

**On the reflexivity of pairs of isometries and
of tensor products of some operator algebras**

by

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Abstract. In the present paper we prove the reflexivity of a WOT-closed algebra generated by a pair of doubly commuting isometries. Our next result is the reflexivity of the tensor product of the algebra of all analytic Toeplitz operators on H^2 with any reflexive algebra.

1. Introduction and preliminaries. $L(K)$ denotes the algebra of all (linear bounded) operators in a complex separable Hilbert space K . I_K or I stands for the identity in K . By a subspace of K we always mean a closed subspace and by an algebra of operators on K we mean a subalgebra of $L(K)$ with unit I_K . If \mathcal{S} is subset of $L(K)$, then $\mathfrak{A}(\mathcal{S})$, $\text{Lat } \mathcal{S}$ stand for the WOT (= weak operator topology)-closed algebra generated by \mathcal{S} and the lattice of all invariant subspaces for \mathcal{S} , respectively. If $T \in L(K)$ then the shorter notation $\mathfrak{A}(T)$, $\mathfrak{A}(T, \mathcal{S})$, $\text{Lat } T$, $\text{Lat}(T, \mathcal{S})$ will be used instead of $\mathfrak{A}(\{T\})$, $\mathfrak{A}(\{T\} \cup \mathcal{S})$, $\text{Lat } \{T\}$, $\text{Lat}(\{T\} \cup \mathcal{S})$. $\text{Alg Lat } \mathcal{S}$ stands for the algebra of all operators on K which leave invariant all subspaces from $\text{Lat } \mathcal{S}$. An algebra \mathcal{A} is called *reflexive* if $\mathcal{A} = \text{Alg Lat } \mathcal{A}$. A family $\mathcal{S} \subset L(K)$ is called *reflexive* if the algebra $\mathfrak{A}(\mathcal{S})$ is reflexive. An operator $T \in L(K)$ is called *reflexive* if so is $\mathcal{S} = \{T\}$.

Sarason [11] proved that every commutative WOT-closed algebra of normal or Toeplitz operators is reflexive. The reflexivity of an isometry was proved by Deddens [2]. Another proof of this fact was given by Wogen [14]. He also proved that quasinormal operators are reflexive. Then Olin and Thomson [9] proved the reflexivity of subnormal operators.

In this paper we study the reflexivity of a pair (two-element family) of isometries. Deddens', Wogen's and Olin and Thomson's proofs needed a sort of canonical decomposition of an isometry, of a quasinormal and of a subnormal operator respectively. But a Wold-type decomposition which was necessary there need not exist for two commuting isometries [12, Example 1]. Słociński showed [12] that the Wold-type decomposition holds for any pair $\{V_1, V_2\}$ of *doubly commuting* isometries (i.e. V_1, V_2 commute and V_1, V_2^* commute). Our first main result is:

THEOREM 1. Every pair $\{V_1, V_2\}$ of doubly commuting isometries is reflexive.

This problem is connected with the reflexivity of the tensor product of reflexive algebras. There are some partial result concerning this problem in [1], [4], [5], [7], [8]. In this paper we prove another result in this direction. Let \mathcal{A}, \mathcal{B} be WOT-closed algebras of operators on Hilbert spaces K, H , respectively. Then, following [8], $\mathcal{B} \overline{\otimes} \mathcal{A}$ denotes the WOT-closure in $L(H \otimes K)$ of the algebraic tensor product $\mathcal{B} \otimes \mathcal{A}$ (for the definition of tensor product see [13, Chap. IV]). We denote by H^2 the classical Hardy space on the unit circle and \mathcal{H}^∞ denotes the collection of all analytic Toeplitz operators on H^2 . Then we can prove

THEOREM 2. For any algebra \mathcal{A}_0 in $L(K)$, $\mathcal{H}^\infty \overline{\otimes} \mathcal{A}_0$ is reflexive whenever \mathcal{A}_0 is \mathcal{A}_0 .

