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On the Recoverability of Nonlinear State Feedback Laws by Extended Linearization Control Techniques

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Abstract

Extended linearization is the process of factoring a nonlinear system into a linear-like structure $\dot{x} = A(x)x + B(x)u$ which contains state-dependent coefficient (SDC) matrices. An extended linearization control technique is any technique which (a) treats the SDC matrices $A(x)$ and $B(x)$ as being constant and (b) uses a linear control synthesis method on the linear-like structure to produce a closed-loop SDC matrix which is pointwise Hurwitz. This paper investigates the recoverability of nonlinear state feedback laws using extended linearization control techniques [1, 2] with particular focus on the state-dependent Riccati equation (SDRE) method [3]. By recoverable it is meant that a given nonlinear state feedback law of the form $u = k(x)$, $k(0) = 0$, can be obtained (or recovered) from a given control design method. Conditions relating to recoverability by extended linearization control methods are provided. An example is then presented where it is attempted to recover an optimal feedback law. It is shown that there exists no extended linearization control technique that is capable of recovering the given law. It is then shown how the feedback law can be recovered by using two control techniques which are variations of the SDRE method. The first uses a state-dependent state weighting matrix which may be negative definite or even indefinite over a subset of the state space while the second uses a nonsymmetric solution of the state-dependent Riccati equation which simultaneously satisfies a symmetry condition. Even though these latter techniques are based on the extended linearization of the system, they are not extended linearization control methods since they do not guarantee that the closed-loop SDC matrix is pointwise Hurwitz.

Notation

$\mathcal{C}^1$	Denotes the class of vector functions which are continuously differentiable
$col\{M\}$	Denotes the columns of a matrix M

1. Introduction

Consider the control–affine nonlinear system

$$\dot{x} = f(x) + B(x)u \quad (1)$$

where $x \in R^n$, $u \in R^m$ and where it is assumed that the vector fields $f(.)$ and $col\{B(.)\}$ are known $\mathcal{C}^1$ functions with $f(0) = 0$. Under these assumptions, it is well known [3, 4] that the nonlinear system (1) can be factored nonuniquely into the following linear–like structure having state-dependent coefficients (SDC):

$$\dot{x} = A(x)x + B(x)u \quad (2)$$

with $f(x) = A(x)x$ and $col\{A(.)\} \in \mathcal{C}^1$.

Definition 1 *The state dependent coefficient (SDC) representation (2) is a stabilizable parameterization of the nonlinear system (1) in a region Ω if the pair $\{A(x), B(x)\}$ is pointwise stabilizable in the linear sense for all $x \in \Omega$.*

Definition 2 *The SDC representation (2) is pointwise Hurwitz in a region Ω if the eigenvalues of $A(x)$ are in the open left half plane $Re(s) < 0$ for all $x \in \Omega$.*

Extended linearization control techniques represent a rather broad class of control design methods. The application of any linear control synthesis method to the linear-like SDC structure (2), where $A(x)$ and $B(x)$ are treated as constant matrices, forms an extended linearization control method. This leads to a control law of the form $u(x) = K(x)x$ that renders the closed-loop SDC matrix $A_{cl}(x) = A(x) + B(x)K(x)$ pointwise Hurwitz. While such techniques only guarantee local asymptotic stability [3] provided that $col\{K(.)\} \in \mathcal{C}^1$, surprisingly, empirical experience shows that in many cases the domain of attraction of these techniques may be as large as the domain of interest, e.g. [5, 6]. Nevertheless, given the lack of an *apriori* guarantee of either semiglobal or global asymptotic stability and given the wealth of well understood and theoretically supported nonlinear synthesis methods, extended linearization control techniques are usually not the method of choice when the only concern is to stabilize the system.

However, the situation changes when in addition to stability, the goal involves minimizing a performance index such as:

$$J(x_o, u) = \frac{1}{2}\int_0^\infty x'Q(x)x + u'R(x)u\,dt, \quad x(0) = x_o \quad (3)$$

where $Q(x) \geq 0$ and $R(x) > 0$ for all x with $col\{Q(.)\}$, $col\{R(.)\} \in \mathcal{C}^1$. In this case a recent workshop on nonlinear control [7] illustrated the fact that the performance of commonly used nonlinear design techniques (such as feedback linearization, control Lyapunov functions (CLF) and recursive backstepping) is highly problem dependent, ranging, for any given method, from near optimal to very poor.

