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# On the range of options prices*

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**Abstract.** In this paper we consider the valuation of an option with time to expiration $T$ and pay-off function $g$ which is a convex function (as is a European call option), and constant interest rate $r$, in the case where the underlying model for stock prices $(S_t)$ is a purely discontinuous process (hence typically the model is incomplete). The main result is that, for "most" such models, the range of the values of the option, using all possible equivalent martingale measures for the valuation, is the interval $(e^{-rT}g(e^{rT}S_0), S_0)$, this interval being the biggest interval in which the values must lie, whatever model is used.

**Key words:** Contingent claim valuation, incomplete model, purely discontinuous process, martingale measures

**JEL classification:** G13

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## 1. Introduction

In a complete financial market any contingent claim is attainable and can be valued on the basis of the unique equivalent martingale measure (see Harrison and Pliska (1981) for terminology). The most prominent example of a complete model is the Black-Scholes model, where stock prices evolve according to a geometric Brownian motion. Despite of its popularity this model has serious deficiencies: from the point of view of the distribution of returns as well as from the point of view of its path properties. If a model is based on daily

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returns of a stock, statistical tests clearly reject the normality assumption made in the Black-Scholes case. For a more recent empirical study of distributions using German stock price data see Eberlein and Keller (1995). References to a number of classical studies of the US-market are given there. Looking at paths on an intraday time-scale, that is looking at the microstructure of stock price movements, Fig. 3 of the same paper shows, that a more realistic model should be a purely discontinuous process instead of a continuous one.

Since the returns are usually defined as increments of log stock prices, that is as $\log S_t - \log S_{t-1}$, we choose as a model for stock prices

$$S_t = S_0 \exp(X_t) \tag{1}$$

with $X = (X_t)_{t \geq 0}$ as the corresponding return process. If $X$ is a semimartingale - it is not easy to find a process which is not in this class; fractional Brownian motions are an example - by Ito's formula $(S_t)$ is the solution of the stochastic differential equation

$$dS_t = S_{t-} \left[ dX_t + \frac{1}{2} d\langle X^c \rangle_t + \left( e^{\Delta X_t} - 1 - \Delta X_t \right) \right]. \tag{2}$$

Here $X^c$ denotes the continuous martingale part of $X$ and $\Delta X_t = X_t - X_{t-}$ the jump at time $t$ (see Jacod and Shiryaev (1987) for terminology). Note that the simpler equation $d\tilde{S}_t = \tilde{S}_{t-} dX_t$ would result in the Doléans-Dade exponential

$$\tilde{S}_t = \tilde{S}_0 \exp\left( X_t - \frac{1}{2} \langle X^c \rangle_t \right) \prod_{0 < s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

The returns of this process are not the statistically observable quantities.

In view of the empirical facts mentioned above we are looking for a model with vanishing continuous martingale part $X^c$, which means that equation (2) reduces to

$$dS_t = S_{t-} \left[ dX_t + \left( e^{\Delta X_t} - 1 - \Delta X_t \right) \right]. \tag{3}$$

In the following we assume that $X$ is a Lévy process under some probability measure $P$, that is a process with stationary, independent increments starting at 0. A typical example for such a process whose continuous martingale part vanishes, is the hyperbolic Lévy motion defined in Eberlein and Keller (1995).

Coming back to the question of contingent claim valuation we have to find an equivalent martingale measure. Unfortunately under the assumptions made above, we entered the realm of incomplete models. Instead of a unique equivalent martingale measure typically there is a large class of such measures. This fact alone would not pose a problem as far as contingent claim valuation is concerned. Real markets know at least two prices: the bid and the ask price. It would be satisfactory if the values computed on the basis of the equivalent martingale measures would span an interval corresponding to the bid-ask spread. In the following we describe the relevant class of measures and show that the corresponding values span a much wider interval. In the case of a European call option with strike $\Gamma$ and time to expiration $T$ and with constant interest rate $r$, the values span the whole interval from $(S_0 - e^{-rT}\Gamma)^+$ to $S_0$, which is an "absolute" interval in which all prices must lie, whatever model is used, for arbitrage reasons. Similarly, if the pay-off function of an option is $g$, a function satisfying the set of assumptions (7) below, the values span the whole interval from $e^{-rT}g(e^{rT}S_0)$ to $S_0$.

