\pi V(P)\,V(p_0)}{I(E_r)} \sqrt{\frac{2\mu}{\varDelta - E_r}} \exp{(-\sqrt{2\mu(\varDelta - E_r)}\,r)} \sin Pr \tag{7.4}$$

at the resonance. The first term in (7·4) is the incoming and the outgoing spherical waves of momentum $p_0(E_r)$ with the phase shift $\pi/2$. The second term

represents a bound state formed by the waves near the minimum momentum P. We notice that the waves near P interfere constructively to form the bound state. The change of the interference nature from above to below the threshold reflects upon the analytic behavior of the reaction matrix K as explained in § 6.

§ 8. Discussion

We have shown that the final state interaction between the pair of excitations has a drastic influence on the Raman spectrum at the threshold, and in general the spectrum has no peaks just at the thresholds contrary to the infinite density of states. We remark, however, that the final state interaction makes little change to the integrated spectrum,

$$\mathscr{J} = \int_{E_1}^{E_2} \mathscr{S}(E)\,dE, \tag{8·1}$$

where the energy range $E_2 - E_1$ is wider than the strength of the pair interaction, $E_2 - E_1 > |W(p, p')|$, because the interaction $W(p, p')$ induces a unitary transformation which makes $\Psi_\lambda^{(-)}$ with energy E_λ be a superposition of noninteracting two-particle states $B_p^+ B_{-p}^+ \Psi_0$ with energy about in the range $E_\lambda - |W|$ $\sim E_\lambda + |W|$, and we have from (3·7)

$$\mathscr{J} \propto \sum_{E_1 < E_\lambda < E_2} |\langle \Psi_\lambda^{(-)}, \mathcal{O}\Psi_0 \rangle|^2 \doteqdot \sum_{E_1 < \epsilon(p) < E_2} |\langle B_p^+ B_{-p}^+ \Psi_0, \mathcal{O}\Psi_0 \rangle|^2. \tag{8·2}$$

Experiment[2] suggests $s_+ > 0$, $s_- < 0$ at the threshold $\varDelta$ and $s_+ > 0$, $s_- > 0$ at $\varDelta'$, and indicates no sharp resonances. It is highly desirable to perform the experiment at lower temperature and with sharper energy resolution in order to have informations about the interactions between phonon-rotons.

Experiment[2] shows that the scattered light is highly depolarized. As noticed by Stephen,[3] this property is probably due to the dipole-dipole interaction between helium atoms; the dipole-dipole interaction leads to the D-wave form factor $\varGamma_2(p)$. In this connection we remark that measurements on the angular distribution of the scattered light irrespective of its polarization will also tell us the relative magnitudes of the S-wave and D-wave form factors.

Raman scattering gives us informations about the states of total momentum Q almost zero. However, there is a high-density band of two-particle states with mean energy about 20°K extending from $Q = 0$ to around $Q = 4\text{Å}^{-1}$. This band seems to be found by the neutron scattering experiment of Cowley and Woods.[7] Analysis of the band is an interesting problem. At nonzero Q the angular momentum is no longer a good quantum number. As Q increases from zero, S-, D-, $\cdots$-waves are mixed with each other. Such an analysis will shed light on the structure of phonon-rotons as well as the high density band.

The author has profited from conversations with Dr. M. Kato and with Dr. T. Terasawa on some aspects on the scattering theory.

Appendix

A) *Asymptotic form of the wave function.* We derive Eq. (2·16). By inserting the well-known formula

$$\frac{1}{E-\epsilon(p)+i\eta} = \frac{\mathscr{P}}{E-\epsilon(p)} - i\pi\delta(E-\epsilon(p)) \tag{A·1}$$

into (2·11) and by using the asymptotic form $i^l p r j_l(pr) \sim \sin pr$ for even l, Eq. (2·15) is written as

$$u_\lambda(r) \sim \sin p_\lambda r - i\pi \sum_{\mu=1}^{3} \left| \frac{dp}{d\epsilon} \right|_{p=p_\mu} T_{\mu\lambda} \sin p_\mu r + \mathscr{P}\int \frac{\sin pr\, T(p,p_\lambda,E)}{E-\epsilon(p)} dp. \tag{A·2}$$

The range of the integral in (A·2) is divided into three parts, each containing $p_\mu (\mu=1,2,3)$. One part of the integral containing p_μ in its range is

$$\frac{1}{r}\mathscr{P}\int \frac{(\sin p_\mu r \cos x + \cos p_\mu r \sin x) T(p_\mu+x/r, p_\lambda, E)}{E-\epsilon(p_\mu+x/r)} dx \tag{A·3}$$

by using the integral variable $x=(p-p_\mu)r$. For large r the range of the x-integral becomes $-\infty \sim +\infty$, and we have

$$(A·3) = -\pi\left(\frac{dp}{d\epsilon}\right)_{p=p_\mu} T_{\mu\lambda} \cos p_\mu r \tag{A·4}$$

by making use of $\int_{-\infty}^{\infty}(\sin x/x)dx=\pi$. Combining (A·2) and (A·4) we have Eq. (2·16).

B) *Proof of Eqs. (2·25) and (6·4).* We use a lemma: The integral equation for $\psi(p,p')$,

$$\psi(p,p') = W(p,p') + \int W(p,p'')\{G_1(p'')+G_2(p'')\}\psi(p'',p')dp'', \tag{B·1}$$

is rewritten with the solution $\chi(p,p')$ of the integral equation

$$\chi(p,p') = W(p,p') + \int W(p,p'')G_1(p'')\chi(p'',p')dp'' \tag{B·2}$$

as

$$\psi(p,p') = \chi(p,p') + \int \chi(p,p'')G_2(p'')\psi(p'',p')dp''. \tag{B·3}$$

The lemma is easily proved by putting $p=q$ in (B·3), multiplying $\delta(p-q) - W(p,q)G_1(q)$, integrating with respect to q, and using (B·2).