It is well known [8] that the solution of the above problem can be obtained by solving the following Hamilton–Jacobi–Bellman (HJB) partial differential equation:

$$\frac{\partial V}{\partial x}f(x) - \frac{1}{2}\frac{\partial V}{\partial x}B(x)R^{-1}(x)B'(x)\frac{\partial V'}{\partial x} + \frac{1}{2}x'Q(x)x = 0 \quad (4)$$

with $V(0) = 0$. If this equation admits a $\mathcal{C}^1$ nonnegative solution V, then the optimal control is given by

$$u = -R^{-1}(x)B'(x)\frac{\partial V'}{\partial x} \quad (5)$$

and $V(x)$ is the corresponding optimal cost (or storage function), i.e.,

$$V(x) = \min_u \frac{1}{2}\int_0^\infty x'Q(x)x + u'R(x)u \; dt \quad (6)$$

Unfortunately, the complexity of equation (4) prevents its solution except in some very simple, low dimensional cases. This has prompted the search for alternative, suboptimal approaches to the problem, such as the SDRE approach [3]. Based on the linear-like parameterization (2), the SDRE method consists of solving pointwise along the trajectory the state-dependent algebraic Riccati equation:

$$A'(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B'(x)P(x) + Q(x) = 0 \quad (7)$$

After obtaining $P(x)$, the positive definite (pointwise stabilizing) solution of (7), the suboptimal control law is given by

$$u_{sdre} = -R^{-1}(x)B'(x)P(x)x \quad (8)$$

In [9, 10], Johnson addresses the benefits of allowing the state weighting matrix Q to be negative definite or even indefinite in linear quadratic regulator (LQR) design. The only requirement on Q is that it has to produce a stable closed-loop system, i.e., the closed-loop coefficient matrix has to be Hurwitz. Thus, Johnson's relaxed LQR design procedure applied to equation (2) would qualify as an extended linearization control technique. We will refer to this method as the relaxed Q–SDRE method. Additionally, we will consider an LQR synthesis applied to (2) which imposes no requirements on Q and will refer to it as the general Q–SDRE method. The general Q–SDRE method is not an extended linearization control technique since the closed-loop SDC matrix is not guaranteed to be pointwise Hurwitz.

Another alternative to the SDRE method [11] is to relax to required symmetry of the $P(x)$ matrix and solve the following state-dependent Riccati equation

$$A^{'}(x)P(x) + P^{'}(x)A(x) - P^{'}(x)B(x)R^{-1}(x)B^{'}(x)P(x) + Q(x) = 0 \quad (9)$$

in conjunction with the symmetry condition

$$P_{ij}(x) + \sum_{k=1}^{n}\frac{\partial P_{ik}(x)}{\partial x_j}x_k = P_{ji}(x) + \sum_{k=1}^{n}\frac{\partial P_{jk}(x)}{\partial x_i}x_k \quad (10)$$

with the control again being in the form of (8). Defining $\frac{\partial V'}{\partial x} = P(x)x$, it is easy to show that this is equivalent to solving the HJB equation and, at the present time, is as computationally difficult as solving equation (4). We will refer to this method as the nonsymmetric–SDRE method which is also not an extended linearization control technique.

The intent of this paper is to address the "recoverability" a given nonlinear state feedback law. In the next section, we define what is meant by "recoverable" and provide the necessary and sufficient conditions that must exist to make recoverability possible by some extended linearization control technique, and in particular, the SDRE control technique. In Section 3, we illustrate the recoverability concept with an example. The paper is then concluded with a summary section.

2. Recoverability by Extended Linearization Techniques

Consider the system (1), the factorization (2), and a given nonlinear state feedback control law of the form $u = k(x)$, $k(0) = 0$, where $k(.) \in \mathcal{C}^1$. Since we assume that $k(.)$ is $\mathcal{C}^1$, we know that $k(x)$ can be represented nonuniquely as

$$k(x) = K(x)x \quad (11)$$

Definition 3 *A continuously differentiable ($\mathcal{C}^1$) control law $u = k(x)$, $k(0) = 0$ is said to be recoverable by extended linearization control in a region Ω (i.e., is said to be EL–recoverable in a region Ω) if there exists a pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$ and an underlying linear control synthesis method which, based on $\{A(x), B(x)\}$, is capable of producing a state-dependent gain $K(x)$ satisfying $k(x) = K(x)x$ for all $x \in \Omega$.*

In particular, we have the following.