## 2. Results

Let $r$ denote the constant interest rate. We write $\mathscr{M}_r$ for the (possibly empty) class of measures locally equivalent to $P$, under which $e^{-rt}S_t$ is a martingale, and $\mathscr{M}'_r$ for the subclass of all $Q \in \mathscr{M}_r$ under which $X$ is again a Lévy process.

As a preliminary result we wish to examine under which conditions $\mathscr{M}_r$ or $\mathscr{M}'_r$ are not empty. This will be expressed in terms of the interest rate $r$, the drift $b$ and the Lévy measure $F$ of $X$ under $P$.

By convention, the drift will be computed with the truncation function $\varphi(x) = x1_{\{|x|\leq 1\}}$, so that if $\mu$ is the jump measure of $X$,

$$\mu(\omega, dt, dx) = \sum_{s\geq 0} 1_{\{\Delta X_s(\omega)\neq 0\}}\varepsilon_{(s,\Delta X_s(\omega))}(dt, dx),$$

and $\nu(dt, dx) = dtF(dx)$, under $P$ we may use the canonical representation

$$X_t = bt + \int_0^t \int \varphi(x)(\mu - \nu)(ds, dx) + \int_0^t \int (x - \varphi(x))\mu(ds, dx). \qquad (4)$$

Introduce also the class $\mathscr{Y}_r$ of functions $y : \mathbb{R} \to (0, \infty)$ such that

$$\left.\begin{aligned} &\int \left(\sqrt{y(x)} - 1\right)^2 F(dx) + \int_{\{x>1\}}(e^x - 1)y(x)F(dx) < \infty \\ &b - r + \int((e^x - 1)y(x) - \varphi(x))F(dx) = 0. \end{aligned}\right\} \qquad (5)$$

**Proposition 1.** *If $\mathscr{Y}_r = \emptyset$ then $\mathscr{M}_r = \mathscr{M}'_r = \emptyset$. If $\mathscr{Y}_r \neq \emptyset$ then both $\mathscr{M}_r$ and $\mathscr{M}'_r$ are non empty, and for each $y \in \mathscr{Y}_r$ there is a measure $Q \in \mathscr{M}'_r$ under which $X$ is a Lévy process with drift $b'$ and Lévy measure $F'$ given by*

$$\left.\begin{aligned} &b' = b + \int \varphi(x)(y(x) - 1)F(dx), \\ &F'(A) = \int_A y(x)F(dx). \end{aligned}\right\} \qquad (6)$$

Now let $g$ be the pay-off function of our option. We assume the following on this function:

$$g \text{ is convex}, \quad \lim_{x\to\infty} \frac{g(x)}{x} = 1, \quad 0 \leq g(x) < x \quad \text{for all } x > 0. \qquad (7)$$

These assumptions are quite natural. For a European call option with strike $\Gamma > 0$, $g(x) = (x - \Gamma)^+$.

Under the measure $Q \in \mathscr{M}_r$ the value of the option is then

$$\gamma(Q) = E_Q \left[ e^{-rT} g(S_T) \right] . \tag{8}$$

We consider the range sets $I_r = \{\gamma(Q) | Q \in \mathscr{M}_r\}$ and $I'_r = \{\gamma(Q) | Q \in \mathscr{M}'_r\}$. We are naturally interested in $I_r$, but the smaller set $I'_r$ also has some interest and it will be the key technical ingredient to our proof.

