

## ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM IN BANACH SPACES

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Let  $E$  denote a separable Banach space and let  $X_i, i \in \mathbb{N}$ , be a sequence of i.i.d.  $E$ -valued random vectors having finite third moment such that the central limit theorem holds. We prove that the convergence rate in the central limit theorem is  $O(n^{-1/2})$  for regions  $\{x \in E: F(x) < r\}$  which are defined by means of a smooth real valued function  $F$  on  $E$ , provided that the limiting distribution of the gradient of  $F$  fulfills a variance condition.

Using this result we prove that the rate of convergence in the functional limit theorem for empirical processes is of order  $O(n^{-1/2})$ .

**1. Introduction and results.** Let  $E$  denote a separable Banach space normed by  $\|\cdot\|$ . Assume that  $X_i, i \in \mathbb{N}$ , is a sequence of i.i.d.  $E$ -valued random vectors. Let  $Q$  denote the common probability distribution of the  $X_i$ 's defined on the Borel  $\sigma$ -field, say  $\mathcal{B}$ , of  $E$ . Let  $Q_n$  denote the probability distribution of  $R_n = n^{-1/2}(X_1 + \cdots + X_n)$ . Suppose that for some  $\gamma \geq 0$

$$(1.1) \quad E\|X_1\|^{3+\gamma} < \infty \quad \text{and} \quad EX_1 = 0.$$

Furthermore, assume that the central limit theorem holds for  $Q_n$ , i.e.,

$$(1.2) \quad Q_n, n \in \mathbb{N}, \text{ converges weakly to some Gaussian Borel measure } N \text{ defined on } \mathcal{B}.$$

The results of Hoffmann-Jørgensen and Pisier (1976) show that the central limit theorem holds for every i.i.d. sequence  $X_i, i \in \mathbb{N}$ , satisfying  $E\|X_1\|^2 < \infty$  and  $EX_1 = 0$  if  $E$  is a Banach space of type 2 (which includes for example the  $L^p$  function space with  $2 \leq p < \infty$ ).

Let  $F: E \rightarrow \mathbb{R}$  denote a measurable functional. We shall study the speed of convergence in the central limit theorem for regions  $\{x \in E: F(x) < r\}$ ,  $r \in \mathbb{R}$ , where  $F$  is differentiable. From another point of view this means that we study the speed of convergence in the functional limit theorem for  $F(R_n)$  with respect to the Kolmogoroff distance of distribution functions.

For this purpose it will be convenient to use Frechét differentiability of  $F$  with respect to  $\|\cdot\|$ , i.e.,

$$|F(x+h) - F(x) - DF(x)[h]| = o(\|h\|),$$

where  $h \rightarrow DF(x)[h]$  denotes a continuous linear functional on  $E$ . The  $k$ th derivative of  $F$  at  $x$ , say  $D^k F(x)$ , is a  $k$ -linear symmetric functional  $(h_1, \dots, h_k) \rightarrow D^k F(x)[h_1, \dots, h_k]$  with finite norm

$$\|D^k F(x)\| = \sup\{|D^k F(x)[h_1, \dots, h_k]|: \|h_1\|, \dots, \|h_k\| \leq 1\}.$$

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Assume the derivatives of  $F$  satisfy the

DIFFERENTIABILITY CONDITION  $D_\beta$ .

$$(1.3) \quad \begin{aligned} \|D^k F(x)\| &\leq c_F(1 + \|x\|^p), \quad 0 \leq k \leq 3, \\ \|D^3 F(x) - D^3 F(y)\| &\leq c_F(1 + \|x\|^p + \|y\|^p)\|x - y\|^\beta \end{aligned}$$

for every  $x, y \in E$  and some fixed constants  $c_F > 0$ ,  $p \geq 0$ , and  $1 \geq \beta > 0$ . In this case  $F$  is said to be of class  $C^{3+\beta}$ .

Let  $v(x) = E(DF(x)[X_1]^2)$ . The crucial condition on  $F$  and  $X_1$  is the following ‘‘variance’’ condition which guarantees the existence of a smooth density of the distribution of  $F(R)$ , where  $R$  has Gaussian distribution  $N$  on  $E$ .

VARIANCE CONDITION.

Let  $k_\beta = 6 + 4/\beta$ . Let  $\varphi_\eta = \prod_1^r I(\eta_j \|X_j\| \leq 1)$  and  $X_\eta = \eta_1 X_1 + \dots + \eta_r X_r$ ,  $r = [k_\beta/2] + 2$ . (Here  $[x]$  denotes the largest integer smaller or equal to  $x$ .) Assume there exists  $\eta > 0$  such that

$$(1.4) \quad E\varphi_\eta I(v(R + X_\eta) \leq \varepsilon) = o(\varepsilon^{k_\beta}), \quad \varepsilon \rightarrow 0 \text{ and } |\eta_j| \leq \eta, 1 \leq j \leq r.$$

REMARK. When

$$E\|X_1\|^{3+\gamma} < \infty, \quad \gamma > 0,$$

we may choose  $\eta = 0$  in (1.4) and replace (1.4) by

$$(1.4') \quad P(v(R) \leq \varepsilon) = O(\varepsilon^{k_\beta}), \quad \text{where } k_\beta = 4/\gamma + 4/\beta + 6.$$

These assumptions combine conditions on the covariance structure of  $X_1$  (or  $R$ ) and conditions on the ‘‘regular’’ behaviour of the region where the gradient of  $F$  vanishes, which will be discussed later.

The main result of this paper is

1.5 THEOREM. Assume that conditions (1.1)–(1.4) hold for some  $c_F$ ,  $p$ , and  $\beta$ . Then

$$\sup_{r \in \mathbb{R}} |P(F(R_n) \leq r) - P(F(R) \leq r)| = O(n^{-1/2}).$$

The validity of the variance condition (1.5) is essential for the convergence speed. A class of Hilbert space examples constructed by Rhee and Talagrand (1984) shows that for functions  $F$  which are ‘‘flat’’ at some points [ $\|DF(x)\| = O(\|x - x_0\|^q)$  for every  $q > 0$ ] and with a sufficiently ‘‘singular’’ covariance structure of  $X_1$  the rate of convergence can be made slower than any prescribed order.

The speed of convergence for functionals  $F(x) = \|x\|$  which are differentiable has been investigated first by Kandelaki (1965) [obtaining  $O(1/\log n)$ ], Kuelbs

and Kurtz (1974) who obtained  $O(n^{-1/8})$  in Hilbert space, and Paulauskas (1976) who improved the rate for three times differentiable norms in Banach space to  $O(n^{-1/6})$ . See also the survey articles of Paulauskas (1979) and Rachkauskas (1980) and the monograph of Sazonov (1981).

For functionals of “polynomial type” (i.e., the  $q$ th derivative of  $F$  vanishes identically) the distribution of  $F(R_n)$  can be approximated in regular cases by smooth expansions up to the order  $O(n^{-q/2+\epsilon})$  (Götze, 1984, 1985). Similar results for general differentiable functionals  $F$  such that the  $q$ th derivative satisfies a variance condition like (1.4) will be treated in a forthcoming paper.

In the following we discuss some applications of Theorem 1.5.

**1.6 COROLLARY.** *Suppose  $E$  is a separable Hilbert space and  $C$  denotes a measurable set which is star-shaped with respect to  $0 \in C$ . Assume that*

- (i)  $E\|X_1\|^{3+\gamma} < \infty$ ,  $\gamma > 0$  and  $EX_1 = 0$ .
- (ii)  $F(x) = \inf\{r > 0: xr^{-1} \in C\}^4$  satisfies differentiability condition (1.3).
- (iii)  $P(R \in \epsilon C) = O(\epsilon^{gk_\beta})$ ,  $g > 10$ ,  $k_\beta$  as in (1.4’).
- (iv) The (ordered) eigenvalues of the covariance operator of  $N$ , say  $\lambda_j$ , are decreasing such that

$$\sum_{i=m}^{\infty} \lambda_i = o(m^{-\lambda}/\log m) \quad \text{as } m \rightarrow \infty,$$

where  $\lambda = 10/(g - 10)$ . Then

$$(1.7) \quad \sup_{r>0} |P(R_n \in rC) - P(R \in rC)| = O(n^{-1/2}).$$

**1.8 REMARKS.** Condition (iii) holds if  $C$  is bounded in norm and more than  $gk_\beta$  eigenvalues are different from zero.

Our conjecture is that condition (ii) can be replaced by a Lipschitz condition on  $F$  when we restrict the supremum in (1.7) to  $r > \delta > 0$ .

For norms in  $l_2$  Nagaev and Chebotarev (1978) obtained  $O(n^{-1/2})$  for the case of r.v.’s with independent components. The author proved (1979) that for a large class of  $U$ -statistics including symmetric quadratic functionals in Hilbert space the actual rate of convergence is  $O(n^{-1+\epsilon})$  in nondegenerate situations. Furthermore, asymptotic expansions are possible. The moment conditions for the norm in Hilbert space have been weakened in papers by Yurinskii (1982) using a similar approach together with strong exponential estimates for sums (Yurinskii, 1976) to the assumption of a third moment for the rate  $O(n^{-1/2})$ . For the symmetric case Zaleskii (1982) obtained  $O(n^{-(1+\delta)/2})$  assuming a moment of order  $3 + \delta$ ,  $\delta < 1$ . Further papers by Sazonov and Zaleskii (1985), Nagaev (1985), and Opsipov and Rotar (1984) focus on weaker variance conditions and the nonidentical case for the norm in Hilbert space.