Definition 4 *A $\mathcal{C}^1$control law $u = k(x)$, $k(0) = 0$ is said to be SDRE–recoverable in a region Ω if there exist a pointwise stabilizable SDC parametrization $\{A(x), B(x)\}$, a pointwise nonnegative definite state weighting matrix $Q(x)$, and a pointwise positive definite control weighting matrix $R(x)$ such that the resulting state-dependent gain $K(x) = -R^{-1}(x)B(x)'P(x)$ satisfies $k(x) = K(x)x$ for all $x \in \Omega$.*

Theorem 1 *A C^1 control law $u = k(x)$, $k(0) = 0$ is EL-recoverable in a region Ω if and only if there exist a control law SDC parameterization $k(x) = K(x)x$ and a pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$ such that $A(x) + B(x)K(x)$ is pointwise Hurwitz in Ω.*

Proof: Necessity follows from the fact that extended linearization control laws are of the form $u(x) = K(x)x$ with $A_{cl}(x) = A(x) + B(x)K(x)$ being pointwise Hurwitz for some pointwise stabilizable pair $\{A(x), B(x)\}$ satisfying (2). On the other hand, if $k(x)$ can be written as $K(x)x$ with $A(x) + B(x)K(x)$ being pointwise Hurwitz for some pair $\{A(x), B(x)\}$ satisfying (2), then the control law $u(x)$ can be recovered by simply using pole placement pointwise for the pair $\{A(x), B(x)\}$. □

Note that Theorem 1 gives the necessary and sufficient conditions for the existence of an extended linearization control method which is capable of recovering a given control law $u = k(x)$, $k(0) = 0$. However, this does not mean that u can be recovered using a specific extended linearization control technique, since the underlying linear control synthesis method for a specific EL-control technique may restrict pole placement, as is the case for the SDRE method. To obtain SDRE-recoverability, a given control law must additionally satisfy the following pointwise minimum-phase property ([12], page 108) inherited from Kalman's Inequality [13] in the underlying LQR synthesis method.

Pointwise Minimum–Phase Property : Pointwise, the zeros of the loop gain $K(x)\left[sI - A(x)\right]^{-1}B(x)$ lie in the closed left half plane $Re(s) \leq 0$. Thus, we have the following:

Theorem 2 *A C^1 control law $u = k(x)$, $k(0) = 0$ is SDRE-recoverable in a region Ω if and only if (1) there exist a control law SDC parameterization $k(x) = K(x)x$ and a pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$ such that $A(x) + B(x)K(x)$ is pointwise Hurwitz in Ω and (2) the gain $K(x)$ satisfies the pointwise minimum-phase property in Ω.*

Proof: For necessity, due to the underlying LQR synthesis method, we have the fact that for any pointwise stabilizable pair $\{A(x), B(x)\}$, SDRE control is of the form $u(x) = K(x)x$ where $K(x) = -R^{-1}(x)B'(x)P(x)$ and can only produce (1) an $A_{cl}(x) = A(x) + B(x)K(x)$ that is pointwise Hurwitz and (2) a gain $K(x)$ that satisfies the pointwise minimum-phase property. Going the other way, if $k(x)$ can be written as $K(x)x$ with $A(x)+B(x)K(x)$ being pointwise Hurwitz in Ω and with $K(x)$ satisfying the pointwise minimum-phase property in Ω for some pair $\{A(x), B(x)\}$ which is pointwise stabilizable in Ω, then there exist a pointwise nonnegative definite $Q(x)$ and a pointwise positive definite $R(x)$ such that $-R^{-1}B'(x)P(x) = K(x)$ in Ω. □

Theorems 1 and 2 provide the necessary and sufficient conditions for recoverability. However, it may be difficult to apply these theorems due to the fact that there are an infinite number of SDC parameterizations. A more practical concept of recoverability follows.