Now, let us fix a separable Hilbert space K with the inner product (\cdot, \cdot) . If $\mathcal{L} \subset K$ then $[\mathcal{L}]$ denotes the smallest subspace of K containing \mathcal{L} . Let $T \in L(K)$ and let L belong to $\text{Lat } T$. Then $T|_L$ denotes the restriction of T to L . If n is a positive integer, then $K^{(n)}$ denotes the direct sum of n copies of K , $T^{(n)}$ denotes the direct sum of n copies of T acting on $K^{(n)}$. If $\mathcal{T} \subset L(K)$ then we denote $\mathcal{T}^{(n)} = \{T^{(n)} : T \in \mathcal{T}\}$. If $x \in K$ then $C(\mathcal{T}, x)$ denotes the smallest subspace of K containing x and invariant for \mathcal{T} . Following [10, Chap. 3] we recall some definitions. A function f from the unit circle C into K will be called measurable if, for each $x \in K$, the function $z \rightarrow (f(z), x)$ is Lebesgue measurable. $L^2(K)$ denotes the collection of all measurable functions f from C to K such that $\int \|f(z)\|^2 dm < \infty$, where m is the normalized Lebesgue measure on C . We identify functions which are equal m -a.e. $H^2(K)$ denotes the set of functions $f \in L^2(K)$ such that $\int (f(z), x) z^n dm = 0$ for all $x \in K$ and $n \geq 1$. If K is the field of complex numbers C , then $H^2(C)$ equals H^2 .

We will also study operator-valued functions. Let $\mathcal{A} \subset L(K)$ be a WOT-closed algebra. A function F from C into \mathcal{A} is said to be measurable if, for every $x \in K$, the K -valued function $z \rightarrow F(z)x$ is measurable. For such an F , let $\|F\|_\infty$ denote the essential supremum of $\|F(z)\|$ on C , i.e. $\|F\|_\infty = \inf \{\sup \{\|F(z)\| : z \in \sigma\} : \sigma \text{ is a Borel subset of } C, m(C - \sigma) = 0\}$. $L^\infty(\mathcal{A})$ denotes the collection of all measurable functions F from C to \mathcal{A} such that $\|F\|_\infty$ is finite (we identify functions equal m -a.e.).

We can treat elements of $L^\infty(\mathcal{A})$ as operators in $L(L^2(K))$ as follows. Let $F \in L^\infty(\mathcal{A})$ and $f \in L^2(K)$. Then Ff is the function in $L^2(K)$ such that $(Ff)z = F(z)f(z)$. Since $\|F\|_\infty$ is finite, F is bounded as an operator on $L^2(K)$. $H^\infty(\mathcal{A})$ denotes the set of all elements $F \in L^\infty(\mathcal{A})$ such that $F(H^2(K)) \subset H^2(K)$; such F can be treated as operators on $H^2(K)$ and $H^\infty(\mathcal{A})$ is the collection of all such operators F on $H^2(K)$.

The unilateral shift (shortly, shift) in $H^2(K)$ will play the main role in

this paper. Let us recall that the shift in $H^2(K)$ is defined as $(Sf)z = zf(z)$ for $f \in H^2(K)$. In what follows S will always denote such a shift.

2. Properties C and C_∞ . A subset \mathcal{T}_0 of $L(K)$ has property C if, for each positive integer n and each y in $K^{(n)}$, there is $x \in K$ and a unitary operator $U: C(\mathcal{T}_0^{(n)}, y) \rightarrow C(\mathcal{T}_0, x)$ such that

$$UT^{(n)}|_{C(\mathcal{T}_0^{(n)}, y)} U^* = T|_{C(\mathcal{T}_0, x)}$$

for every T in \mathcal{T}_0 . Property C was introduced by Wogen [14]. He applied it to prove the reflexivity of an isometry. But we will also need another property. Let $\mathcal{T}_0 \subset L(K)$. If $T_0 \in \mathcal{T}_0$ then T denotes the element of $H^\infty(L(K))$ defined by $T(z) = T_0$ for z in C . Put $\mathcal{T} = \{T : T_0 \in \mathcal{T}_0\}$. We will say that \mathcal{T}_0 has property C_∞ if for each f in $H^2(K)$ there is x in K and a unitary operator $U: C(\mathcal{T}, f) \rightarrow C(\mathcal{T}_0, x)$ such that

$$UT|_{C(\mathcal{T}, f)} U^* = T_0|_{C(\mathcal{T}_0, x)}$$

for every T_0 in \mathcal{T}_0 .