There are obvious bounds on $\gamma(Q)$: first by the convexity of $g$ and the bound (7), the process $M_t = g(e^{r(T-t)} S_t)$ is a $Q$-submartingale for each $Q \in \mathscr{M}_r$, so $\gamma(Q) = e^{-rT} E_Q[M_T] \geq e^{-rT} g(e^{rT} S_0)$. Second we have $e^{-rT} g(S_T) < e^{-rT} S_T$ by (7), so $\gamma(Q) < S_0$. Thus

$$I'_r \subset I_r \subset \left[ e^{-rT} g(e^{rT} S_0), S_0 \right) . \tag{9}$$

Our main theorem is as follows:

**Theorem 2.** *Assume that the Lévy measure $F$ of the Lévy process $X$ under $P$ has the following properties:*

(i) $F((-\infty, a]) > 0$ *for all* $a \in \mathbb{R}$.
(ii) $F$ *has no atom and satisfies* $\int_{[-1,0)} |x| F(dx) = \int_{(0,1]} x F(dx) = \infty$.

*Then $\mathscr{M}_r$ is not empty, $I_r$ is the full interval $\left( e^{-rT} g(e^{rT} S_0), S_0 \right)$ and $I'_r$ is dense in this interval.*

**Remark 3.** It can be shown that under the assumptions considered above $I'_r$ as well is the full interval. This will be proved in a forthcoming paper, where various assumptions will be discussed.

**Remark 4.** The above assumptions on $F$ imply that the process $X$ has both positive and negative jumps, and indeed negative jumps of arbitrary large size; it also implies that it has infinitely many jumps, and even infinite variation, over every non void time interval. In fact we can replace (ii) by a weaker property, more complicated to state, but which proves to be more natural in the proof. For this, we need the following notation, for $z > 0$:

$$\delta(z) = \int_{(-\infty,-z)} (1 - e^x) F(dx), \quad \delta'(z) = \int_{(z,\infty)} (1 - e^{-x}) F(dx). \tag{10}$$

These are two non-increasing right-continuous functions with limit 0 at $+\infty$, and (ii) above may be replaced by:

(ii') *For any $A \in \mathbb{R}$ there are two sequences $(\varepsilon_n)$, $(\varepsilon'_n)$ decreasing to 0 such that $\delta(\varepsilon_n) - \delta'(\varepsilon'_n) \to A$.*

**Remark 5.** The assumptions of Theorem 2 are obviously satisfied by all nonnormal stable processes. More interestingly for our concern, it is shown in Eberlein and Keller (1995), that the centered symmetric hyperbolic distributions fit financial data quite well, and in order to get a good model for stock prices one can choose $X$ as a hyperbolic Lévy motion plus a drift: that is to say $X$ is given by (4) with an arbitrary drift $b$ and a Lévy measure $F_{\zeta,\delta}$ depending on a shape parameter $\zeta > 0$ and a scale parameter $\delta > 0$. This Lévy measure admits a density w.r.t. Lebesgue measure given by

$$f_{\zeta,\delta}(x) \;=\; \frac{e^{-|x|\zeta/\delta}}{|x|} + \frac{1}{\pi^2|x|}\int_0^\infty \frac{\exp\left(-|x|\sqrt{2y+(\zeta/\delta)^2}\right)}{y(J_1^2(\delta\sqrt{2y})+Y_1^2(\delta\sqrt{2y}))}dy,$$

where $J_1$ and $Y_1$ are the Bessel functions of the first and second kind. By formulae 9.1.7,9 and 9.2.1,2 in Abramowitz and Stegun (1968), the denominator of the integrand above is asymptotically equivalent to a constant as $y \to 0$ and to $y^{-1/2}$ as $y \to \infty$, hence $f_{\zeta,\delta}(x)$ behaves like $1/x^2$ at the origin: therefore all assumptions of Theorem 2 are satisfied.
The density of the infinitely divisible distribution on which the hyperbolic Lévy motion is based, is in the centered symmetric case given by

$$\mathrm{hyp}_{\zeta,\delta}(x) \;=\; \frac{1}{2\delta K_1(\zeta)}\exp\left(-\zeta\sqrt{1+(\frac{x}{\delta})^2}\right),$$

where $K_1$ denotes the modified Bessel function of the third kind with index 1. Eberlein and Keller (1995) contains more analytical details on this process as well as an explicit option pricing formula.