Definition 5 *Given a selected pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$, a C^1 control law $u = k(x)$, $k(0) = 0$ is said to be $EL_{\{A,B\}}$–recoverable in a region Ω if there exists an underlying linear control synthesis method which, based on $\{A(x), B(x)\}$, is capable of producing a state-dependent gain $K(x)$ satisfying $k(x) = K(x)x$ for all $x \in \Omega$.*

Definition 6 *Given a selected pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$, a C^1 control law $u = k(x)$, $k(0) = 0$ is said to be $SDRE_{\{A,B\}}$–recoverable in a region Ω if there exist a pointwise nonnegative definite state weighting matrix $Q(x)$ and a pointwise positive definite control weighting matrix $R(x)$ such that the resulting state-dependent gain $K(x) = -R^{-1}(x)B(x)'P(x)$ satisfies $k(x) = K(x)x$ for all $x \in \Omega$.*

Corollary 1 *Given a selected pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$, a C^1 control law $u = k(x)$, $k(0) = 0$ is $EL_{\{A,B\}}$–recoverable in a region Ω if and only if there exists a control law SDC parameterization $k(x) = K(x)x$ such that $A(x) + B(x)K(x)$ is pointwise Hurwitz in Ω.*

Corollary 2 *Given a selected pointwise stabilizable SDC parameterization $\{A(x), B(x)\}$, a C^1 control law $u = k(x)$, $k(0) = 0$ is $SDRE_{\{A,B\}}$–recoverable in a region Ω if and only if (1) there exist a control law SDC parameterization $k(x) = K(x)x$ such that $A(x) + B(x)K(x)$ is pointwise Hurwitz in Ω and (2) the gain $K(x)$ satisfies the pointwise minimum-phase property in Ω.*

The proofs of Corollaries 1 and 2 are similar to those of Theorems 1 and 2, respectively, differing only due to the fact that the SDC pair $\{A(x), B(x)\}$ is now given. Finally, we provide the following lemma and theorem which will be used in Section 3.

Lemma 1 *Given an SDC matrix $A_o(x), col\{A_o(.)\} \in \mathcal{C}^1$, any other SDC matrix $A_1(x), col\{A_1(.)\} \in \mathcal{C}^1$ can be written in the form $\tilde{A}(x) = A_o(x) + \hat{A}(x)$ where $\hat{A}(x)x = 0$.*

Proof: We can rewrite $A_1(x)$ as $A_o(x) + [A_1(x) - A_o(x)]$. Letting $\hat{A}(x) = [A_1(x) - A_o(x)]$ we have that $\hat{A}(x)x = 0$ since $A_1(x)x = A_o(x)x = f(x)$. □

Theorem 3 *Assume that $A_o(x)$, $col\{A_o(.)\} \in \mathcal{C}^1$ is an SDC parametrization of (1) such that $A_o(x)$ is pointwise non–Hurwitz in a region Ω. Let $\lambda_i(x), i = 1, \ldots, r$ be the unstable eigenvalues of $A_o(x)$ having the respective multiplicities of $m_i(x), i = 1, \ldots, r$ with $\sum_{i=1}^{r} m_i(x) \leq n$. Let $E_i(x)$ be the set of all linearly independent eigenvectors corresponding to each $\lambda_i(x)$. Finally, let $\mathcal{S}_i(x) \doteq span\{E_i(x)\}$. Then, if for some $x \in \Omega$, $x \neq 0$, we have that $x \in \Omega \bigcap \mathcal{S}_i(x)$ for some $1 \leq i \leq r$, there is no SDC parametrization of (1) that is pointwise Hurwitz in all of Ω.*

Proof: Without loss of generality, assume that there exists $x^* \in \Omega \bigcap span\{E_1(x^*)\}$ and consider any other SDC matrix of the form:

$$\tilde{A}(x) = A_o(x) + \hat{A}(x) \tag{12}$$

with $\hat{A}(x)x = 0$ for all x. After evaluating (12) at the point x^*, postmultiplication of (12) by x^* yields

$$\begin{aligned} \tilde{A}(x^*)x^* &= A_o(x^*)x^* + \hat{A}(x^*)x^* \\ &= A_o(x^*)x^* = \lambda_1(x^*)x^* \end{aligned} \qquad (13)$$