It is clear that property C_∞ implies property C and that \mathcal{T}_0 contained in $L(K)$ has property C (or C_∞) if and only if $\mathfrak{A}(\mathcal{T}_0)$ does. Now, we have

Remark 3. (1) Each algebra of normal operators has property C_∞ .

(2) Each shift of arbitrary multiplicity has property C_∞ .

The proof that any algebra of normal operators has property C is essentially contained in the proof of the reflexivity of such an algebra [10, Theorem 9.21]. Taking $H^2(K)$ instead of $K^{(n)}$ and following the idea of the above-mentioned proof, we get property C_∞ for an algebra of normal operators. (2) is easy to see because a shift of arbitrary multiplicity restricted to its cyclic invariant subspace is unitarily equivalent to the shift of multiplicity 1 [10, Theorem 3.33].

The following proposition will be needed:

PROPOSITION 4. Let S be the shift on $H^2(K)$ and $T_0 \in L(K)$. T denotes the element of $H^\infty(L(K))$ defined by $T(z) = T_0$ for z in C . If T_0 has property C_∞ then $\{T, S\}$ has property C.

Wogen [14, proof of Lemma 2] proved Proposition 4 when T_0 was a normal operator. If T_0 has only property C_∞ Wogen's proof also applies with small modifications (we first apply property C_∞ and next property C which results from property C_∞). Theorem 1 in [14] can be generalized to

THEOREM 5. Let $\mathcal{T} = \{T_i : (i \in I)\}$ be a set of operators in a separable Hilbert space K_1 and let $\mathcal{R} = \{R_i : (i \in I)\}$ be a set of operators in a separable Hilbert space K_2 . $\mathcal{T} \oplus \mathcal{R}$ denotes the collection of all operators $T_i \oplus R_i$ on $K_1 \oplus K_2$ for $i \in I$. If the algebras $\mathfrak{A}(\mathcal{T})$, $\mathfrak{A}(\mathcal{R})$ are reflexive and have property C then $\mathfrak{A}(\mathcal{T} \oplus \mathcal{R})$ is reflexive and has property C.

The proof can be done in the same way as the proof of Theorem 1 in [14].

3. Reflexivity of $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$. The unitary isomorphism between $H^2 \otimes K$ and $H^2(K)$ is well known. Hence, operators on $H^2 \otimes K$ correspond unitarily to operators on $H^2(K)$. In particular, if $\varphi \in \mathcal{H}^\infty$ and $A \in L(K)$ then $\varphi \otimes A$ corresponds to the operator-valued function $\varphi(\cdot)A$ acting as an operator on $H^2(K)$ as follows: $(\varphi(\cdot)A)f(z) = \varphi(z)Af(z)$ for each $f \in H^2(K)$. For the sake of convenience, we will study operator-valued functions instead of tensor products.

Let us fix for this section a WOT-closed algebra \mathcal{A}_0 in $L(K)$. Then \mathcal{A} denotes the collection of all $A \in H^\infty(L(K))$ such that there is $A_0 \in \mathcal{A}_0$ with $A(z) = A_0$ for almost all z . It is easy to see that $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$ regarded as an algebra of operators on $H^2 \otimes K$ corresponds unitarily to $\mathfrak{A}(S, \mathcal{A})$ as an algebra on $H^2(K)$. Thus Theorem 2 becomes

THEOREM 2'. If \mathcal{A}_0 is reflexive then $\mathfrak{A}(S, \mathcal{A})$ is reflexive.

Proof. Suppose that $B \in L(H^2(K))$ and $\text{Lat}(S, \mathcal{A}) \subset \text{Lat} B$. Let \mathcal{A}_0^* denote the set $\{A_0^*: A_0 \in \mathcal{A}_0\}$ and let \mathcal{A}^* denote the set $\{A^*: A \in \mathcal{A}\}$. Then $\text{Lat}(S^*, \mathcal{A}^*) \subset \text{Lat} B^*$. For a complex number a of modulus less than 1, we define the function h_a on C as follows: $h_a(z) = (1-az)^{-1}$ for $z \in C$. The function h_a is an eigenvector of the adjoint of the shift of multiplicity 1 on H^2 corresponding to the eigenvalue a . Let $L \in \text{Lat} \mathcal{A}^*$ and let $x \in L$. Thus the function $z \rightarrow h_a(z)x$, shortly denoted by $h_a x$, is an eigenvector of S^* with the same eigenvalue a . Thus $h_a L$ is invariant for S^* . Let $A^* \in \mathcal{A}^*$. Then