## 3. Proof of Proposition 1

**1)** Set $S'_t = e^{-rt}S_t$, so that $S'_t = S_0 e^{X'_t}$ with $X'_t = X_t - rt$. Then $X'$ is again a Lévy process under $P$, with the same Lévy measure $F$ and drift $b - r$. Hence, up to replacing $b$ by $b - r$ we can and will assume that $r = 0$.

**2)** Suppose now that $\mathscr{M}_0 \neq \emptyset$ and let $Q \in \mathscr{M}_0$. Girsanov's Theorem (Jacod and Shiryaev (1987), III.3.24) shows that $X$ is a $Q$-semimartingale with characteristics $(\overline{B}, 0, \overline{\nu})$ given by

$$\left.\begin{aligned} \overline{B}_t &= bt + \textstyle\int_0^t ds \int \varphi(x)(Y(s,x)-1)F(dx) \\ \overline{\nu}(dt,dx) &= Y(t,x)dtF(dx) \end{aligned}\right\} \tag{11}$$

where $Y = Y(\omega, s, x)$ is a positive predictable function. The Hellinger process $h(P,Q)$ of order $1/2$ between $P$ and $Q$ has a version given by (cf. Jacod and Shiryaev (1987), IV.3.28):

$$h_t(P,Q) \;=\; \frac{1}{2}\int_0^t ds \int \left(\sqrt{Y(s,x)}-1\right)^2 F(dx) < \infty \quad (P+Q)-a.s. \tag{12}$$

so the integral in the first formula (11) makes sense, because $\int (x^2 \wedge 1) F(dx) < \infty$ and $|(Y-1)\varphi| \leq 2|\varphi||\sqrt{Y}-1| + (\sqrt{Y}-1)^2$. Observe also that (12) implies that $\int_0^t ds \int (x^2 \wedge 1) Y(s,x) F(dx) < \infty \ (P+Q)-a.s.$, as it should be, since $Y \leq 4 + 3(\sqrt{Y}-1)^2 1_{\{Y>4\}}$. Further, if we write by Ito's formula

$$e^{X_t} = 1 + \int_0^t e^{X_{s-}} dX_s + \int_0^t \int e^{X_{s-}} (e^x - 1 - x) \mu(ds, dx)$$

and express $dX_s$ in the first integral in terms of the canonical representation with respect to $Q$

$$X_t = \overline{B}_t + \int_0^t \int \varphi(x)(\mu - \overline{\nu})(ds, dx) + \int_0^t \int (x - \varphi(x)) \mu(ds, dx),$$

then $e^{X_t}$ is the sum of a local martingale under $Q$, plus the following process of locally bounded variation:

$$D_t = \int_0^t e^{X_{s-}} d\overline{B}_s + \int_0^t \int e^{X_{s-}} (e^x - 1 - \varphi(x)) \mu(ds, dx).$$

Then $D_t$ is also a $Q$-local martingale iff its Q-compensator is well-defined and equal to 0 (see Jacod and Shiryaev (1987), Chapter II, fore more details). Observing that the $Q$-compensator of $D_t$ is of the same form, with $\mu$ replaced by $\overline{\nu}$, we see that $e^{X_t}$, hence $S_t = S_0 e^{X_t}$ as well, is a local martingale under $Q$ iff we have $Q$ - a.s. for all $t \geq 0$:

$$\left.\begin{aligned} &\int_0^t ds \int_{\{x>1\}} Y(s,x)(e^x - 1) F(dx) < \infty \\ &bt + \int_0^t ds \int (Y(s,x)(e^x - 1) - \varphi(x)) F(dx) = 0. \end{aligned}\right\} \quad (13)$$

Observe that (12) and the first property of (13) imply that the second integral in (13) exists.

Now, putting together (12) and (13), there clearly exists $(\omega, s)$ such that $y(x) = Y(\omega, s, x)$ satisfies (5): hence $\mathscr{Y}_0 \neq \emptyset$.

