QUARTERLY OF APPLIED MATHEMATICS
JANUARY, 1975

ON THE REDUCTION OF THE HODOGRAPH EQUATIONS FOR ONE-DIMENSIONAL ELASTIC-PLASTIC WAVE PROPAGATION

By C. ROGERS and D. L. CLEMENTS (*University of Western Ontario*)

1. Introduction. One-dimensional wave propagation in an elastic-plastic bar is governed by a system of two first-order partial differential equations. In the elastic region, this system is hyperbolic and linear with constant coefficients; in the plastic region the system is quasilinear but may be linearized by a hodograph transformation. The object of the present paper is to ascertain under what conditions a canonical form of the resulting hodograph equations may be further reduced to a form associated with the wave equation. To this end, finite Baecklund transformations are introduced, and it emerges that such reduction may be achieved subject to the stress-strain law of the material under consideration adopting certain multiparameter forms. It is interesting to observe that one of these forms has already been found useful for the explicit integration of the equations for elastic-plastic wave propagation (see Courant and Friedrichs [1, p. 246]). Here, such results are placed in the broader context of Baecklund transformation theory. It should be noted that the forms of the stress-strain relations given here can be considered of practical use only if it is possible to fit these relations to the stress-strain curves describing the behaviour of real materials. In Sec. 3, we demonstrate, as an example, that close approximation to the stress-strain curve for a cylinder of cold-rolled steel may be readily achieved using one of the forms.

2. The hodograph equations. One-dimensional wave propagation in an elastic-plastic bar assumed to exhibit isotropic strain-hardening behavior, in the absence of strain-rate and Bauschinger effects, is governed in the plastic region by (Lee [12])

$$\Omega_t = \begin{bmatrix} 0 & \rho_0 c^2(\sigma) \\ \rho_0^{-1} & 0 \end{bmatrix} \Omega_x , \tag{2.1}$$

where

$$\Omega = \begin{bmatrix} \sigma \\ v \end{bmatrix} , \tag{2.2}$$

together with the loading condition $\sigma_t > 0$ for tension or $\sigma_t < 0$ for compression, σ being assumed to be positive in tension. Here σ represents the force per unit initial cross-sectional area and $v = \partial u/\partial t$ where u denotes the particle displacement from its initial position x in the unstrained rod. Further, ρ_0 represents the density of the undisturbed medium and $c(\sigma)$ is the speed of propagation of a disturbance. This speed is related to the gradient of the stress-strain curve by the equation

$$d\sigma/d\epsilon = \rho_0 c^2(\sigma) \tag{2.3}$$

* Received June 18, 1973.

where the strain ϵ is change in length per unit initial length. For the elastic region, the wave propagation is governed by

$$\Omega_t = \begin{pmatrix} 0 & \rho_0 c_0^2 \\ \rho_0^{-1} & 0 \end{pmatrix} \Omega_x , \tag{2.4}$$

where c_0 is the speed of propagation in the elastic region. The system (2.1) can be linearized by introducing

$$\phi = \frac{1}{\rho_0 c_0} \int_0^\sigma \frac{c_0}{c(\sigma)} \, d\sigma, v, \tag{2.5}$$

as a new independent and x, t as new dependent variables. Subject to this hodograph transformation, the following relations hold:

$$\begin{aligned} \sigma_x &= J^{-1} t_v \sigma_\phi , & \sigma_t &= -J^{-1} x_v \sigma_\phi , \\ v_x &= -J^{-1} t_\phi , & v_t &= J^{-1} x_\phi , \end{aligned} \tag{2.6}$$

where $0 < |J| < \infty$ and J is defined by

$$J = \begin{vmatrix} x_\phi & x_v \\ t_\phi & t_v \end{vmatrix} .$$

The transformations (2.6) take (2.1) to

$$\Lambda_\phi = H \Lambda_v , \tag{2.7}$$

where

$$\Lambda = \begin{pmatrix} t \\ x \end{pmatrix} , \qquad H = \begin{pmatrix} 0 & K(\phi)^{-1} \\ K(\phi) & 0 \end{pmatrix} , \qquad K(\phi) = c(\sigma). \tag{2.8}$$

The object of this note is to determine stress-strain relations for which the system defined by (2.7) and (2.8) may be transformed by finite Baecklund transformations to one associated with the wave equation, namely,

$$\Lambda' = \begin{pmatrix} t' \\ x' \end{pmatrix} , \qquad H' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.9}$$