For a typical application of the functional version of Theorem 1.5 consider the empirical process

$$x_n(t) = n^{-1/2} \sum_{i=1}^n (I(X_i \leq t) - t)$$

based on an i.i.d. sample  $X_1, \dots, X_n$  taken from the uniform distribution on  $[0, 1]$ . Consider functionals of the following additive type,

$$F(x(\cdot)) = \int_0^1 V(t, x(t)) dt,$$

such that for  $0 \leq i \leq 4$

$$(1.9) \quad \sup_{0 \leq t \leq 1} |D_x^i V(t, x)| \leq c(1 + |x|^p).$$

Furthermore, define  $f(x(\cdot), s) = \int_0^1 (I(s \leq t) - t) D_x V(t, x(t)) dt$  and suppose that for a Brownian bridge  $w(t)$ ,  $0 \leq t \leq 1$ ,

$$(1.10) \quad P\left(\int_0^1 f(w(\cdot), s)^2 ds \leq \varepsilon\right) = O(\varepsilon^{k_\beta}), \quad k_\beta \text{ as in (1.4)}.$$

We have

1.11 COROLLARY. *Suppose that conditions (1.9) and (1.10) hold. Then*

$$\sup_z |P(F(x_n(\cdot)) < r) - P(F(w(\cdot)) < r)| = O(n^{-1/2}).$$

1.12 REMARK. Condition (1.10) holds, in particular, if

$$(1.13) \quad |V(t, x) - V(s, x)| \leq c|t - s|^{1/2}(1 + |x|^p) \quad \text{for some } p > 0,$$

$$(1.14) \quad P\left(\int_0^1 D_x V(t, w(t))^2 dt \leq \varepsilon\right) = O(\varepsilon^{5.1k_\beta}), \quad k_\beta \text{ as in (1.4)}.$$

1.15 EXAMPLES. It can be shown by means of Remark 1.12 that the order of convergence is  $O(n^{-1/2})$  in Corollary 1.11 for the following functions  $V(t, x)$ :

- (i)  $V(t, x) = c(t)x^p$ ,  $p \in \mathbb{N}$ ,  $c(t)$  Lipschitz continuous of exponent 1,
- (ii)  $D_x V(0, 0) \neq 0$  or  $D_x V(1, 0) \neq 0$ ,  $V(t, x)$  satisfies (1.13),
- (iii)  $D_x V(t, x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $t \in [0, 1]$  and  $P(\int_\delta^{1-\delta} D_x V(t, w(t))^2 dt \leq \varepsilon) = O(\varepsilon^{k_\beta})$  for small  $\delta > 0$ .

In example (iii) condition (1.10) follows immediately (interchanging integrations) since the covariance kernel of the Brownian bridge is bounded from below by  $c(\delta) > 0$  on  $(s, t) \in [\delta, 1 - \delta] \times [\delta, 1 - \delta]$ .

The previous result shows that convergence rates  $O(n^{-1/2} \log n)$  based on the strong approximation techniques of Komlós, Major, and Tusnády (1975, 1976) can be improved for some classes of functionals.

We believe that for the particular case of the empirical process the conditions (1.9) and (1.10) are still too restrictive.

The method of proof is a kind of “partial integration” scheme for sums of i.i.d. random vectors in  $E$  (see Lemma 3.7) which reduces to Stein’s method of partial integration (or differential equations) in the case of linear functionals  $F$ . In order to guarantee the existence of the terms obtained by partial integration an obvious condition is  $Ev(R)^{-q} < \infty$ , hence condition (1.4) for  $\eta = 0$  [compare (3.23)].

**2. Proof of the results.**

PROOF OF COROLLARY 1.6. Since the region is star-shaped the functional  $M(x) = F(x)^{1/4}$  is homogeneous, i.e.,  $M(\lambda x) = \lambda M(x)$  for  $\lambda > 0$ . Hence

$$M(x) = \langle DM(x), x \rangle, \quad x \neq 0,$$

where  $\langle x, y \rangle$  denotes the scalar product in  $E$ .

Denote by  $C$  the covariance operator of  $N$  with nonnegative eigenvalues  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$  and  $\lambda_0 = 0$ . Let  $\{e_k, k = 1, 2, \dots\}$  denote an orthonormal system of eigenvectors of  $C$  corresponding to  $\lambda_k, k \in \mathbb{N}$ , which spans the supporting subspace of  $N$ . Denote by  $P_m$  resp.  $P_m^c$  the projection on the space  $\langle e_1, \dots, e_m \rangle$  resp.  $\langle e_1, \dots, e_m \rangle^c$ . Let  $C_m = P_m C P_m$ . When  $\lambda_m > 0$  then

$$(2.1) \quad \begin{aligned} M(x) &= \langle C_m^{1/2} DM(x), C_m^{-1/2} P_m x \rangle + \langle DM(x), P_m^c x \rangle \\ &\leq \|C_m^{1/2} DM(x)\| \|C_m^{-1/2} P_m x\| + \|DM(x)\| \|P_m^c x\|. \end{aligned}$$

Since  $DM(\lambda x) = DM(x), x \neq 0, \lambda > 0$ , we have by condition (1.3)

$$\|DF(x)\| = 4 \|DM(x)\| M(x)^3 \leq c_F (1 + \|x\|^p).$$

By definition

$$(2.2) \quad \begin{aligned} v(x) &= \langle CDF(x), DF(x) \rangle \\ &= 16M(x)^6 \langle CDM(x), DM(x) \rangle \\ &\geq 16M(x)^6 \|C_m^{1/2} DM(x)\|^2. \end{aligned}$$

We have

$$\{v(x) \leq \varepsilon\} = A_\varepsilon \cup B_\varepsilon \cup D_\varepsilon \cup E_\varepsilon,$$

where

$$\begin{aligned} B_\varepsilon &= \{M(x) \leq \varepsilon^{1/g}\}, \\ D_\varepsilon &= \{\|C_m^{-1/2} P_m x\|^2 \geq m \log \varepsilon^{-k}\}, \\ E_\varepsilon &= \{\|x\| \geq \varepsilon^{-1/(gp)}\}, \\ A_\varepsilon &= \{v(x) \leq \varepsilon\} \setminus (A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon). \end{aligned}$$

In the following arguments write  $k$  for  $k_\beta$ .

By assumption  $P(R \in B_\varepsilon) = O(\varepsilon^k)$ , and using Lemma 3.12 we have  $P(R \in E_\varepsilon) = O(\varepsilon^k)$ . Furthermore,  $P(R \in D_\varepsilon) = P(\eta_1^2 + \dots + \eta_m^2 \geq m \log \varepsilon^{-k})$ , where  $\eta_1, \dots, \eta_m$  denote i.i.d.  $N(0, 1)$ -variates. Chebyshev's inequality yields

$$\begin{aligned} P(R \in D_\varepsilon) &\leq \exp[-m(\log \varepsilon^{-k})/m] (1 - 2/m)^{-m/2} \\ &= O(\varepsilon^k). \end{aligned}$$

Hence it remains to prove  $P(R \in A_\varepsilon) = O(\varepsilon^k)$ . If  $x \in A_\varepsilon$  then by (2.1) and (2.2)

$$\begin{aligned} \varepsilon^{1/g} &\leq M(x), \\ M(x) &\leq c\varepsilon^{1/2-3/g}(m \log \varepsilon^{-k})^{1/2} + c\|P_m^c x\| \varepsilon^{-3/g}(1 + \varepsilon^{-p/(gp)}). \end{aligned}$$

Choosing  $m = \lceil \varepsilon^{-1+10/g} \rceil$ , this yields a lower bound

$$\|P_m^c x\| \geq c\varepsilon^{1/g} \varepsilon^{3/g} \varepsilon^{p/(2g)} \quad \text{for } \varepsilon \text{ sufficiently small.}$$

By Chebyshev's inequality we obtain (since  $g > 10$ ) from this lower bound

$$\begin{aligned} P(R \in A_\varepsilon) &\leq P(\|P_m^c R\|^2 \geq c\varepsilon^{10/g}) \\ &\leq \exp(-t\varepsilon^{10/g}) \prod_{k=m+1}^\infty (1 - 2t\lambda_k)^{-1/2}, \end{aligned}$$

provided  $t > 0$ . Choose  $t = \varepsilon^{-10/g} \log(\varepsilon^{-k})$ . Hence by condition 1.6(iv) and the choice of  $m$

$$P(R \in A_\varepsilon) = o(\varepsilon^k). \quad \square$$

**PROOF OF REMARK 1.8.** Let  $K$  denote a positive constant such that  $x \in C$  implies  $\|x\| \leq K$ . Then

$$\begin{aligned} P(R \in \varepsilon C) &\leq P(\|R\| \leq K\varepsilon) \\ &\leq eE \exp(-\|R\|^2 K^{-1} \varepsilon^{-1}) \\ &\leq e \prod_{i=1}^\infty (1 + 2\lambda_i / (K\varepsilon))^{-1/2} \\ &= O(\varepsilon^{gk_\mu}). \quad \square \end{aligned}$$

**PROOF OF COROLLARY 1.11.** Condition (i) guarantees that the functional  $F(x(\cdot))$  is well defined for every  $x(t) \in L^q([0, 1])$ , where  $q = p + 4$ .