$$(A^*(h_a x))(z) = A_0^* h_a(z)x = h_a(z)A_0^* x = (h_a y)(z)$$

where $y \in L$, because $L \in \text{Lat} \mathcal{A}_0^*$. Thus $h_a L$ is invariant for each element of \mathcal{A}_0^* . Hence, $h_a L$ is invariant for B^* . Let $x \in K$ and let a be as above. Then there is $y \in K$ such that $B^*(h_a x) = h_a y$. Thus

$$B^* S^*(h_a x) = B^*(ah_a x) = aB^*(h_a x) = ah_a y = S^*(h_a y) = S^* B^*(h_a x).$$

The set $\{h_a x: x \in K, |a| < 1\}$ is linear dense in $H^2(K)$ because $\{h_a: |a| < 1\}$ is linear dense in H^2 . Thus B^* commutes with S^* , i. e. B commutes with S . Hence, $B \in H^\infty(L(K))$ [10, Corollary 3.20].

Now, we prove that $B \in H^\infty(\mathcal{A}_0)$. Let $L \in \text{Lat} \mathcal{A}_0$. Then $H^2(L) \in \text{Lat}(S, \mathcal{A}) \subset \text{Lat} B$. Let $x \in L$ and let \tilde{x} denote the function in $H^2(L)$ defined by $\tilde{x}(z) = x$ for all $z \in C$. Thus $B\tilde{x} \in H^2(L)$, so $L \ni (B\tilde{x})(z) = B(z)\tilde{x}(z) = B(z)x$ for almost all $z \in C$. Hence, $\text{Lat} \mathcal{A}_0 \subset \text{Lat} B(z)$ for almost all $z \in C$. Since \mathcal{A}_0 is reflexive, $B(z) \in \mathcal{A}_0$ for almost all $z \in C$. Hence, B is an element of $H^\infty(\mathcal{A}_0)$. The following crucial lemma will finish the proof of Theorem 2'. It is separated from the whole proof, because it may be of independent interest.

LEMMA 6. If \mathcal{A}_0 is reflexive then $\mathfrak{A}(S, \mathcal{A}) = H^\infty(\mathcal{A}_0)$.

Proof. Since $\{S\} \cup \mathcal{A}$ is contained in $H^\infty(\mathcal{A}_0)$, to prove the inclusion

\subset it is enough to show that $H^\infty(\mathcal{A}_0)$ is WOT-closed. Let A be a WOT-limit of a sequence of elements of $H^\infty(\mathcal{A}_0)$. It is obvious that A is a measurable operator-valued function. $A \in L^\infty(L(K))$ because $\|A\|$ as norm of an operator is equal to $\|A\|_\infty$ [3, Chap. II, § 2, Proposition 2]. It is easy to see that $A \in H^\infty(L(K))$. Now $A \in H^\infty(\mathcal{A}_0)$ because \mathcal{A}_0 is WOT-closed.

The important part of the proof is the proof of the inclusion \supset . For the sake of convenience, we will study vector-valued or operator-valued functions defined on the interval $[-\pi, \pi]$ instead of C . Let B be an element of $H^\infty(\mathcal{A}_0)$ and let k be an integer. If $x, y \in K$ then

$$\left| (2\pi)^{-1} \int_{-\pi}^{\pi} (B(t)x, y) e^{-ikt} dt \right| \leq \|B\|_\infty \|x\| \|y\|.$$

Hence, the function $(x, y) \rightarrow (2\pi)^{-1} \int_{-\pi}^{\pi} (B(t)x, y) e^{-ikt} dt$ is a bounded sesquilinear form on K . By [13, Chap. II, Theorem 1.3], there is a bounded operator B_k on K such that

$$(B_k x, y) = (2\pi)^{-1} \int_{-\pi}^{\pi} (B(t)x, y) e^{-ikt} dt.$$

$H^2(K)$ is invariant for B , so it is easy to see that $B_k = 0$ for $k = -1, -2, \dots$

Now, we prove that $B_k \in \mathcal{A}_0$ for all integers k . Let $L \in \text{Lat} \mathcal{A}_0$ and $x \in L$, $y \in L^\perp$ (the orthogonal complement of L in K). Then $(B(t)x, y) = 0$ for almost all t , thus $(B_k x, y) = 0$. This means that $B_k x \in L$. Hence L is invariant for B_k . Since \mathcal{A}_0 is reflexive, $B_k \in \mathcal{A}_0$.