3. The finite Baecklund transformations. Transformations $\xi_j \to \xi_j^*$, $j = 1, 2$, $U \to U^*$ defined by relations of the form

$$\mathcal{B}_i(\xi_1 , \xi_2 , U, U_{\xi_1} , U_{\xi_2} ; \xi_1^*, \xi_2^*, U^*, U_{\xi_1^*}^*, U_{\xi_2^*}^*) = 0, i = 1, 2, \cdots , 4, \tag{3.1}$$

where $U_{\xi_i} \equiv \partial U/\partial \xi_i$, $U_{\xi_i^*}^* \equiv \partial U^*/\partial \xi_i^*$, were introduced by Baecklund [3] in connection with the transformation of surfaces between three-dimensional spaces with coordinates (ξ_1 , ξ_2 , U) and (ξ_1^*, ξ_2^*, U^*) respectively. More generally, Baecklund-type transformations of the form

$$\begin{gathered} \mathcal{B}_i(\xi_1 , \xi_2 , U_1 , U_2 , \cdots , U_m , U_{1,\xi_1} , \cdots , U_{m,\xi_1} , U_{1,\xi_2} , \cdots , U_{m,\xi_2} ; \\ \xi_1^*, \xi_2^*, U_1^*, U_2^*, \cdots , U_m^*, U_{1,\xi_1^*}^*, \cdots , U_{m,\xi_1^*}^*, U_{1,\xi_2^*}^*, \cdots , U_{m,\xi_2^*}^*) = 0, \\ i = 1, 2, \cdots , 2m + 2, \end{gathered}$$

where $U_{m,\xi_i} \equiv \partial U_m/\partial \xi_i$, $U_{m,\xi_i^*}{}^* \equiv \partial U^*/\partial \xi_i^*$, may be applied to general systems of m linear first-order partial differential equations.

$$\sum_{k=1}^{m} \alpha_{ik} U_{k,\xi_2} + \sum_{k=1}^{m} \beta_{ik} U_{k,\xi_1} + \sum_{k=1}^{m} \gamma_{ik} U_k + \delta_i = 0, \qquad i = 1, 2, \cdots, m, \tag{3.3}$$

for the m functions in the two independent variables ξ_j , $j = 1, 2$ (Rogers [4]). In the present context, restricting attention to the case $m = 2$, it is noted that Baecklund transformations may be applied to a quasilinear system of the form

$$\sum_{k=1}^{2} a_{ik} U_{k,\xi_2} + \sum_{k=1}^{2} b_{ik} U_{k,\xi_1} = 0 \qquad i = 1, 2, \tag{3.4}$$

where $a_{ik} \equiv a_{ik}(U_j)$, $b_{ik} \equiv b_{ik}(U_j)$, $0 < |J(U_1 , U_2 ; \xi_1 , \xi_2)| < \infty$, since then the role of dependent and independent variables may be interchanged and a linear system obtained by using the hodograph transformation. Such an interchange has been performed in the preceding section for the characterizing set of equations under investigation and Baecklund transformations of the type (3.2) may now be applied. In fact, we shall be concerned with a subclass of transformations of the latter form, namely, with Baecklund transformations defined by

$$\Lambda'_{\phi'} = A\Lambda_\phi + B\Lambda + C\Lambda', \ |A| \neq 0, \tag{3.5}$$

$$\Lambda'_{v'} = \tilde{A}\Lambda_v + \tilde{B}\Lambda + \tilde{C}\Lambda', \ |\tilde{A}| \neq 0, \tag{3.6}$$

$$\phi' = \phi, \qquad v' = v, \tag{3.7}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, C are in turn 2×2 matrices $[\tilde{a}_j^{\,i}]$, $[\tilde{b}_j^{\,i}]$, $[\tilde{c}_j^{\,i}]$, $[a_j^{\,i}]$, $[b_j^{\,i}]$, $[c_j^{\,i}]$, i, $j = 1, 2$, with entries functions of ϕ and v. Transformations of this type are sought which transform

$$\Lambda_\phi = H\Lambda_v \rightarrow \Lambda_{\phi'}{}' = H'\Lambda_{v'}{}', \tag{3.8}$$

where H and H' are defined by (2.8) and (2.9) respectively. Transformations of the type (3.5)–(3.7) were introduced by Loewner [5] in connection with the reduction to cononical form of the hodograph equations in subsonic, transonic and supersonic flow. It emerges that reduction may be achieved subject to the real gas pressure-density relation being approximated by certain multiparameter forms. Various important approximations of gasdynamics such as the well-known Kármán–Tsien relation may be extracted as particular cases of the theory. In analogous fashion, in this paper, it is shown that reduction of the system (2.8) to one associated with the wave equation may be made provided the prevailing stress-strain relation may be approximated by certain three-parameter expressions.