By the dominated convergence theorem and Hölder's inequality  $F$  admits four Fréchet derivatives with respect to the norm of  $L^q([0, 1])$ . Hence the differentiability condition (1.3) holds.

The invariance principle in  $C[0, 1]$  implies  $\lim P(F(x_n(\cdot)) \leq r) = P(F(w(\cdot)) \leq r)$  for all continuity points  $r$ . Hence the CLT for  $L^q$  [compare the remarks following condition (1.1)] together with the continuity of the limit distribution function of  $F(R)$ ,  $R \in L^q$  implies  $P(F(w(\cdot)) \leq r) = P(F(R) \leq r)$  for every  $r$ . The same remark applies to condition (1.10).  $\square$

**PROOF OF REMARK 1.12.** Let  $\delta > 0$ , arbitrarily small, and  $\alpha > 0$  be determined later. Let  $A_\varepsilon$  denote the set of sample paths  $w(t)$  of the Brownian bridge satisfying for  $\varepsilon$  sufficiently small

- (i)  $\sup_{|t-s| \leq \varepsilon^\alpha} |w(t) - w(s)| \leq \varepsilon^{-\delta + \alpha/2},$
- (ii)  $\sup_{0 \leq t \leq 1} |w(t)| \leq \log(\varepsilon^{-1}),$
- (iii)  $v(w(\cdot)) = \int_0^1 f(w(\cdot), s)^2 ds \leq \varepsilon.$

By well-known properties of the Brownian bridge we have

$$(2.3) \quad P(v(w(\cdot)) \leq \varepsilon) \leq P(A_\varepsilon) + O(\varepsilon^{k_\mu}).$$

Define  $g(s) = f(w(\cdot), s)$  for  $0 \leq s < 1$  and extend it by  $g(k + s) = g(s)$ ,  $k \in \mathbb{Z}$  outside  $[0, 1]$ . Let  $\chi$  denote a r.v. independent of  $w(t)$ ,  $t \in [0, 1]$  with symmetric distribution around zero such that  $\chi^2 \leq \varepsilon^{2\alpha}$  a.s. and  $E\chi^2 \geq c\varepsilon^{2\alpha}$ . Then

$$(2.4) \quad \begin{aligned} \varepsilon \geq E \left( \int_0^1 g(s)^2 ds - \int_0^1 g(s + \chi)^2 ds | w(\cdot) \right) \\ + \int_0^1 \text{Var}(g(s + \chi) | w(\cdot)) ds. \end{aligned}$$

Conditioning on  $\chi$  in the first integral on the r.h.s. of (2.4) shows that this integral vanishes since  $g(t)$  is defined to be periodic. If  $A_\varepsilon$  occurs and  $[s, s + \chi]$  does not contain 0 or 1 we have

$$g(s + \chi) = g(s) + Dg(s)\chi + O(\log^p(\varepsilon^{-1})(\varepsilon^{-\delta + \alpha/2} + \varepsilon^{\alpha/2})|\chi|).$$

Hence

$$\begin{aligned} \varepsilon &\geq \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} \text{Var}(g(s + \chi) | w(\cdot)) ds \\ &\geq E\chi^2 \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} Dg(s)^2 ds + O(\varepsilon^{-\delta + 5\alpha/2}) \log^p(\varepsilon^{-1}). \end{aligned}$$

Choosing  $\alpha = 2/5 + \delta/2$  we conclude that  $A_\varepsilon$  implies

$$\int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} D_x V(s, w(s))^2 ds \leq \varepsilon^{-\delta + 1/5}.$$

Restoring integration to  $[0, 1]$  this immediately proves Remark 1.12.  $\square$

**PROOF OF THEOREM 1.5.** The first step is to replace  $X_j$  by its truncation  $Z_j$  at  $n^\alpha$  where  $\alpha = 3/(6 + 2\gamma)$ . Let

$$Z_j = X_j \quad \text{when } \|X_j\| \leq n^\alpha \text{ and } Z_j = 0 \text{ otherwise.}$$

Define

$$S_n = n^{-1/2}(Z_1 + \dots + Z_n).$$

Furthermore, replace the indicator function  $x \rightarrow I(x \leq z)$  by

$$g_{nz}(x) = P(x \leq z + Un^{-1/2}),$$

where  $U$  denotes a random variable which is symmetric around 0,  $|U| \leq 1$  a.e. and which admits a Lebesgue density that is infinitely often differentiable.

We have

$$(2.5) \quad \begin{aligned} \sup_z |P(F(R_n) \leq z) - P(F(S_n) \leq z)| \\ \leq P(F(R_n) \neq F(S_n)) \leq \sum_{k=1}^n P(\|X_k\| > n^\alpha) \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} P(F(S_n) \leq z) - P(F(R) \leq z) \\ \leq P(F(S_n) \leq \bar{z} + Un^{-1/2}) - P(F(R) \leq \bar{z} + Un^{-1/2}) + I_0, \end{aligned}$$

where

$$I_0 = 2 \sup_z P(F(R) \in [z - n^{-1/2}, z + n^{-1/2}]), \quad \bar{z} = z + n^{-1/2}.$$

By Lemma 3.13 we have

$$(2.7) \quad I_0 = O(n^{-1/2}).$$

From (2.5) together with Chebyshev's inequality and (2.7) as well as its analogous lower bound it follows that

$$(2.8) \quad \begin{aligned} & \sup_z |P(F(R_n) \leq z) - P(F(R) \leq z)| \\ & \leq \sup_z |Eg_{nz}(F(S_n)) - Eg_{nz}(F(R))| + I_0 + O(n^{-1/2})E\|X_1\|^{3+\gamma} \\ & = I_1 + O(n^{-1/2}), \quad \text{say.} \end{aligned}$$

We are going to expand the difference  $I_1$  by means of derivatives of  $g_{nz}$ . Since we want to replace the  $r$ th derivative [which near  $z$  is unbounded of order  $O(n^{r/2})$ ] by the bounded function  $g_{nz}$  using "partial integration" (Lemma 3.7) we have to insert factors of order  $1 + O_p(n^{-1/2})$  in  $I_1$ . These factors provide us with functions which make the application of Lemma 3.7 possible.

Define

$$(2.9) \quad \Delta_n = n^{-1/2}(\theta_1 Z_{n+1} + \dots + \theta_B Z_{n+B}),$$

where  $B$  is a number independent of  $n$  and  $\theta_1, \dots, \theta_B$  are i.i.d., uniformly distributed in  $[0, 1]$  and independent of  $X_j, j \in \mathbb{N}$ . In the following, random variables  $t, \tau, \mu$  are understood to be independent and uniform in  $[0, 1]$  and independent of  $X_j, \theta_j$ . Define the factors mentioned above by

$$(2.10) \quad e(x) = \prod_1^B (e(Z_{n+j}, x) + n^{-\delta} \sigma(x)^{-2}), \quad x \in E,$$

$$(2.11) \quad e(z, x) = DF(x)[z]^2 \sigma(x)^{-2}, \quad \sigma(x)^2 = n^{-\delta} + E(DF(x)[Z_1])^2$$

for some  $0 < \delta < \min(\beta, 1/4)/4$  and moreover  $\delta < \gamma/(6 + 2\gamma)$  when  $\gamma > 0$ .

Returning to (2.8) again we shall write  $g$  instead of  $g_{nz}$  and remark that the following inequalities hold uniformly for all  $z \in \mathbb{R}$ . Then

$$(2.12) \quad \begin{aligned} I_1 & \leq |Eg(F(S_n + \Delta_n))e(S_n) - Eg(F(R))| \\ & \quad + |Eg(F(S_n + \Delta_n))e(S_n) - Eg(F(S_n))| \\ & = I_2 + I_3, \quad \text{say.} \end{aligned}$$

Since the estimation of the term  $I_3$  is particularly involved we describe it in detail.

ESTIMATION OF  $I_3$ . Write  $F_1 = F(S_n + \Delta_n)$  and  $\epsilon_n = F(S_n) - F_1$ . Then

$$(2.13) \quad \begin{aligned} I_3 & = Ee(S_n)|g(F_1) - g(F_1 + \epsilon_n)| \\ & \leq Ee(S_n)I(|F_1 - z| \leq |\epsilon_n|) \end{aligned}$$

by definition of  $g$ . (Note that  $g(z + \delta) = 1$  or  $0$  for  $|\delta| \geq n^{-1/2}$ .)