If we put in the lemma

$$G_1(p) = \frac{\mathscr{P}}{E-\epsilon(p)},$$

$$G_2(p) = -i\pi\delta(E-\epsilon(p)) = -i\pi\sum_\mu \frac{1}{v_\mu}\delta(p-p_\mu),$$

$$\tag{B·4}$$

and identify $\psi = T$ and $\chi = K$, we have at once the Heitler equation (2·25).

By taking in the lemma

$$G_1(p) = \frac{\mathscr{P}}{\varDelta - \zeta^2 - \epsilon(p)} + \sqrt{\frac{\mu}{2}} \frac{\pi}{\zeta} \{\delta(p - P + i\sqrt{2\mu}\,\zeta) + \delta(p - P - i\sqrt{2\mu}\,\zeta)\},$$

$$G_2(p) = -\sqrt{\frac{\mu}{2}} \frac{\pi}{\zeta} \{\delta(p - P + i\sqrt{2\mu}\,\zeta) + \delta(p - P - i\sqrt{2\mu}\,\zeta)\},$$

(B·5)

Eq. (6·4) is derived.

C) *Analytic properties of the principal value integral.* We show that the integral

$$J(E) = \mathscr{P} \int \frac{f(p)}{\epsilon(p) - E} dp \qquad\qquad (\text{C·1})$$

is a regular function of E near $\varDelta$ when defined at $E > \varDelta$, assuming $f(p)$, $\epsilon(p)$ to be regular functions of p. On the contrary, when defined at $E < \varDelta$, $J(E)$ is an analytic function of $\sqrt{\varDelta - E}$ except a term proportional to $1/\sqrt{\varDelta - E}$.

We first consider a part $0 < p < P'$. For this part the integral is rewritten as

$$J_1(E) = \mathscr{P} \int_0^{\varDelta'} \frac{F(\epsilon)}{\epsilon - E} d\epsilon, \qquad\qquad (\text{C·2})$$

where $F(\epsilon) = f(p) dp/d\epsilon$ is a regular function of ϵ. We write $J_1(E)$ as

$$J_1(E) = F(E)\mathscr{P} \int_0^{\varDelta'} \frac{d\epsilon}{\epsilon - E} + \int_0^{\varDelta'} \frac{F(\epsilon) - F(E)}{\epsilon - E} d\epsilon. \qquad\qquad (\text{C·3})$$

The first term is $F(E)\ln(\varDelta' - E/E)$, which is regular near $\varDelta$. The integrand of the second term is a regular function of $\epsilon - E$, so the integral is also regular with respect to E.

Next we choose q_0 such that $\epsilon(P - q_0)$ and $\epsilon(P + q_0)$ are slightly above E. For the part $P' < p < P - q_0$ and $p > P + q_0$, we have

$$J_2(E) = \int \frac{f(p)}{\epsilon(p) - \varDelta} \sum_{n=0}^{\infty} \left(\frac{E - \varDelta}{\epsilon(p) - \varDelta}\right)^n dp. \qquad\qquad (\text{C·4})$$

Because $(E - \varDelta)/(\epsilon(p) - \varDelta) < 1$ for this part, $J_2(E)$ is regular.

Finally the rest part, $P - q_0 < p < P + q_0$, is written as

$$J_3(E) = 2\mu\mathscr{P} \int_{-q_0}^{+q_0} \frac{f(P + q)}{q^2 - 2\mu(E - \varDelta)} dq, \qquad\qquad (\text{C·5})$$

where we approximate $\epsilon(p) = \varDelta + (p - P)^2/2\mu$, and put $q = p - P$. Expanding $f(P + q)$ with respect to q, we have, when $E > \varDelta$,

$$J_3(E) = \{f(P) + \mu(E - \varDelta)f''(P) + \cdots\} \sqrt{\frac{2\mu}{E - \varDelta}} \ln \frac{q_0 - \sqrt{2\mu(E - \varDelta)}}{q_0 + \sqrt{2\mu(E - \varDelta)}}$$

$$+ 2\mu q_0 f''(P) + \cdots, \qquad\qquad (\text{C·6})$$

which is regular with respect to $E-\Delta$. When $E<\Delta$, we have

$$J_3(E) = \{f(P) - \mu(\Delta - E)f''(P) + \cdots\}\sqrt{\frac{2\mu}{\Delta - E}}\left(\pi - 2\tan^{-1}\frac{\sqrt{2\mu(\Delta - E)}}{q_0}\right)$$
$$+ 2\mu q_0 f''(P) + \cdots, \qquad\qquad\qquad (C\cdot7)$$

which contains

$$\pi\sqrt{\frac{2\mu}{\Delta - E}}\{f(P) - \mu(\Delta - E)f''(P) + \cdots\}$$

apart from the analytic continuation of (C·6).

D) *Analytic property of* $K(p, p', E>\Delta)$. To show that $K(p, p', E>\Delta)$ is regular, we have from Eq. (2·24)

$$K(p, p', E>\Delta) = W(p, p') + \mathscr{P}\int\frac{W(p, p'')W(p'', p')}{E - \epsilon(p'')}dp''$$
$$+ \int W(p, p'')\frac{\mathscr{P}}{E - \epsilon(p'')}W(p'', p''')\frac{\mathscr{P}}{E - \epsilon(p''')}W(p''', p')dp''dp''' + \cdots.$$
$$\qquad\qquad\qquad (D\cdot1)$$

Appendix C says that each term in (D·1) is regular with respect to E, and the series will converge uniformly, at least, when the interaction W is sufficiently weak.

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