It is assumed that t, X, t', X' have continuous mixed second-order derivatives with respect to the hodograph variables ϕ, v, so that the commutativity conditions

$$\Lambda_{\phi v} = \Lambda_{v\phi} , \ \Lambda_{\phi' v'}{}' = \Lambda_{v'\phi'}{}', \tag{3.9}$$

hold. Thus, employing these conditions and the Baecklund transformation relations (3.5)–(3.7), it is seen that conditions (3.9) will hold if

$$(A - \tilde{A})\Lambda_{\phi v} + (A_v - \tilde{B})\Lambda_\phi + (B - \tilde{A}_\phi)\Lambda_v + (B_v - \tilde{B}_\phi)\Lambda + C_v\Lambda' + C\Lambda_v' - \tilde{C}_\phi\Lambda' - \tilde{C}\Lambda_{\phi'}{}' = 0. \tag{3.10}$$

Using (3.5) and (3.6) to substitute for $\Lambda_{v'}'$ and $\Lambda_{\phi'}'$ in (3.10), we obtain

$$(A - \tilde{A})\Lambda_{\phi v} + (A_v - \tilde{B} - \tilde{C}A)\Lambda_\phi + (B + C\tilde{A} - \tilde{A}_\phi)\Lambda_v + (B_v - \tilde{B}_\phi - \tilde{C}B + C\tilde{B})\Lambda + (C_v - \tilde{C}_\phi + C\tilde{C} - \tilde{C}C)\Lambda' = 0, \tag{3.11}$$

which, since $\Lambda_\phi = H\Lambda_v$, is identically satisfied by setting

$$A = \tilde{A}, \tag{3.12}$$

$$(A_v - \tilde{B} - \tilde{C}A)H + (B + C\tilde{A} - \tilde{A}_\phi) = 0, \ |A_v - \tilde{B} - \tilde{C}A| \neq 0, \tag{3.13}$$

$$B_v - \tilde{B}_\phi - \tilde{C}B + C\tilde{B} = 0, \tag{3.14}$$

$$C_v - \tilde{C}_\phi + C\tilde{C} - \tilde{C}C = 0. \tag{3.15}$$

Returning to (3.8) it follows, from (3.5), (3.6) and (3.12), that

$$\Lambda_{\phi'}' - H'\Lambda_{v'}' = A[\Lambda_\phi - A^{-1}H'A\Lambda_v] + (B - H'\tilde{B})\Lambda + (C - H'\tilde{C})\Lambda', \ |A| \neq 0,$$

so that, setting

$$A^{-1}H'A = H \tag{3.16}$$

$$B = H'\tilde{B}, \ C = H'\tilde{C}, \tag{3.17}$$

the system $\Lambda_\phi = H\Lambda_v$ is transformed to the associated system $\Lambda_{\phi'}' = H'\Lambda_{v'}'$ and conversely via the Baecklund transformations defined by (3.5)–(3.7).

Summarizing, it has been established that

$$\Lambda_\phi = H\Lambda_v \leftrightarrow \Lambda_{\phi'}' = H'\Lambda_{v'}'$$

via the Baecklund transformations (3.5)–(3.7) subject to the conditions (3.12)–(3.17). This result was first obtained by Loewner [5] in a gasdynamic context and has recently been extended by Rogers [4] to general systems of m linear first-order partial differential equations. We now proceed to investigate the system (3.12)–(3.17). If we choose A $(= \tilde{A})$ and B, $\tilde{B}$ independent of v and ϕ respectively and $C = \tilde{C} = 0$, it follows that (3.15) is satisfied while (3.14) will be satisfied if B is a constant matrix. With B constant, it is immediately apparent from (3.17) that $\tilde{B}$ is also a constant matrix. Eq. (3.13) now reduces to

$$\tilde{A}_\phi - H'\tilde{B} + \tilde{B}\tilde{A}^{-1}H'\tilde{A} = 0 \tag{3.18}$$

where (3.16), (3.17) and (3.12) have been used to eliminate B, H and A. It is now necessary to specialize the matrix $A = \tilde{A}$ so that the property of zero principal diagonal elements is preserved under the mapping $H \to H'$. From (3.16) it is clear that this property is invariant if (but not only if) A adopts the diagonal form

$$A = \begin{pmatrix} a_1^{\ 1} & 0 \\ 0 & a_2^{\ 2} \end{pmatrix}, \tag{3.19}$$

in which case

$$H' = AHA^{-1} = \begin{pmatrix} 0 & a_1^{\ 1}h_2^{\ 1}/a_2^{\ 2} \\ a_2^{\ 2}h_1^{\ 2}/a_1^{\ 1} & 0 \end{pmatrix}. \tag{3.20}$$