Using  $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ ,  $a, b \geq 0$ , we have by condition (1.3)

$$|\epsilon_n| \leq c(1 + \|S_n^{(1)}\|^p + \|S_n^{(2)}\|^p)\|\Delta_n\|,$$

where  $S^{(1)} = n^{-1/2}\sum_1^M Z_j$ ,  $S^{(2)} = S_n - S_n^{(1)}$ , and  $M = [n/2]$ . Let  $b = B - 2$ . We have

$$\|\Delta_n\| \leq n^{-1/2}\left\{\sum' \|Z_{n+j}\| + \sum'' \|Z_{n+j}\|\right\},$$

where the sum  $\Sigma'$  extends over  $j = 1, \dots, [B/2]$  and  $\Sigma''$  over  $j = [B/2] + 1, \dots, B$  and

$$(2.14) \quad I(|F_1 - z| \leq (s_1 + s_2)(z_1 + z_2)) \leq \sum I(|z_1 - z| \leq 4s_i z_j),$$

where  $s_i, z_j \geq 0$  and the summation is over  $i = 1, 2$  and  $j = 1, 2$ . Hence we obtain

$$(2.15) \quad I_3 \leq 4EE\left(e(S_n)I(|F_1 - z| \leq n^{-1/2}A_n)\right|\mathcal{C})$$

with

$$A_n = c(1 + \|S_n^{(2)}\|^p)\sum'' \|Z_{n+j}\|$$

and

$$\mathcal{C} = \sigma(Z_j, n/2 \leq j \leq n, n + B/2 \leq j \leq n + B).$$

In order to apply the partial integration, Lemma 3.7, note that

$$(2.16) \quad \begin{aligned} e(x) &= \sum_k^* n^{-\delta k} \sigma(x)^{-2B} \prod_{(j)}' DF(x)[z_{n+j}]^2 \\ &= e_1(x) + r_n, \quad \text{say,} \end{aligned}$$

where  $\Sigma^*$  denotes the summation over all  $(B - k)$ -tuples of integers between 1 and  $B$  and  $\Pi'$  denotes the product over such  $(B - k)$ -tuples of indices  $(j)$  and  $e_1(x)$  resp.  $r_n$  denotes the sum from  $k = 1$  to  $b - 1$  resp.  $k = b$  to  $B$ . We have by (2.15)

$$(2.17) \quad \begin{aligned} I_3 &\leq cEE\left(e_1(S_n)I(|F_1 - z| \leq n^{-1/2}A_n)\right|\mathcal{C}) + Er_n \\ &= I_4 + Er_n, \quad \text{say.} \end{aligned}$$

Let  $\Sigma^{**}$  denote summation over all  $k$  between  $b$  and  $B$  and  $\Pi'$  as in (2.16). By (2.20) and (2.11) we have

$$(2.18) \quad \begin{aligned} Er_n &\leq c \sum_k^{**} n^{-\delta k} E\sigma(S_n)^{-2k} \prod_{(j)}' e(Z_{n+j}, S_n) \\ &\leq c \max\{n^{-\delta k} E\sigma(S_n)^{-2k} : k = b, b+1, \dots, B\} \end{aligned}$$

$$(2.19) \quad = O(n^{-1/2})$$

by Lemma 3.15 with  $B = q/2 = [I(\gamma < 0)4/\gamma + 4/\beta] + 6$  such that  $\delta b > \frac{1}{2}$ .

Let

$$G_n(x) = \int_{-\infty}^{x-z} I(|a| \leq n^{-1/2}A_n) da.$$

While applying the “partial integration” Lemma 3.7 replace  $F$  by  $F_S(x) = F(x + S_n^{(2)})$  and let  $g(y) = G_n(y)$  and

$$\Delta = \Delta_n - \theta_1 n^{-1/2} Z_{n+1}.$$

Furthermore, let

$$H(x) = n^{-\delta k} \sigma(x + S_n^{(2)})^{-2k-2} \prod_{j \neq 1}' e(Z_{n+j}, x + S_n^{(2)}).$$

Let  $M = [n/2]$  and identify  $Z_{M+1}$  in Lemma 3.7 with  $Z_{n+1}$ . Hence by appropriate choice of the splitting (2.17) (using the i.i.d. assumption) we can estimate a typical summand  $DF(x + S_n^{(2)})[Z_{n+1}]^2 H(x)$  of  $e_1(x)$  conditioned on  $\mathcal{D} = \sigma(Z_j, M + 1 \leq j \leq n, \Delta)$  using

$$(2.20) \quad Ee(Z_{n+j}, x) \leq 1 \quad \text{and} \quad \sigma(x)^{-2} n^{-\delta} \leq 1.$$

Lemma 3.7 yields

$$(2.21) \quad \begin{aligned} I_4 \leq & cEG_n(F(S_{n+1} + \Delta)) |DF(S_{n+1} + \Delta)[S_{n+1}]H(S_{n+1}, \Delta)| \\ & + \|\alpha_n\| EG_n(F(S_n + \Delta))(1 + \|S_n\|^q)H(S_n, \Delta) \\ & + EG_n(F(S_{n+\tau} + \Delta)) |H_1(S_{n+\tau}, \Delta)[Z_{n+1}^2]| \\ & + O(n^{-1/2}) |E(H_2(S_{n+\mu\tau}, \Delta))[Z_{n+1}^3]|. \end{aligned}$$

By definition  $H_1$  and  $H_2$  consist of derivatives of  $F$  up to the second order and a first derivative of  $e_1(x)$ . The following estimate for this derivative will be frequently used: [notation of (2.16)]

$$(2.22) \quad |De_1(x)| \leq c(1 + \|x\|^{4p}) \sum_k^* \|Z_{n+k}\|^2 \sigma(x)^{-2k-4} n^{-\delta k} \prod_{j \neq k}' e(Z_{n+j}, x).$$

We have  $G_n(x) \leq cn^{-1/2} A_n$  for every  $x$  (with  $A_n$  constant given  $\mathcal{D}$ ) and  $\|\alpha_n\| = O(n^{-1/2})$ . By expansion of  $\Sigma''$  in  $A_n$  and the i.i.d. assumption, together with bounds for  $H_1$  and  $H_2$ , relations (2.21) and (2.20) yield for some  $q > 0$ :

$$(2.23) \quad \begin{aligned} I_4 \leq & [O(n^{-1/2}) + o(\|\alpha_n\|n^{-1/2})] E\sigma(S_n)^{-4}(1 + \|S_n\|^q)(1 + \|Z_{n+B}\|^3) \\ & + O(n^{-1/2}) E(1 + \|S_n\|^q)(1 + \|Z_{n+1}\|^2 + \|Z_{n+1}\|^3)\sigma(S_{n+\tau})^{-4} \\ & + O(n^{-1/2}) E(1 + \|S_n\|^q)(1 + \|Z_{n+1}\|^3 + \|Z_{n+B}\|^2)\sigma(S_{n+\tau\mu})^{-4} \\ = & O(n^{-1/2}) \end{aligned}$$

by Remark 3.24 with  $q = 4$  using similar arguments as in (2.18). Hence (2.17)–(2.23) yield

$$(2.24) \quad I_3 = O(n^{-1/2}).$$

Recall that

$$I_2 = |Eg(F(S_n + \Delta_n))e(S_n) - Eg(F(R))e(R)|.$$

Let  $W_{nt} = S_n \sin(\pi t/2) + R \cos(\pi t/2)$ . We may rewrite the difference in  $I_2$  as an integral over the derivative of the function  $t \rightarrow g(F(\Delta_n \sin(\pi t/2) + W_{nt}))e(W_{nt})$

and obtain with Proposition 3.1(i) and the notation used there,

$$\begin{aligned}
 (2.25) \quad I_2 &\leq |EDg(F(S_{nt}))DF(S_{nt})[T_{nt}]e(W_{nt})| \\
 &\quad + |Eg(F(S_{nt}))De(W_{nt})[T_{nt}]| \\
 &\quad + |EDg(F(S_{nt}))DF(S_{nt})[\Delta_n]e(W_{nt})| \\
 &= I_5 + I_6 + I_7, \quad \text{say,}
 \end{aligned}$$

where  $T_{nt} = (d/dt)S_{nt}$  and  $t$  is uniformly distributed in  $[0, 1]$ .

ESTIMATION OF THE TERM  $I_6$  OF (2.25). Notice that by definition of  $e(x)$  as a conditional density

$$E(De(W_{nt})[T_{nt}]|Z_j, j \neq n + r) = 0 \quad \text{a.s. for } r = 1, \dots, B.$$

Hence

$$\begin{aligned}
 (2.26) \quad I_6 &\leq E|g(F(S_{nt})) - g(F(W_{nt}))||De(W_{nt})[T_{nt}]| \\
 &\leq cEI(|F(S_{nt}) - z| \leq |F(S_{nt}) - F(W_{nt})|)|De(W_{nt})[T_{nt}]|
 \end{aligned}$$

by arguments similar to those used in (2.13). Using (2.22) the estimation of (2.26) is very similar to the estimations of (2.14)–(2.23), when we condition on  $t$ , split  $e(x)$  with  $b = B - 3$ , and apply Lemma 3.12 together with Remark 3.24. Here again the splitting can be chosen such that at most third-order powers of  $\|X_{n+j}\|$  occur in a final estimate similar to (2.23) together with factors of the type  $\sigma(W_{(n+1)t})^{-6}$ . The result is again

$$(2.27) \quad I_6 = O(n^{-1/2})$$

by Remark 3.24 with  $q = 6$ .