Let us denote

$$\begin{aligned} \bar{\sigma}_n &= \sum_{k=0}^n B_k S^k = \sum_{k=-n}^n B_k S^k, \quad n = 1, 2, \dots, \\ \sigma_n &= n^{-1} (\bar{\sigma}_0 + \dots + \bar{\sigma}_{n-1}). \end{aligned}$$

If k_n denotes the n th Fejér's kernel [6, Chap. II] then as in [6, Chap. II] we can prove that

$$\begin{aligned} (*) \quad (\sigma_n(s)x, y) &= (2\pi)^{-1} \int_{-\pi}^{\pi} (B(t)x, y) k_n(s-t) dt \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} (B(s-t)x, y) k_n(t) dt \end{aligned}$$

for all $x, y \in K$, $s \in [-\pi, \pi]$ and $n = 1, 2, \dots$. In the second integral B is periodically extended to the function on the real line.

To finish our proof, it is enough to show that σ_n converges to B in WOT. Let $f, g \in H^2(K)$. We can periodically extend f, g to the whole real line and define for real t the functions $f_t, g_t \in H^2(K)$ as follows:

$f_t(s) = f(s+t)$, $g_t(s) = g(s+t)$ for $s \in [-\pi, \pi]$. We also define the functions $B_t \in H^\infty(\mathcal{A}_0)$ for real t by $B_t(s) = B(s-t)$ for $s \in [-\pi, \pi]$. Our first step is to show that $(B_t f, g) \rightarrow (Bf, g)$ as $t \rightarrow 0$. It is easy to prove that $(B_t f, g) = (Bf_t, g_t)$. Hence, we have

$$\begin{aligned} |(B_t f, g) - (Bf, g)| &= |(Bf_t, g_t) - (Bf, g)| \\ &\leq |(Bf_t, g_t) - (Bf_t, g)| + |(Bf_t, g) - (Bf, g)| \\ &\leq |(Bf_t, g_t - g)| + |(B(f_t - f), g)| \\ &\leq \|B\|_\infty \|f_t\| \|g_t - g\| + \|B\|_\infty \|f_t - f\| \|g\|. \end{aligned}$$

Because $\|f_t\| = \|f\|$ and $f_t \rightarrow f$, $g_t \rightarrow g$ as $t \rightarrow 0$ the desired result is proved.

The equality (*) is satisfied for all $x, y \in K$, in particular for the vectors $f(s), g(s)$. Now, using the inner product in K we can carry out the estimation in the same way as in [6, Chap. II, p. 19]. For all $\delta > 0$ we get the inequality

$$\begin{aligned} |(\sigma_n f, g) - (Bf, g)| &\leq \sup_{|t| < \delta} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} ((B(s-t) - B(s)) f(s), g(s)) ds \right| \\ &\quad + \sup_{|t| \geq \delta} k_n(t) \cdot 2 \|B\|_\infty \|f\| \|g\| \\ &= \sup_{|t| < \delta} |(B_t - B)f, g| + 2 \|B\|_\infty \|f\| \|g\| \sup_{|t| \geq \delta} k_n(t). \end{aligned}$$

$(B_t f, g)$ converges to (Bf, g) as $t \rightarrow 0$ and $\sup_{|t| \geq \delta} k_n(t)$ converges to 0 as $n \rightarrow \infty$ as the Fejér's kernel. Hence $\sigma_n \rightarrow B$ as $n \rightarrow \infty$ in WOT. ■

Hadwin and Nordgren [5] called an operator algebra *super-reflexive* if every its WOT-closed subalgebra is reflexive. From the previous results we get immediately:

PROPOSITION 7. *If $\mathcal{A}_0 \subset L(K)$ is reflexive and has property C_∞ then $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$ is super-reflexive.*

Proof. The algebra $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$ is reflexive by Theorem 2 and has property C by Proposition 4. Thus, by Proposition 2.5 in [5], $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$ is super-reflexive. ■

Proposition 7 and Remark 3 imply immediately the following

COROLLARY 8. (1) *The algebra $\mathcal{H}^\infty \widehat{\otimes} \mathcal{H}^\infty$ is super-reflexive.*

(2) *If \mathcal{A}_0 is an algebra of normal operators, then $\mathcal{H}^\infty \widehat{\otimes} \mathcal{A}_0$ is super-reflexive.*

(3) *If $\mathcal{A} \subset L(K)$ is an algebra of normal operators then the algebra $\mathcal{H}^\infty \widehat{\otimes} L^\infty(\mathcal{A})$ is super-reflexive as an algebra of operators on $H^2 \widehat{\otimes} L^2(K)$.*

(3) was also proved in [5].

4. Reflexivity of a pair of isometries. Before we present the proof of Theorem 1, we show the following proposition concerning "acting in orthogonal directions" which will be useful.

PROPOSITION 9. *Let U be a normal operator in $L(H^2(K))$ which commutes with the shift S on $H^2(K)$. Then $U \in H^\infty(L(K))$ and there is a normal operator $U_0 \in L(K)$ such that $U(z) = U_0$ for almost all z in C .*

Proof. By Corollary 3.20 in [10], $U \in H^\infty(L(K))$ and $U(z)$ is normal for almost every z by Theorem 3.17 from [10]. For each $x \in K$ we define the function $\tilde{x} \in H^2(K)$ putting $\tilde{x}(z) = x$ for all z in C . Now, we prove that

(**) *for each x in K , there is y_0 in K such that*

$$U\tilde{x} = \tilde{y}_0.$$

Let $x \in K$. Then there is $y_0 \in K$ and $f \in H^2(K)$ such that $U\tilde{x} = \tilde{y}_0 + Sf$. Thus $U\tilde{x} - \tilde{y}_0 = Sf$. Hence

$$S^*(U\tilde{x} - \tilde{y}_0) = S^*Sf = f.$$

By Putnam's theorem [10, Corollary 1.19]

$$f = US^*\tilde{x} - S^*\tilde{y}_0 = 0.$$

Thus (**) is fulfilled.

For $x, y \in K$ and nonnegative integers n, m we denote by \tilde{x}_n, \tilde{y}_m the functions defined as follows: $\tilde{x}_n(z) = z^n x$ and $\tilde{y}_m(z) = z^m y$. Since (**) is fulfilled, if $m \neq n$ we have

$$(U\tilde{x}_n, \tilde{y}_m) = (US^n \tilde{x}, S^m \tilde{y}) = (S^n U\tilde{x}, S^m \tilde{y}) = 0.$$

It is obvious that the function $(x, y) \rightarrow (U\tilde{x}, \tilde{y})$ is a bounded sesquilinear form on K . By Theorem 1.3 in [13, Chap. II], there is an operator U_0 on K such that $(U_0 x, y) = (U\tilde{x}, \tilde{y})$ for all $x, y \in K$. \tilde{U}_0 denotes the operator in $H^\infty(L(K))$ defined by $\tilde{U}_0(z) = U_0$ for all z . To complete the proof, the equality $(\tilde{U}_0 f, g) = (Uf, g)$ for $f, g \in H^2(K)$ is needed. Because \tilde{U}_0, U are bounded, it is enough to show that $(\tilde{U}_0 \tilde{x}_n, \tilde{y}_m) = (U\tilde{x}_n, \tilde{y}_m)$ for all x, y in K and for all nonnegative integers n, m . Let $x, y \in K$. If $m \neq n$ then $(U\tilde{x}_n, \tilde{y}_m) = 0$ and

$$(\tilde{U}_0 \tilde{x}_n, \tilde{y}_m) = \int (U_0 x z^n, y z^m) dm = (U_0 x, y) \int z^n z^{-m} dm = 0.$$

If $m = n$ then

$$\begin{aligned} (U\tilde{x}_n, \tilde{y}_m) &= (U\tilde{x}_n, \tilde{y}_n) = \int (U(z) x z^n, y z^n) dm \\ &= \int (U(z) x, y) dm = (U\tilde{x}, \tilde{y}) \\ &= (U_0 x, y) = (U_0 x, y) \int z^n z^{-n} dm \\ &= \int (U_0 x z^n, y z^n) dm = (\tilde{U}_0 \tilde{x}_n, \tilde{y}_m). \end{aligned}$$

Hence, the above necessary equality is proved. ■

A result like Proposition 9 concerning "acting in orthogonal directions" for two doubly commuting shifts was proved by Stociński [12, Theorem 1].