Hence, $\lambda_1(x^*)$ is also an eigenvalue of $\tilde{A}(x^*)$. Since $x^* \in \Omega$, $Re[\lambda_1(x^*)] \geq 0$. Thus, $\tilde{A}(x)$ cannot be pointwise Hurwitz in all of Ω, and from Lemma 1, there is no SDC parameterization that can be pointwise Hurwitz in all of Ω. □

3. Recoverability Example

Consider the following nonlinear regulator problem:

$$\min_u \left\{ J = \frac{1}{2}\int_0^\infty x_2^2 + u^2 dt \right\} \qquad (14)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 e^{x_1} + \tfrac{1}{2}x_2^2 + e^{x_1}u \end{aligned} \qquad (15)$$

It can be shown that the optimal control law is given by:

$$u_{opt} = -x_2 \qquad (16)$$

with the corresponding optimal storage function:

$$V(x) = \frac{1}{2}[x_1^2 + x_2^2 e^{-x_1}] \qquad (17)$$

Using the SDC parameterization:

$$A_o(x) = \begin{bmatrix} 0 & 1 \\ -e^{x_1} & \frac{1}{2}x_2 \end{bmatrix}; \quad B(x) = \begin{bmatrix} 0 \\ e^{x_1} \end{bmatrix} \qquad (18)$$

where $col\{A_o(.)\}, B(.) \in \mathcal{C}^1$, it can be shown that the solution of the corresponding SDRE is given by:

$$P(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 1 \end{bmatrix} p \qquad (19)$$

where

$$p = 0.5 * \left(x_2 + \sqrt{x_2^2 + 4e^{2*x_1}} \right) e^{-2x_1} \qquad (20)$$

with associated control action:

$$u_{sdre} = -x_2 \left[\left(\frac{x_2}{2e^{x_1}} \right) + \sqrt{1 + \left(\frac{x_2}{2e^{x_1}} \right)^2} \right] \qquad (21)$$

Thus $u_{sdre} \cong u_{opt}$ only when $|.5\, x_2 e^{-x_1}| \ll 1$. On the other hand, if $|.5\, x_2 e^{-x_1}| \gg 1$ then $|\frac{u_{sdre}}{u_{opt}}| \gg 1$. For instance, starting from the initial condition $x(0) = [-2 \quad 2]$ we have that $J_{optimal} = 33.55$ and $J_{sdre} = 154.73$. Note here that initially $.5\, x_2 e^{-x_1} = e^2 \cong 7.4$ so the difference between J_{sdre} and $J_{optimal}$ is not surprising.

Further insight can be gained by using Theorems 1 and 3. From the necessity part of Theorem 1, we know that for the given control law (16) to be recoverable by any extended linearization control method, there has to exist both $K(x)$ such that $K(x)x = -x_2$ and $\{A(x), B(x)\}$ such that $A(x)+B(x)K(x)$ is pointwise Hurwitz. Selecting the SDC parameterization (18) and $K = [0 \quad -1]$ yields:

$$A_o(x) + B(x)K(x) = \begin{bmatrix} 0 & 1 \\ -e^{x_1} & \frac{1}{2}x_2 - e^{x_1} \end{bmatrix} \qquad (22)$$

It can be easily seen that

$$A_{cl}(x) \doteq A_o(x) + B(x)K(x) \qquad (23)$$

is non–Hurwitz in the region $.5\, x_2 e^{-x_1} \geq 1$.

We will now use Theorem 3 to show that there exist no parameterizations $\{A(x), B(x)\}$ of the above problem that will yield a closed-loop $A_{cl}(x)$ which is Hurwitz everywhere. Let

$$\Omega \doteq \left\{ x = [x_1 \quad x_2]' : \frac{x_2}{2e^{x_1}} = \epsilon \right\}$$

where $\epsilon \gg 1$. For $x \in \Omega$, the eigenvalues of $A_{cl}(x)$ in (23) are $\lambda_1 = \frac{1}{2}x_2$ and $\lambda_2 = \frac{1}{2}x_2\epsilon^2$. It can be easily verified that

$$x^* \doteq \begin{bmatrix} 2 \\ 2\epsilon e^2 \end{bmatrix}, \ \epsilon \gg 1 \qquad (24)$$

is such that $x^* \in \Omega \bigcap span\{E_1(x^*)\}$. From Theorem 3 we have that there exixts no SDC parametrization of the closed–loop system that can be Hurwitz in all of Ω. Hence the optimal control law cannot be globally recovered by any extended linearization control method, including the SDRE method and the relaxed Q–SDRE method.