Now

$$\tilde{A}^{-1}H'\tilde{A} = \begin{bmatrix} 0 & a_2{}^2h_2{}^{1'}/a_1{}^1 \\ a_1{}^1h_1{}^{2'}/a_2{}^2 & 0 \end{bmatrix}$$

so in (3.18)

$$\begin{bmatrix} a_1{}^1 & 0 \\ 0 & a_2{}^2 \end{bmatrix}_\phi - \begin{bmatrix} 0 & h_2{}^{1'} \\ h_1{}^{2'} & 0 \end{bmatrix} \begin{bmatrix} \tilde{b}_1{}^1 & \tilde{b}_2{}^1 \\ \tilde{b}_1{}^2 & \tilde{b}_2{}^2 \end{bmatrix} + \begin{bmatrix} \tilde{b}_1{}^1 & \tilde{b}_2{}^1 \\ \tilde{b}_1^2 & \tilde{b}_2{}^2 \end{bmatrix} \begin{bmatrix} 0 & a_2{}^2h_2{}^{1'}/a_1{}^1 \\ a_1{}^1h_1{}^{2'}/a_2{}^2 & 0 \end{bmatrix} = 0.$$

so the constant vector $\tilde{B}$ is necessarily of the form

$$\tilde{B} = \begin{bmatrix} 0 & \tilde{b}_2{}^1 \\ \tilde{b}_1{}^2 & 0 \end{bmatrix},$$

and the above matrix equation reduces to

$$(a_1{}^1)_\phi - h_2{}^{1'}\tilde{b}_1{}^2 + h_1{}^{2'}\tilde{b}_2{}^1(a_1{}^1/a_2{}^2) = 0, \tag{3.21}$$

$$(a_2{}^2)_\phi - h_1{}^{2'}\tilde{b}_2{}^1 + h_2{}^{1'}\tilde{b}_1{}^2(a_2{}^2/a_1{}^1) = 0. \tag{3.22}$$

Eqs. (3.21) and (3.22) combine to show that

$$\det A = a_1{}^1a_2{}^2 = \text{constant} = \lambda, \lambda \neq 0,$$

whence the system (3.21), (3.22) may be reduced to a single Riccati-type equation in either $a_1{}^1$ or $a_2{}^2$. In particular, the Riccati equation in $a_1{}^1$ is, on setting $h_2{}^{1'} = h_1{}^{2'} = 1$,

$$(a_1{}^1)_\phi + \alpha(a_1{}^1)^2 + \beta = 0, \alpha = \tilde{b}_2{}^1\lambda^{-1}, \beta = -\tilde{b}_1{}^2. \tag{3.23}$$

Thus:

a) If $\alpha = 0$, $a_1{}^1 = -\beta\phi + \delta$,
b) If $\beta = 0$, $a_1{}^1 = 1/(\alpha\phi + \epsilon)$,
c) If $\beta/\alpha > 0$, $a_1{}^1 = (\beta/\alpha)^{1/2} \cot\{(\beta/\alpha)^{1/2}(\alpha\phi + \zeta)\}$,
d) If $\beta/\alpha < 0$, $a_1{}^1 = (-\beta/\alpha)^{1/2} \coth\{(-\beta/\alpha)^{1/2}(\alpha\phi + \eta)\}$,

where ϕ, ϵ, ζ, η are arbitrary constants of integration. Now, upon choosing $h_2{}^{1'} = h_1{}^{2'} = 1$, it follows from (3.20) and (2.8) that

$$h_2{}^1 = a_2{}^2/a_1{}^1 = K(\phi)^{-1}, h_1{}^2 = a_1{}^1/a_2{}^2 = K(\phi). \tag{3.24}$$

Hence

$$K(\phi) = a_1{}^1/a_2{}^2 = (a_1{}^1)^2/\lambda, \tag{3.25}$$

whence, in the cases (a)–(d) above, in turn,

a) $K(\phi) = \lambda^{-1}[-\beta\phi + \delta]^2$,
b) $K(\phi) = \lambda^{-1}[\alpha\phi + \epsilon]^{-2}$,
c) $K(\phi) = \lambda^{-1}(\beta/\alpha) \cot^2\{(\beta/\alpha)^{1/2}(\alpha\phi + \xi)\}$,
d) $K(\phi) = \lambda^{-1}(-\beta/\alpha) \coth^2\{(-\beta/\alpha)^{1/2}(\alpha\phi + \eta)\}$.