**3)** Conversely assume that $y \in \mathscr{Y}_0$. Exactly as above, the measure $F'$ defined by (6) integrates $x^2 \wedge 1$ and $(y-1)\varphi$ is $F$-integrable and the first formula (6) makes sense. Thus there is a measure $Q$ under which $X$ is a Lévy process with drift $b'$ and Lévy measure $F'$. Further $Q$ is locally equivalent to $P$ in virtue of Theorem IV.4.39 of Jacod and Shiryaev (1987) (in which (ii) is in fact implied by (v), by the argument above).

Further we have (13) with $Y(\omega, s, x) = y(x)$, so $S$ is a $Q$-local martingale. If we prove that $S$ is in fact a $Q$-martingale, we will have $Q \in \mathscr{M}'_r$, and the theorem will be proved.

Since $S$ is a nonnegative $Q$-local martingale, by Fatou's Lemma it is also a $Q$-supermartingale and it remains to prove that $E_Q[e^{X_t}] = 1$ for all $t$. By (5) and (6) we have $\int_{\{x>1\}} e^x F'(dx) < \infty$. Thus by classical results on infinitely divisible distributions the variable $e^{X_t}$ is integrable and

$$E_Q[e^{\lambda X_t}] \;=\; \exp\left[t\left(\lambda b' + \int (e^{\lambda x} - 1 - \lambda\varphi(x))F'(dx)\right)\right]$$

is well-defined for all $\lambda \in \mathbb{C}$ with $0 \leq \mathrm{Re}(\lambda) \leq 1$. Taking (6) and the last property in (5) (with $r = 0$) into account yields for $\lambda$ as above:

$$E_Q[e^{\lambda X_t}] \;=\; \exp\left[t \int y(x)(e^{\lambda x} - 1 - \lambda(e^x - 1))F(dx)\right]. \tag{14}$$

Applying this with $\lambda = 1$ gives the result.

## 4. Proof of Theorem 2

**1)** We will first prove that the set $I_r'$ is dense in the interval defined in (9), under the assumptions (i) and (ii') (see Remark 4).

Introduce the functions

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ e^{-x} & \text{if} \quad x \geq 0, \end{cases} \qquad\qquad k(x) = f(x)(e^x - 1) - \varphi(x).$$

$k$ is bounded and behaves like $x^2/2$ near 0, so that $\alpha = \int k(x)F(dx)$ is well-defined. Fix $\beta > 0$. Set $A = \beta - b + r - \alpha$ and consider the double sequence $(\varepsilon_n, \varepsilon_n')$ associated by (ii') with $A$. Let $\upsilon_n$ denote a sequence of positive numbers which will be fixed later, and set $B_n = [-n, -\varepsilon_n) \cup (\varepsilon_n', \infty)$ and

$$y_n(x) = f(x)\left(\upsilon_n 1_{(-\infty,-n)}(x) + \frac{1}{n} 1_{B_n}(x) + 1_{[-\varepsilon_n,\varepsilon_n']}(x)\right).$$

The first condition in (5) is obviously met by $y_n$, and the second will be iff

$$\begin{aligned} & b - r - \upsilon_n\delta(n) - \tfrac{1}{n}\left(\delta(\varepsilon_n) - \delta(n)\right) - \textstyle\int_{[-n,-\varepsilon_n)} \varphi(x)F(dx) \\ & + \textstyle\int_{[-\varepsilon_n,\varepsilon_n']} k(x)F(dx) + \tfrac{1}{n}\delta'(\varepsilon_n') - \textstyle\int_{(\varepsilon_n',\infty)} \varphi(x)F(dx) \quad = \quad 0, \end{aligned}$$

which amounts to saying that

$$\upsilon_n\delta(n) \;=\; b \;-\; r \;+\; \alpha \;+\; \frac{n-1}{n}\left(\delta(\varepsilon_n) - \delta'(\varepsilon_n')\right) \;+\; \frac{\delta(n)}{n}. \tag{15}$$

The right-hand side of (15) converges to $b - r + \alpha + A = \beta > 0$, so it is positive for $n$ large enough. By (i) we also have $\delta(n) > 0$, hence (15) defines the number $\upsilon_n > 0$. Then $y_n$ belongs to $\mathscr{Y}_r$, so $\mathscr{M}_r' \neq \emptyset$, and we denote by $Q_n$ the measure in $\mathscr{M}_r'$ associated with it by Proposition 1.