Proof of Theorem 1. Let V_1, V_2 be two doubly commuting isometries on a separable Hilbert space H . By a result of Słociński [12, Theorem 3], there exist subspaces $H_{uu}, H_{us}, H_{su}, H_{ss}$ such that:

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss},$$

$H_{uu}, H_{us}, H_{su}, H_{ss}$ reduce V_1 and V_2 ,

$V_1|_{H_{uu}}$ and $V_2|_{H_{uu}}$ are unitary operators,

$V_1|_{H_{us}}$ is unitary and $V_2|_{H_{us}}$ is a shift,

$V_1|_{H_{su}}$ is a shift and $V_2|_{H_{su}}$ is unitary,

$V_1|_{H_{ss}}$ and $V_2|_{H_{ss}}$ are shifts.

This decomposition will be called the *Słociński decomposition*.

The pair $\{V_1|_{H_{uu}}, V_2|_{H_{uu}}\}$ is reflexive and has property C as a pair of commuting normal operators [10, Theorem 9.21]. The pair $\{V_1|_{H_{us}}, V_2|_{H_{us}}\}$ is reflexive by Proposition 9 and Theorem 2' and has property C by Remark 3 (1) and Proposition 4. The pair $\{V_1|_{H_{su}}, V_2|_{H_{su}}\}$ is reflexive and has property C by the same reason. $V_1|_{H_{ss}}, V_2|_{H_{ss}}$ are doubly commuting shifts. Thus, they act in "orthogonal directions" by [12, Theorem 1]. This means that H_{ss} is unitarily isomorphic to $H^2(H^2(K))$. Then $V_1|_{H_{ss}}$ corresponds to the shift S in $H^2(H^2(K))$ and $V_2|_{H_{ss}}$ corresponds to the operator T defined by $T(z) = T_0$ for all z in C , where T_0 is a shift on $H^2(K)$. Hence, this pair is reflexive by Theorem 2' and has property C by Remark 3 (2) and Proposition 4. Now, we apply Theorem 5 and we get the reflexivity of the pair $\{V_1, V_2\}$. ■

Let $\{V_1, V_2\}$ be a pair of (not necessarily doubly) commuting isometries and suppose it has the above Słociński decomposition (which need not exist in general [12, Example 1]). Putnam's theorem [10, Corollary 1.19] shows that the double commutativity of parts of V_1, V_2 can fail only on H_{ss} . Hence, the following theorem can be deduced by the same techniques as in the proof of Theorem 1.

THEOREM 10. *Let $\{V_1, V_2\}$ be a pair of commuting isometries which has the Słociński decomposition. If the pair $\{V_1|_{H_{ss}}, V_2|_{H_{ss}}\}$ is reflexive and has property C, then the pair $\{V_1, V_2\}$ is reflexive and has property C.*

We end up with the following example. Let $\{V_1, V_2\}$ be a pair of isometries having the Słociński decomposition and $H_{ss} = H^2$, $V_1|_{H_{ss}}$ is the shift on H^2 , i.e. $(V_1|_{H_{ss}}f)z = zf(z)$ for all f in H^2 , $V_2|_{H_{ss}}$ is the Toeplitz operator of multiplication by $\varphi \in H^\infty$, i.e. $(V_2|_{H_{ss}}f)z = \varphi(z)f(z)$, but φ is not a constant function. Then V_1, V_2 commute, but do not doubly commute. The pair $\{V_1|_{H_{ss}}, V_2|_{H_{ss}}\}$ is reflexive and has property C. Hence, the pair $\{V_1, V_2\}$ is reflexive by Theorem 10.

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