ESTIMATION OF THE TERM  $I_7$  IN (2.25). We have by (2.37)

$$\begin{aligned}
 I_7 &\leq cn^{-1/2}|Ee(W_{nt})Dg(F(S_{nt}))DF(S_{nt})[Z_{n+1}]| \\
 &\leq c|Ee(W_{nt})I(|F(S_{nt}) - z| \leq n^{-1/2})(1 + \|S_{nt}\|^p)\|Z_{n+1}\|.
 \end{aligned}$$

Conditioning on  $Z_{n+1}$  we split  $e(x) = e_1(x) + r_n$  with  $b = B - 2$  and may proceed as in (2.13)–(2.24) by partial integration (Lemma 3.7 together with Lemmas 3.12 and 3.24), obtaining  $I_7 = O(n^{-1/2})$ .

Combining this result with (2.23), (2.24), and (2.26) it follows from (2.11) and (2.12) that

$$(2.28) \quad I_1 = I_5 + O(n^{-1/2}).$$

In view of (2.8) it is sufficient to show that  $I_5 = O(n^{-1/2})$  in order to prove the theorem.

ESTIMATION OF THE TERM  $I_5$  OF (2.12). Split  $e(x) = e_1(x) + r_n$  as in (2.16) with  $b = B - 3$ . We have in a manner similar to (2.17)–(2.23) with  $\delta b > 1$ ,

$$(2.29) \quad I_5 \leq E|Dg(F(S_{nt}))DF(S_{nt})[T_{nt}]e_1(W_{nt})| + O(n^{-1/2}).$$

Define  $G(x)[y] = Dg(F(x + \Delta_{nt}))DF(x + \Delta_{nt})[y]e_1(x)$ , suppressing the dependence on  $\Delta_{nt}$  in this definition.

Applying Lemma 3.1(ii) (which does a second-order Taylor expansion with integral remainder term with respect to the i.i.d. summands  $Z_j$  of  $T_{nt}$ ) with this function  $G(x)[y]$  yields with the notation used there ( $U_{nt} = S_{nt} - n^{-1/2}Z_{1t}$  and  $U'_{nt} = S_{nt} - (1 - \tau)n^{-1/2}Z_{1t}$ )

$$(2.30) \quad \begin{aligned} I_5 &= EG(U_{nt})[a_{nt}] + EL_{nt}[DG(U_{nt})] + O(n^{-1/2}) \\ &\quad + ED^2G(U'_{nt})[Z_{1t}^2, Y_{1t}](1 - \tau)n^{-1/2}. \end{aligned}$$

Here we have

$$(2.31) \quad \begin{aligned} D^2G(x)[z^2, y] &= \{ D^3g(F(\bar{x}))DF(\bar{x})[y]DF(\bar{x})[z]^2 \\ &\quad + 2D^2g(F(\bar{x}))D^2F(\bar{x})[y, z]DF(\bar{x})[z] \\ &\quad + Dg(F(\bar{x}))D^3F(\bar{x})[z^2, y] \} e_1(x) \\ &\quad + D^2g(F(\bar{x}))DF(\bar{x})[y]DF(\bar{x})[z]De_1(x)[z] \\ &\quad + Dg(F(\bar{x}))D^2F(\bar{x})[y, z]De_1(x)[z] \\ &\quad + Dg(F(\bar{x}))DF(\bar{x})[y]D^2e_1(x)[z^2], \end{aligned}$$

where

$$\bar{x} = x + \Delta_{nt}.$$

The expressions for  $DG(x)[z, y]$  are similar.

Summarizing we have

$$(2.32) \quad \begin{aligned} &D^iG(x)[z^i, y] \\ &= \sum'' D^p g(F(\bar{x}))M_{pr}(x, \Delta_{nt})[z^i, y]n^{-\delta k}\sigma(x)^{-2k-2r} \prod_j e(x, Z_{n+j}), \end{aligned}$$

where  $i, p = 0, 1, 2$ , the sum  $\Sigma''$  is over all integral numbers  $k, p, r$ , and  $(B - k)$ -tuples of numbers  $n + 1$  to  $n + b$  such that  $p + r \leq i + 1, r \geq 0, k \leq B - 3$ . The product  $\prod'$  extends over  $B - k$  indices  $j \geq 1$  of a  $(B - k)$ -tuple. The functions  $M_{pr}(x, \bar{x})$  are of class  $C^{3+\beta-(i-p+1)}$  for  $i \geq 1$  (conditionally on  $\Delta_{nt}$ ).

The next step is to reduce  $D^p g(x)$  to  $g(x)$  by repeated application of Lemma 3.7. This lemma can be applied up to three times since (2.32) shows that there are at least six factors  $e(Z_{n+j}, x)^2$  in every summand of (2.32) which (even after application of Lemma 3.7) always allows us to rewrite the resulting term in the form required by (3.8).

We shall demonstrate this procedure for a typical term in the sum  $e_1(x)$  (or its derivative) involving  $D^3g$  [see (2.31) resp. (2.32)] and we denote this term by  $I$ . The expectations of other terms occurring in (2.31) resp. (2.32) can be treated similarly.

In Lemma 3.7 let

$$H(x, \Delta_{nt}) = M_{3r}(x, \Delta_{nt})[Z_{1t}^2, Y_{1t}]n^{-\delta k}\sigma(x)^{-2r-2k} \prod_j e(x, Z_{n+j}),$$

where the product  $\prod''$  extends over  $B - k - 1$  indices  $2 \leq j \leq B$ . Furthermore, we replace expectations by conditional expectations given  $\mathcal{F} = \sigma(Z_{n+j}, j > 1, Z_{1t}, t, \tau)$  and identify  $\theta$  with  $\theta_1$ . Let  $M = n - 1$  and replace  $Z_j$  by  $Z_{jt}$  as well as  $g$  by  $D^2g$  and  $\Delta$  by  $\Delta_{nt}$  in Lemma 3.7. The particular term  $I$  can now be written as

$$\begin{aligned}
 (2.33) \quad I = n^{-1/2} E \left\{ E \left( \left[ D^2g(F(S'_{nt})) DF(S'_{nt}) [S'_{nt}] H(S'_{nt}, \Delta_{nt}) n / (n - 1) \right. \right. \right. \\
 \left. \left. \left. + D^2g(F(U'_{nt})) DF(U'_{nt}) [Z_{n+1}]^2 H_1(U'_{nt}, \Delta_{nt}) \right. \right. \right. \\
 \left. \left. \left. + D^2g(F(S''_{nt})) DF(S''_{nt}) [a_{nt}] H(S''_{nt}, \Delta_{nt}) \right. \right. \right. \\
 \left. \left. \left. + n^{-1/2} D^3g(F(U'_{nt})) H_2(U'_{nt}, \Delta_{nt}) \right] \middle| \mathcal{F} \right) \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 (2.34) \quad S'_{nt} &= U'_{nt} + n^{-1/2} (1 - \theta_1) Z_{(n+1)t}, \\
 S''_{nt} &= U'_{nt} - n^{-1/2} \theta_1 Z_{(n+1)t},
 \end{aligned}$$

and  $H_j, j = 1, 2$  denote functions of class  $C^{1+\beta}$  involving  $\sigma(x)^{-2r-2k} \times \prod^*_j e(x, Z_{n+j})$ , where the product extends over at least  $B - k - 1 > 4$  different indices  $n + j > n + 1$ .

Applying Lemma 3.7 to the r.h.s. of (2.33) again with  $M = n$  resp.  $n - 1$  and  $g$  replaced by  $Dg$  resp.  $n^{-1/2} D^2g$  yields

$$\begin{aligned}
 (2.35) \quad I \leq n^{-1/2} \left| E \left[ Dg(F(S'^*_{nt})) H_4(S'^*_{nt}, \Delta_{nt}) \right. \right. \\
 \left. \left. + \text{similar terms involving } S''^*_{nt}, S^*_{nt} \right. \right. \\
 \left. \left. + n^{-1/2} D^2g(F(U'_{nt})) H_5(U'_{nt}, \Delta_{nt}) + \text{similar terms} \right. \right. \\
 \left. \left. + n^{-1} D^3g(F(U'_{nt})) H_6(U'_{nt}, \Delta_{nt}) + \text{similar terms} \right] \right|,
 \end{aligned}$$

where  $S'^*_{nt}, S^*_{nt}$  etc., are derived from the quantities in (2.34) as before by conditioning on  $\theta_2 = 0, 1$ .