Observing that $\delta(n)/F((-\infty,-n)) \to 1$, we deduce from $\upsilon_n\delta(n) \to \beta$ that $\upsilon_n F((-\infty,-n)) \to \beta$ as well.

Now we set $X_t'^n = \sum_{s \leq t} \Delta X_s 1_{\{\Delta X_s < -n\}}$ and $X''^n = X - X'^n$, which are two independent Lévy processes under $Q_n$. Set $U_n' = e^{X_T'^n}$ and $U_n'' = e^{X_T''^n}$, so that $S_T = S_0 U_n' U_n''$. On the one hand, under $Q_n$ the process $X'^n$ is a compound Poisson process with Lévy measure $F_n'(dx) = \upsilon_n F(dx) 1_{(-\infty,-n)}(x)$. Therefore we have either $X'^n_T < -n$ and $U'_n < \exp(-n)$, or $X'^n_T = 0$ and $U'_n = 1$. Further $Q_n[U'_n = 1] = \exp(-Tv_n F((-\infty, -n)))$. Summarizing these results, we get:

$$0 < U'_n \leq 1, \quad Q_n(U'_n = 1) \to e^{-\beta T}, \quad Q_n(U'_n < e^{-n}) \to 1 - e^{-\beta T}. \tag{16}$$

On the other hand, for any $\lambda \in \mathbb{C}$ with $0 \leq \mathrm{Re}(\lambda) \leq 1$, we have by (14) (where $X_t$ has to be replaced by $X_t - rt$ again) and with $g_\lambda(x) = e^{\lambda x} - 1 - \lambda(e^x - 1)$:

$$E_{Q_n}[e^{\lambda(X_T - rT)}] =$$
$$\exp\left[T \int g_\lambda(x) f(x) \left(v_n 1_{(-\infty,-n)}(x) + \tfrac{1}{n} 1_{[-n,-\varepsilon_n)\cup(\varepsilon'_n,\infty)}(x) + 1_{[-\varepsilon_n,\varepsilon'_n]}(x)\right) F(dx)\right].$$

Since $E_{Q_n}[e^{\lambda X'^n_T}] = \exp\left[T \int (e^{\lambda x} - 1) v_n 1_{(-\infty,-n)}(x) F(dx)\right]$ the independence between $X'^n$ and $X''^n$ yields

$$E_{Q_n}[e^{\lambda X''^n_T}] = e^{T(\lambda v_n \delta(n) + a_n(\lambda) + r\lambda)}, \tag{17}$$

where $a_n(\lambda) = \int g_\lambda(x) y_n(x) 1_{[-n,\infty)}(x) F(dx)$.

The function $g_\lambda$ is continuous, equivalent to $x^2(\lambda^2 - \lambda)/2$ near 0, bounded near $-\infty$, and smaller than $Ce^x$ near $+\infty$. On the other hand $y_n(x) \to 0$ for all $x \neq 0$ and $y_n(x) 1_{[-n,\infty)}(x) \leq 1_{(-\infty,0]}(x) + e^{-x} 1_{(0,\infty)}(x)$. Therefore $a_n(\lambda) \to 0$ as $n \to \infty$. Thus (17) and $v_n \delta(n) \to \beta$ imply that $E_{Q_n}[e^{\lambda X''^n_T}] \to e^{\lambda T(\beta + r)}$. This with $\lambda = 1 + iu$ where $u \in \mathbb{R}$ yields

$$E_{Q_n}[U''_n e^{iuX''^n_T}] \to e^{T(\beta+r)(1+iu)}.$$

Note that the left side is $\int e^{iux} d\nu_n(x)$ where, with $\mu_n$ denoting the law of $X''^n_T$ under $Q_n$, $d\nu_n(x) = e^x d\mu_n(x)$ and $\sup_n \nu_n(\mathbb{R}) < \infty$ by (17) and the arguments following it. Therefore we get