The (σ, ϵ) relations in cases (a) and (b) are given below; in the cases (c) and (d) and (η, ϵ) laws may be generated parametrically via $\sigma = \sigma(\phi)$, $\epsilon = \epsilon(\phi)$ relations.

a) $K(\phi) = \lambda^{-1}[-\beta\phi + \delta]^2$. From (2.8), $c(\sigma) = K(\phi) = \lambda^{-1}[-\beta\phi + \delta]^2$, so that, on employing (2.5),

$$d\phi/d\sigma = \lambda\rho_0{}^{-1}[-\beta\phi + \delta]^{-2}, \tag{3.26}$$

whence

$$\phi = \{\delta + [3\beta\lambda(\sigma + \gamma)\rho_0{}^{-1}]^{1/3}\}\beta^{-1},$$

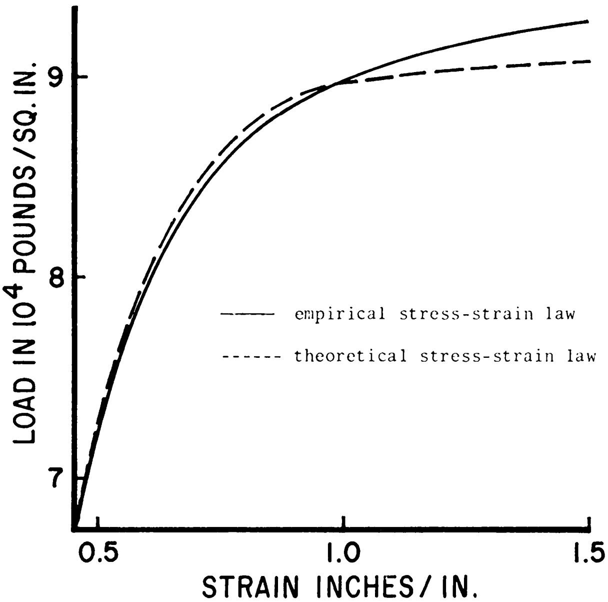

FIG. 1. $\lambda^2\rho_0/(3\beta^4) = -1223.994$, $\xi = -0.1$, $\gamma = -91181.203$.

where γ is a constant of integration. Relations (3.26) and (2.3) now show that $d\sigma/d\epsilon = \lambda^{-2}\rho_0[3\beta\lambda(\sigma + \gamma)\rho_0^{-1}]^{4/3}$, and hence

$$\sigma = -\tfrac{1}{3}\lambda^2\rho_0\beta^{-4}(\epsilon + \xi)^{-3} - \gamma, \tag{3.27}$$

where ξ is a further constant of integration.

b) $K(\phi) = \lambda^{-1}[\alpha\phi + \epsilon]^{-2}$. In a similar way, we may show that, in this case, the stress-strain relation adopts the form

$$\sigma = -\rho_0[(3\lambda^2\alpha^4)(\epsilon + \xi)]^{-1/3} + \delta \tag{3.28}$$

where ξ, δ are arbitrary constants.

In Fig. 1, the theoretical stress-strain law (3.27) has been employed to approximate an empirical stress-strain curve for a cylinder of cold-rolled steel. In fitting (3.27) to the actual stress-strain curve, the approximating curve was required to pass through the point at which plastic deformation began and also to have the same gradient at this point. Further, the theoretical model curve was required to pass through another convenient point on the real stress-strain curve with a view to obtaining acceptable alignment for the early stages of the plastic deformation; other ways of fitting the curves are available, of course, depending on the requirements of a specific problem. The theoretical stress-strain laws (b)–(d) are similarly available for approximation purposes

REFERENCES

[1] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience, New York (1948)

[2] E. H. Lee, *A boundary value problem in the theory of plastic wave propagation*, Quart. Appl. Math. **10**, 335–346 (1953)

[3] A. V. Baecklund, *Om ytor med konstant negativ krökning*, Lunds Universitets Årsskrift 19 (1883)

[4] C. Rogers, *Iterated Baecklund-type transformations and the propagation of disturbances in non-linear elastic materials*, J. Math. Anal. Appl., to appear

[5] C. Loewner, *A transformation theory of partial differential equations of gasdynamics*, NACA Technical Note 2065, 1–56 (1950)