Here  $H_j(x, \Delta)$  denote differentiable functions of class  $C^\beta$  [see (1.3)] such that

$$\begin{aligned}
 (2.36) \quad |\tilde{H}_j(x, \Delta)| &\leq c(1 + \|\Delta\|^q + \|x\|^q)(1 + \sigma(x)^{-8})n^{-\delta k} \\
 &\times (1 + \|Z_{n+2}\|^2 + \|Z_1\|^3 + \|Y_1\|^3) \\
 &\times \sigma(x)^{-2k-2r} \prod^{**} e(x, Z_{n+j}),
 \end{aligned}$$

where  $\prod^{**}$  denotes a product over at least one index  $n + j$  out of  $B - k - 2$ . By the choice of the function  $g$

$$\begin{aligned}
 (2.37) \quad Dg(x) &\leq 0, \\
 n^{-i/2} |D^{i+1}g(x)| &\leq c |Dg(x)| \leq cn^{1/2} I(|x - z| \leq n^{-1/2}).
 \end{aligned}$$

Let  $S^{**}_{nt} = S_{nt} + O(n^{-1/2})(Z_{n+1} + Z_{n+2})$  be one of the sums obtained by the partial integrations above. Then

$$I \leq cEI(F(S^{**}_{nt}) - z| \leq n^{-1/2}) \sum_j \|H_j\|.$$

Using (2.37) together with (2.22) and the splitting arguments of (2.16)–(2.23) we obtain (with  $q = 8$ )

$$I = O(n^{-1/2})$$

for the term in (2.25) involving  $D^3g$ . The other terms can be treated similarly. Hence

$$I_5 = O(n^{-1/2})$$

and the theorem holds.  $\square$

**3. Lemmas.** Let  $R_j, j = 1, \dots, n$ , denote i.i.d. Gaussian random vectors with the same distribution as  $R$ .

**3.1 PROPOSITION.** (i) Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  denote a function of class  $C^1$ , let  $F$  denote a functional on  $E$  satisfying (1.3), and let  $H: E \rightarrow \mathbb{R}$  denote a functional of class  $C^1$ . Let  $t \in [0, 1]$  be uniformly distributed and independent of  $X_j, j \in \mathbb{N}$ . Define

$$(3.2) \quad S_{nt} = n^{-1/2}\{Z_{1t} + \dots + Z_{nt} + \Delta_{nt}\} \quad \text{and} \quad T_{nt} = \frac{d}{dt}S_{nt}, \quad \text{where}$$

$$\Delta_{nt} = [\theta_1 Z_{n+1} + \dots + \theta_B z_{n+B}] \sin \frac{\pi t}{2}, \quad \Delta_n = \Delta_{n0},$$

$$Z_{jt} = Z_j \sin \frac{\pi t}{2} + R_j \cos \left( \frac{\pi t}{2} \right) \quad \text{and} \quad Y_{jt} = \frac{d}{dt}Z_{jt}.$$

Then we have by Taylor expansion with integral remainder term

$$(3.3) \quad \begin{aligned} & Eg(F(S_n + \Delta_n))H(S_n + \Delta_n) - Eg(F(R))H(R) \\ &= E\{Dg(F(S_{nt}))DF(S_{nt})[T_{nt}]H(S_{nt}) \\ &\quad + g(F(S_{nt}))DH(S_{nt})[T_{nt}]\}. \end{aligned}$$

(ii) Let  $G(x)[v], x, v \in E$  denote a real valued function of class  $C^2$  with respect to  $x$  and which is linear and continuous in  $v$ . Let  $U_{nt} = S_{nt} - n^{-1/2}Z_{1t}$ . Then ( $\Delta'_{nt} = d/dt \Delta_{nt}$ )

$$(3.4) \quad \begin{aligned} & EG(S_{nt})[T_{nt} - \Delta'_{nt}] \\ &= E\left\{G(U_{nt})[a_{nt}] + L_{nt}(DG(U_{nt})) \right. \\ &\quad \left. + \sum_{j=1}^n D^2G(S_{nt} - \tau n^{-1/2}Z_{jt})[Z'_{jt}, Y_{jt}] \tau n^{-3/2}\right\}. \end{aligned}$$

Here  $\tau \in [0, 1]$  is uniformly distributed and independent of  $X_j$  and  $t$  and

$$a_{nt} = n^{1/2}EZ_1 \frac{\pi}{2} \cos \frac{\pi t}{2}, \quad \text{such that } \|a_{nt}\| = O(n^{-1/2}).$$

The function  $L_{nt}$  is a continuous linear functional on the Banach space of continuous symmetric bilinear forms on  $E$  (endowed with the strong supremum-norm topology for bilinear forms) defined by

$$(3.5) \quad L_{nt}(w) = Ew[X_1, X_1]I(\|X_1\| > n^\alpha) \frac{\pi \sin(\pi t)}{4}$$

on bilinear forms  $w$ , such that  $\|L_{nt}\| = O(n^{-1/2})$ .

PROOF. (i) The assertion immediately follows by Taylor expansion in  $t$  around  $t = 0$  with integral remainder term.

(ii) We have

$$(3.6) \quad EG(S_{nt})[T_{nt}] = \sum_{j=1}^n n^{-1/2}EG(S_{nt})[Y_{jt}]$$

using the identical distribution of  $Z_{jt}$  and a Taylor expansion with integral remainder term with respect to the summand of  $S_{nt}$  which depends on  $Y_{jt}$ , we obtain a three-term expansion for (3.6).

In order to evaluate the first term of this Taylor expansion note that  $EY_n = 0$ . Since  $U_{nt}$  and  $Y_{nt}$  are independent we have

$$EG(U_{nt})[Y_{nt}] = EG(U_{nt})[a_{nt}].$$

The upper bound for  $\|a_{nt}\|$  is a consequence of

$$EZ_n = EX_n - EX_n I(\|X_n\| > n^\alpha).$$

The second expansion term of (3.6) follows from

$$\begin{aligned} Ew[Z_{jt}, Y_{jt}] &= (Ew[Z_j, Z_j] - Ew[R, R]) \frac{\pi}{2} \cos \frac{\pi t}{2} \sin \frac{\pi t}{2} \\ &= \frac{\pi \sin(\pi t)}{4} (Ew[X_j, X_j] - Ew[R, R]) + L_{nt}(w), \end{aligned}$$

where the term in brackets vanishes by assumption (1.3) and Lemma 4.6 in Götze (1981).

The third term of the Taylor expansion of (ii) is

$$n^{-3/2} \sum_{j=1}^n ED^2G(S_{nt} - \tau n^{-1/2}Z_{jt})[Z_{jt}^2, Y_{jt}] \tau/2,$$

thus proving the assertion (3.2).  $\square$

The following lemma provides the “partial integration” tool.

**3.7 LEMMA.** Let  $S_M = n^{-1/2}(Z_1 + \dots + Z_M)$ ,  $M \leq n$ , where  $Z_j$ ,  $j \in \mathbb{N}$ , are i.i.d. and  $E\|Z_1\|^3 < \infty$ . Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  denote a function of class  $C^\infty$  and let  $H: E \times E \rightarrow \mathbb{R}$  be of class  $C^p$ . Denote by  $\Delta$  a linear combination of  $Z_{n+j}$ ,  $2 \leq j \leq B$ , with coefficients of order  $O(n^{-1/2})$  and let  $S_{M+\tau} = \tau Z_{M+1} n^{-1/2} + S_M$ ,

where  $\tau$  is uniformly distributed in  $[0, 1]$  and independent of  $Z_j$ ,  $j \in \mathbb{N}$ . Define  $\alpha_j = EZ_j n^{-1/2}$  and  $F_\Delta(s) = F(\Delta + s)$ . Then

$$\begin{aligned}
 & EDg(F_\Delta(S_{M+\tau}))DF(S_M)[Z_{M+1}]^2H(S_M, \Delta) \\
 &= Eg(F_\Delta(S_{M+1}))DF_\Delta(S_{M+1})[S_{M+1}]H(S_{M+1}, \Delta)nM^{-1} \\
 (3.8) \quad &+ Eg(F_\Delta(S_{M+\tau}))H_1(S_{M+\tau}, \Delta)[Z_{M+1}^2] \\
 &+ Eg(F_\Delta(S_M))H_3(S_M, \Delta)[\alpha_M] \\
 &+ n^{-1/2}EDg(F_\Delta(S_{M+\tau}))H_2(S_{M+\tau}, \Delta)[Z_{M+1}^3],
 \end{aligned}$$

where the functions  $H_j$  are of class  $C^{p-1}$  and given by

$$\begin{aligned}
 H_1(s, \Delta)[z^2] &= -D_s(DF_\Delta(s)[z]H(s, \Delta))[z], \\
 H_2(s, \Delta)[z^3] &= -D_s(DF_\Delta(s))[z]^2H(s, \Delta))[z] \\
 &\quad - D_s(DF_{\Delta, \nu}(s)[z]^2[\Delta n^{1/2}]H(s, \Delta))[z], \\
 H_3(s)[a] &= DF_\Delta(s)[a]H(s, \Delta),
 \end{aligned}$$

where  $\nu$ ,  $\mu$ , and  $\tau$  have the uniform distribution in  $[0, 1]$ , independent of all other r.v.'s.