$$E_{Q_n}\left[U''_n f\left(X''^n_T\right)\right] \to e^{T(\beta+r)} f(T(\beta + r))$$

for every bounded continuous function $f$ and even uniformly in $f$ within the class of functions satisfying $0 \leq f \leq C_0$ and $|f(x) - f(x')| \leq C|x - x'|$. In particular the family of functions $f_z(x) = e^{-x} g(zS_0 e^x)$ for $z \in [0, 1]$ is in this class with $C_0 = S_0$ and $C = 1 + S_0$ by (7), and we deduce that

$$E_{Q_n}\left[g(S_0 U'_n U''_n) | U'_n = z\right] = E_{Q_n}\left[U''_n f_z(X''^n_T)\right] \to g(zS_0 e^{T(\beta+r)})$$

uniformly in $z \in (0, 1]$. Now we have $\gamma(Q_n) = e^{-rT} E_{Q_n}\left[g(S_0 U'_n U''_n)\right]$, so the above fact and (16) show that

$$\gamma(Q_n) \to G(\beta) := e^{-T(\beta+r)} g(S_0 e^{T(\beta+r)}).$$

In other words, the closure of $I'_r$ contains $G(\beta)$ for every $\beta > 0$. But $G$ is a continuous function on $\mathbb{R}_+$, with limit $e^{-rT} g(e^{rT} S_0)$ at 0 and limit $S_0$ at $\infty$ by (7): thus $I'_r$ is dense in the interval $[e^{-rT} g(e^{rT} S_0), S_0]$.

**2)** Next we observe that the map $Q \rightsquigarrow \gamma(Q)$ is linear, while the set $\mathscr{M}_r$ is a convex set of probability measures, so the set $I_r$ is necessarily an interval. In view of (9) and of the previous step, it remains to show that the left endpoint $e^{-rT} g(e^{rT} S_0)$ does not belong to $I_r$.

Suppose that $e^{-rT} g(e^{rT} S_0) = \gamma(Q)$ for some $Q \in \mathscr{M}_r$. This means that if $g'(x) = g(e^{rT} x)$ and $U = e^{-rT} S_T$, then $E_Q[g'(U)] = g'(E_Q[U])$. Hence the convex function $g'$ should be linear on the interval $(a, a')$, where $a$ and $a'$ are the left and right endpoints of the support of the random variable $U$. Now, the Lévy measure $F$ charges both $\mathbb{R}_+$ and $\mathbb{R}_-$, hence the support of $X_T$ under $P$ extends from $-\infty$ to $+\infty$, and the support of $U = S_0\, e^{-rT+X_T}$ extends from 0 to $+\infty$ under $P$. Since $Q$ is equivalent to $P$, the same holds for $Q$, i.e. $a = 0$ and $a' = \infty$. That is, $g'$ and $g$ must be linear on $\mathbb{R}_+$, which contradicts (7).

## 5. Conclusions

The purely discontinuous processes studied in this paper include processes which on the level of the microstructure allow more realistic modeling of stock prices than the usual diffusions. Our result shows that for these incomplete models the no arbitrage approach alone does not suffice to value contingent claims. The class of equivalent martingale measures, which provides the candidates for risk neutral valuation, is by far too large. Additional optimality criteria or preference assumptions have to be imposed.

Various attempts have been made to choose a particular probability. Föllmer and Sondermann (1986) emphasize the hedging aspect and look for strategies which minimize the remaining risk in a sequential sense. Given the initial (historical) probability measure it is natural to look for "closest" elements in the set of martingale measures. Föllmer and Schweizer (1990) study a minimal martingale measure in the sense that it minimizes relative entropy. From this an optimal hedging strategy is derived. Variance-optimality is another approach. This means to choose the martingale measure whose density is minimized in the $\mathscr{L}^2$-sense. We refer to Schweizer (1996). Also the Esscher transform used by Eberlein and Keller (1995) to derive explicit option values seems to be a natural choice. Our main result underlines the importance of research in this direction.

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