We now use two control methods which are not extended linearization control techniques to recover u_{opt}. The first method is the general Q–SDRE method where we allow $Q(x)$ to be negative definite or even indefinite pointwise over part of the state space. Setting

$$Q(x) = 4\begin{bmatrix} 0 & 0 \\ 0 & (1 - 0.5x_2 e^{-x_1}) \end{bmatrix}; \ R = 1 \qquad (25)$$

yields

$$P(x) = 2e^{-x_1}\begin{bmatrix} e^{x_1} & 0 \\ 0 & 1 \end{bmatrix} \qquad (26)$$

and $u = -0.5 * R^{-1}B'(x)P(x)x = -x_2 = u_{opt}$. Note that in this case $Q(x)$ is pointwise nonnegative definite only in the region $\Omega = \{x : 0.5x_2 e^{-x_1} \leq 1\}$ and this is precisely the region where u_{opt} is recoverable via the standard SDRE method.

The second method is the nonsymmetric–SDRE method. For this simple problem, we can obtain a closed-form analytic solution of the Riccati equation (9) in conjunction with the symmetry condition (10) using the symbolic software package Macsyma. Selecting the parameterization (18) and choosing Q and R according to (14), i.e.,

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ R = 1 \tag{27}$$

four symmetric $P(x)$ solutions of (9) were obtained, one of which was pointwise positive definite and represented the SDRE solution for the standard SDRE method. However, the symmetric positive definite solution did not satisfy the symmetry condition (10). Additionally, the following two nonsymmetric $P(x)$ solutions were obtained where c_1 is an arbitrary constant:

$$P_1(x) = \begin{bmatrix} c_1 & -.5(c_1 x_2 e^{-x_1} + 1 - c_1^2) \\ 0 & c_1 e^{-x_1} \end{bmatrix} \tag{28}$$

$$P_2(x) = \begin{bmatrix} c_1 & -.5(c_1 x_2 e^{-x_1} + 1 - c_1^2) \\ -2e^{-x_1} & x_2 e^{-2x_1} - c_1 e^{-x_1} \end{bmatrix} \tag{29}$$

Substituting $P_2(x)$ into (10), it was found that no constant c_1 could satisfy the symmetry condition. Upon substituting $P_1(x)$ into (10), the symmetry condition was satisfied with $c_1 = 1$. Thus,

$$P_1(x) = \begin{bmatrix} 1 & -.5x_2 e^{-x_1} \\ 0 & e^{-x_1} \end{bmatrix} \tag{30}$$

which yields

$$\begin{aligned} u_{opt} &= -R^{-1}B'(x)P_1(x)x & (31) \\ &= -[0 \quad e^{x_1}] \begin{bmatrix} 1 & -.5x_2 e^{-x_1} \\ 0 & e^{-x_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & (32) \\ &= -x_2 & (33) \end{aligned}$$

4. Summary

Necessary and sufficient conditions for the recoverability of a given nonlinear state feedback law by extended linearization control techniques, and in particular, by the SDRE method, have been presented. An example was provided to illustrate the fact that there exist nonlinear state feedback laws which are unrecoverable using EL-control methods. The example also showed that by relaxing the Hurwitz requirements on the closed-loop coefficient matrix of the underlying linear control synthesis method (in this case, the LQR synthesis method), recoverability can be achieved. Finally, it was illustrated in the example that the nonsymmetric SDRE method provides an alternative approach to solving the Hamilton-Bellman-Jacobi equation. While presently this approach is just as difficult as solving the HBJ equation, there are structural differences between the nonsymmetric–SDRE method and the HBJ equation. Therefore it may be worthwhile investigating numerical approaches for solving or approximately solving the nonsymmetric SDRE in conjunction with the symmetry condition.

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