PROOF. Let  $J$  denote the first term on the r.h.s. of (3.8). By the i.i.d. assumption and Taylor expansion with integral remainder term we have

$$\begin{aligned}
 Mn^{-1}J &= Eg(F_\Delta(S_{M+1}))DF_\Delta(S_{M+1})[Z_{M+1}]Mn^{-1/2}H(S_{M+1}, \Delta) \\
 (3.9) \quad &= Eg(F_\Delta(S_M))DF_\Delta(S_M)[\alpha_M]H(S_M, \Delta) \\
 &+ Mn^{-1}E\{Dg(F_\Delta(S_{M+\tau}))DF_\Delta(S_{M+\tau})[Z_{M+1}]^2H(S_{M+\tau}, \Delta) \\
 &\quad + g(F_\Delta(S_{M+\tau}))D(DF_\Delta(S_{M+\tau})[Z_{M+1}]H(S_{M+\tau}, \Delta))[Z_{M+1}]\}.
 \end{aligned}$$

Using an additional Taylor expansion with integral remainder term of the second term of the r.h.s. of (3.9) around  $\Delta = 0$  and  $\tau = 0$  yields the l.h.s. of (3.8) as well as the term involving  $H_2(s, \Delta)$  on the r.h.s. of (3.8). Hence the lemma is proved by (3.9).  $\square$

Let  $V$  denote a Gaussian random vector independent of  $R$  with the same distribution. Similar to Lemma 3.7 we have

3.10 LEMMA. Let  $H: E \rightarrow \mathbb{R}$  denote a function of class  $C^p$ . Then

$$\begin{aligned}
 & EDg(F(R))DF(R)[V]^2H(R) \\
 (3.11) \quad &= Eg(F(R))\{DF(R)[R]H(R) - D^2F(R)[V^2]H(R) \\
 &\quad - DF(R)[V]DH(R)[V]\}.
 \end{aligned}$$

PROOF. Since  $R$  and  $(R_1 + \dots + R_m)m^{-1/2}$  (where  $R_j$  denote independent copies of  $R$ ) have the same distribution the arguments of the proof of Lemma 3.7



immediately entail (3.11) as  $m \rightarrow \infty$ . (Note that arbitrarily high moments of  $\|R\|$  exist by Lemma 3.12.)  $\square$

**3.12 LEMMA.** *Let  $Z_j$  denote the random vector which equals  $X_j$  if  $\|X_j\| \leq n^\alpha$ , where  $\alpha \leq \frac{1}{2}$  and is zero otherwise. Assume that  $X_j$  satisfy (1.1)–(1.2). Let  $S_n = n^{-1/2}(Z_1 + \dots + Z_n)$ . Then*

- (i)  $\sup\{E\|S_n\|^p: n \in \mathbb{N}\} < \infty,$
- (ii)  $E\|R\|^p < \infty$

for every  $p \in \mathbb{N}$ .

**PROOF.** Chebyshev’s inequality yields

$$\begin{aligned} EZ_1 &= EX_1 - EX_1 I(\|X_1\| > n^\alpha) \\ &= E\|X_1\|^3 o(n^{-(2+\gamma)\alpha}). \end{aligned}$$

Hence

$$\|ES_n\| = o(n^{-(2+\gamma)\alpha+1/2}).$$

Using

$$\|S_n\|^p \leq 2^{p-1}(\|ES_n\|^p + \|S_n - ES_n\|^p),$$

we may assume  $EZ_j = 0$ . By assumption the CLT holds for  $X_j$ . Hence, the results of de Acosta and Giné (1979) imply that  $E\|X_1 + \dots + X_n\|^2/n$  is uniformly bounded in  $n$ . Hence the same holds true for  $Z_j$ . Assertion (i) now follows immediately by an explicit estimate of Yurinskii (1976), page 478, (2.8). Assertion (ii) is a well-known result of Fernique (1970).  $\square$

**3.13 LEMMA.** *Let  $R$  denote a Gaussian random vector in  $E$  as in Lemma 3.10. Then*

$$(3.14) \quad P(|F(R) - z| \leq \varepsilon) \leq c\varepsilon.$$

**PROOF.** Let

$$g(x) = \int_{-\infty}^x I(-\varepsilon \leq a - z \leq \varepsilon) da.$$

Then  $0 \leq g(x) \leq 2\varepsilon$ . Writing

$$P(|F(R) - z| \leq \varepsilon) = EI(-\varepsilon \leq F(R) - z \leq \varepsilon)DF(R)[V]^2/v(R),$$

the r.h.s. of (3.14) can be bounded after application of Lemma 3.10 [with  $H(R) = v(R)^{-1}$ ] by

$$\begin{aligned} & Eg(F(R))\{DF(R)[R]v(R)^{-1} - D^2F(R)[V^2]v(R)^{-1} \\ & \quad - DF(R)[V]D(v(R)^{-1})[V]\} \\ & \leq \varepsilon E\{v(R)^{-1}(1 + \|R\|^p)\|R\| + (1 + \|R\|^p)^2 v(R)^{-2}\} \\ & \leq c\varepsilon E^q v(R)^{-2-s}, \quad \text{for some } q > 0, 1 > s > 0 \\ & \leq c\varepsilon \end{aligned}$$

by (3.23) and the choice of  $k_\beta$ , thus proving the assertion.  $\square$

3.15 LEMMA. Let  $Z_{jt} = Z_j \sin(\pi t/2) + R_j \cos(\pi t/2)$  and let  $S_{nt}$  denote the sum  $n^{-1/2}(Z_{1t} + \dots + Z_{nt})$ . Let  $Y_{jt} = d/dt Z_{jt}$ . As in (2.11) let  $\sigma(x)^2 = n^{-S} + E(DF(x)[Z_1])^2$ . Then

$$(i) \sup \{ E\sigma(R + \eta_1 X_1 + \dots + \eta_r X_r)^{-q} \|X_1\|^3 \dots \|X_r\|^3 : n \in \mathbb{N}, |\eta_i| \leq n^{-1/2}, 1 \leq i \leq r \} < D < \infty,$$

where  $r = [q/2] + 1$  and  $\delta < \min(\beta, 1/4)/4$  implies

$$(3.16) \quad \sup \{ E\sigma(S_{ns})^{-q} : n \in \mathbb{N}, s \in [0, 1] \} \leq C(D, \beta) < \infty.$$

When  $E\|X_1\|^{3+\gamma} < \infty, \gamma > 0$  and  $\delta < (\beta \wedge 1/4)/4 \wedge (\gamma/(6 + 2\gamma))$  the condition

$$(ii) \quad \sup_n E\sigma(R)^{-q} < c(1 + D) < \infty$$

is sufficient for condition (i) to hold. Conditions (i) and (ii) hold assuming that the variance condition (1.4) holds with  $\gamma = 0$  and  $\gamma > 0$ , respectively, when  $q < 2k_\beta$ .

PROOF. Let  $S'_{ns} = S_{ns} - n^{-1/2}Z_{ns}$ . Similar to relations (3.2)–(3.3) we have by Taylor expansion with integral remainder term

$$(3.17) \quad \begin{aligned} & E\sigma(S_{nt})^{-q} - E\sigma(R)^{-q} \\ &= \int_t^1 ds \{ ED\sigma(S'_{ns})^{-q} [Z_{ns}] n^{1/2} + ED^2\sigma(S'_{ns})^{-q} [Z_{ns}, Y_{ns}] \\ &\quad + ED^2\sigma(S_{ns} - n^{-1/2}\theta Z_{ns})^{-q} [Z_{ns}, Y_{ns}] \\ &\quad - ED^2\sigma(S'_{ns})^{-q} [Z_{ns}, Y_{ns}] \}, \end{aligned}$$

where  $\theta$  is uniformly distributed in  $[0, 1]$ . Using the relations in the proof of Lemma 3.12 the first term on the r.h.s. of (3.17) can be estimated for fixed  $s$  by

$$(3.18) \quad \begin{aligned} & cE(1 + \|S'_{ns}\|^{2\rho})\sigma(S'_{ns})^{-q-2}\|Z_n\|^3 n^{-(2+\gamma)\alpha+1/2} \\ & \leq CE^\nu \sigma(S'_{ns})^{-(q+2)/\nu} E\|Z_1\|^3 n^{-(2+\gamma)\alpha+1/2}, \end{aligned}$$

for some  $0 < \nu < 1$  arbitrarily close to 1 and  $\alpha = 3/(6 + 2\gamma), \gamma \geq 0$ . Similar to (3.4) we have again by independence and the definition of  $Z_{ns}, Y_{ns}$ , and

$$\begin{aligned} Ew[Z_{ns}, Y_{ns}] &= \frac{\pi}{2} \cos \frac{\pi s}{2} \sin \frac{\pi s}{2} \{ Ew[R, R] - Ew[X_n, X_n] I(\|X_n\| < n^\alpha) \} \\ &= O(n^{-\alpha(1+\gamma)}) \|w\| E\|X_1\|^3 I(\|X_1\| > n^\alpha). \end{aligned}$$

Using this relation in order to estimate the second term of the r.h.s. of (3.17) similar as the first one we get as an upper bound for the second term

$$(3.19) \quad cE(1 + \|S'_{ns}\|^{4\rho})\sigma(S'_{ns})^{-(q+4)} O(n^{-\alpha(1+\gamma)}) E\|X_2\|^{3+\gamma}.$$

Finally, the third term of the r.h.s. needs more elaborate estimation using

$$\begin{aligned}
 |D^2\sigma(x)^{-q} - D^2\sigma(y)^{-q}| &\leq c(1 + \|x\|^p + \|y\|^p)^4 \\
 (3.20) \quad &\times \left\{ \|x - y\|^\beta \sigma(x)^{-q-2} + \|y - y\| \sigma(x)^{-q-4} + \|x - y\| \right. \\
 &\quad \left. \times \int d\theta \sigma(\theta x + (1 - \theta)y)^{-q-6} (1 + \|x\|^p + \|y\|^p)^2 \right\}
 \end{aligned}$$

for every  $x, y \in E$  with  $\|x - y\| < 1$  and integration over  $[0, 1]$ . This inequality follows from condition (1.3) and elementary calculus.

Relations (3.17)–(3.20) together with Hölder’s inequality,  $\|Z_{1s}\| < n^\alpha + \|Y_1\|$  and  $\sup \sigma(x)^{-2} \leq n^\delta$  yield for some  $\nu < 1$  sufficiently close to 1,

$$\begin{aligned}
 E\sigma(S_{nt})^{-q} &\leq E\sigma(R)^{-q} \\
 (3.21) \quad &+ \int_t^1 ds \left\{ O(n^{-\alpha(1)} + n^{-\alpha(2)} + n^{-\alpha(3)} + n^{-\alpha(4)}) E^\nu \sigma(S'_{ns})^{-q/\nu} \right. \\
 &\quad \left. + O(n^{-\alpha(5)}) \int_0^1 d\tau E(1 + \|S'_{ns}\|^{6p})(\|Z_n\|^3 + \|Y_n\|^3) \right. \\
 &\quad \left. \times \sigma(S'_{ns} + \tau n^{-1/2} Z_{ns})^{-q} \right\},
 \end{aligned}$$

where  $\alpha(1) = (2 + \gamma)\alpha - 1/2 - \delta$ ,  $\alpha(2) = (1 + \gamma)\alpha - 2\delta$ ,  $\alpha(3) = \beta/2 - \delta$ ,  $\alpha(4) = 1/2 - 2\delta$ , and  $\alpha(5) = 1/2 - 3\delta$ . Using

$$E^\nu \sigma(S'_{ns})^{-q/\nu} \leq n^{(1-\nu)q\delta/2} E\sigma(S'_{ns})^{-q}$$

and  $E\|Y_n\|^p < \infty$  for arbitrarily large  $p$  by Lemma 3.12(ii) we have

$$\begin{aligned}
 E\sigma(S_{nt})^{-q} &\leq E\sigma(R)^{-q} + O(n^{-\chi}) \int_t^1 ds \left\{ E\sigma(S'_{ns})^{-q} \right. \\
 (3.22) \quad &\quad \left. + \int d\tau E\sigma(S'_{ns} + \tau n^{-1/2} Z_{ns})^{-q} \|Z_n\|^3 \right\} \\
 &\leq E\sigma(R)^{-q} + O(n^{-\chi}) \sup_{0 \leq \tau, s \leq 1} E\sigma(S'_{ns} + \tau n^{-1/2} Z_{ns})^{-q} \|Z_n\|^3,
 \end{aligned}$$

where  $\chi = \min(\alpha(1), \dots, \alpha(5)) - (1 - \nu)q\delta/2$ . Choosing  $\nu$  sufficiently close to 1 and  $\delta < (\beta \wedge 1/4)/4$  we obtain  $\chi > (\beta \wedge 1/4)/4 > \delta$ .

Conditioning on  $Z_n$  we apply (3.22)  $r$  times recursively, obtaining

$$\begin{aligned}
 E\sigma(S_{nt})^{-q} &\leq \sup \left\{ \sum_{j=0}^{r-1} O(n^{-\chi j}) E\sigma(R\chi_{nj} + \Delta_{nj})^{-q} \|Z_n\|^3 \cdots \|Z_{n-j+1}\|^3 \right. \\
 &\quad \left. + O(n^{-\chi r}) \sup_s E\sigma(S'_{ns} + \Delta_{nr})^{-q} \|Z_n\|^3 \cdots \|Z_{n-r}\|^3 \right\},
 \end{aligned}$$

where  $\Delta_{nj} = \sum_{k=0}^j \tau_k Z_{ns_k} n^{-1/2}$ ,  $\chi_{nj}^2 = 1 - j/n$ , and the supremum is over all  $0 \leq \tau_k, s_k \leq 1$ . Since

$$\sigma^2(R\chi_{nj} + \Delta_{nj}) \geq \sigma^2(R + \Delta_{nj})/2,$$

and

$$c(1 + \|R\|^p + \|\Delta_{n,j}\|^p)^2(1 - \chi_{n,j})\|R\| < n^{-\delta}/2$$

hold with probability  $1 - o(n^{-A})$ ,  $A > 0$  arbitrarily large, we have by the assumptions of Lemma 3.15,

$$\sup_{t,n} E\sigma(S_{nt})^{-q} \leq cD + O(n^{-A}) + O(n^{-\chi r + q\delta/2})(E\|Z_n\|^3)^r.$$

Choosing  $r = [q/2] + 1$ , the proof of the first part of Lemma 3.15 is complete.

When  $\gamma > 0$  we have

$$\sigma^2(R + \Delta_{n,j}) \geq \sigma^2(R) - c(1 + \|R\|^p + \|\Delta_{n,j}\|^p)^2\|\Delta_{n,j}\|.$$

Since  $\|\Delta_{n,j}\| \leq cn^{\alpha-1/2}$  we have  $c(1 + \|R\|^p)^2\|\Delta_{n,j}\| \geq \frac{1}{2}n^{-\delta}$  with probability  $1 - O(n^{-A})$ ,  $A > 0$  arbitrarily large, provided  $\alpha - 1/2 < -\delta$ , which follows from  $\delta < \gamma/(6 + 2\gamma)$ . Hence  $\sigma^2(R + \Delta_{n,j}) \geq \frac{1}{2}\sigma^2(R)$  with probability  $1 - O(n^{-A})$ , which immediately shows that

$$E\sigma(R)^{-q} < cD(E\|X_1\|^3)^{-r} + c < \infty$$

is a sufficient condition.

In order to reduce (i) and (ii) to the variance condition (1.4) note that by Chebyshev's inequality

$$\sigma(R)^2 = v(R) + n^{-\delta} + (1 + \|R\|^{2p})O(n^{-(1+\gamma)\alpha})E\|X_1\|^{3+\gamma},$$

i.e.,

$$E\sigma(R)^{-k} \leq Ev(R)^{-k/2} + O(n^{k\delta/2})P(\|R\| > n^{(\alpha-\delta)/(2p)}).$$

By Lemma 3.9 and condition (1.4) we have for  $k > 0$ ,  $\gamma > 0$ ,

$$\begin{aligned} E\sigma(R)^{-k} &\leq c \int_0^1 x^{-k-1}P(v(R) \leq x^2) dx + O(1) \\ (3.23) \quad &\leq c \int_0^1 x^{-k-1}x^{2k_\beta} dx + O(1) < \infty \end{aligned}$$

for  $k < 2k_\beta$ . For  $\gamma = 0$  the proof is similar. This completes the proof of Lemma 3.15.  $\square$

**3.24 REMARK.** Using the notation of Lemma 3.15 we have for arbitrary  $A > 0$  and  $q = k_\beta/(1 + \epsilon)$ , for some  $\epsilon > 0$  sufficiently small

$$(3.25) \quad \sup\{E(1 + \|S_{nt}\|^A)\sigma(S_{nt})^{-q}\|Z_n\|^3: n \in \mathbb{N}, t \in [0, 1]\} < \infty,$$

provided condition (1.4) holds.

**PROOF.** Since  $\|Z_n\| \leq n^\alpha$  we have  $\|S_{nt}\|^A \leq 2^{A-1}(1 + \|S_{nt} - n^{-1/2}Z_{nt}\|^A)$ . Conditioning on  $Z_n$  and  $Y_n$  we may proceed as in Lemma 3.15 using Hölder's inequality to get rid of the factor  $(1 + \|S_{nt} - n^{-1/2}Z_{nt}\|^A)$ , thereby replacing  $q$  by  $q(1 + \epsilon)$ ,  $F$  by  $F(\cdot + n^{-1/2}Z_n)$ , and  $n$  by  $n - 1$ . Using condition (1.4) we can carry out the proof of Lemma 3.15 with the additional factor  $\|Z_n\|^3$  for  $\gamma = 0$  and  $\gamma > 0$  as well.

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