On the regularity to the solutions of the Navier–Stokes equations via one velocity component

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Abstract

We consider the regularity criteria for the incompressible Navier–Stokes equations connected with one velocity component. Based on the method from [4] we prove that the weak solution is regular, provided $u_3 \in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s} \leq \frac{3}{4}+\frac{1}{2s}$, $s > \frac{10}{3}$ or provided $\nabla u_3 \in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s} \leq \frac{19}{12}+\frac{1}{2s}$ if $s \in (\frac{30}{19},3]$ or $\frac{2}{t}+\frac{3}{s} \leq \frac{3}{2}+\frac{3}{4s}$ if $s \in (3,\infty]$. As a corollary, we also improve the regularity criteria expressed by the regularity of $\frac{\partial p}{\partial x_3}$ or $\frac{\partial u_3}{\partial x_3}$.

1 Introduction

We consider the incompressible Navier–Stokes equations in the full three-dimensional space, i.e.

1.1 (1.1)
$$\left.\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } (0,T)\times\mathbb{R}^3,$$
$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \quad \text{in } \mathbb{R}^3,$$

where $\boldsymbol{u} : (0,T)\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $p : (0,T)\times\mathbb{R}^3 \to \mathbb{R}$ is the pressure, $\boldsymbol{f} : (0,T)\times\mathbb{R}^3 \to \mathbb{R}^3$ is the given external force, $\nu > 0$ is the viscosity. In what follows, we consider $\nu = 1$ and $\boldsymbol{f} \equiv \boldsymbol{0}$. The value of the viscosity does not play any role in our further considerations. We could also easily formulate suitable regularity assumptions on $\boldsymbol{f}$ so that the main results remain true. However, it would partially complicate the calculations, thus we skip it.

The existence of a weak solution to (1.1) (provided $\boldsymbol{u}_0$ and $\boldsymbol{f}$ satisfy certain regularity assumptions) is well known since the famous paper by LERAY [11]. Its regularity and

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uniqueness remains still open. However, many criteria ensuring the smoothness of the solution are known. The classical PRODI–SERRIN conditions (see [17], [18] and for $s=3$ [7]) say that if the weak solution $\boldsymbol{u}$ additionally belongs to $L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s}=1$, $s\in[3,\infty]$, then the solution is as regular as the data allow and unique in the class of all weak solutions satisfying the energy inequality. Similar results on the level of the velocity gradient, i.e. $\nabla\boldsymbol{u}\in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s}=2$, $s\in[\frac{3}{2},\infty]$, is due to BEIRÃO DA VEIGA (see [1]). Note that the case $s=\frac{3}{2}$ is a consequence of the Sobolev embedding theorem and [7].

Later on, criteria just for one velocity component appeared. The first result in this direction is due to NEUSTUPA, NOVOTNÝ and PENEL [12], where the authors showed that if $u_3\in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s}=\frac{1}{2}$, $s\in(6,\infty]$, then the solution is smooth. Similar result, for the gradient of one velocity component, is independently due to ZHOU [19] and POKORNÝ [15]. Further criteria, including several components of the velocity gradient, pressure or other quantities can be found e.g. in [14], [16], [5], [3], [6], [2], [13] ....

Recently, two interesting improvements appeared. In [9], KUKAVICA and ZIANE proved that if $u_3\in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s}=\frac{5}{8}$, $s\in(\frac{24}{5},\infty]$ (the authors claim the result for $s\geq\frac{24}{5}$, but their technique does not work for the case $L^\infty(0,T;L^s(\mathbb{R}^3))$) or if $\nabla u_3\in L^t(0,T;L^s(\mathbb{R}^3))$, $\frac{2}{t}+\frac{3}{s}=\frac{11}{6}$, $s\in[\frac{54}{23},\frac{18}{5}]$, the weak solution is regular. Next, in [4], CAO and TITI used different method, instead of technical estimates they applied the multiplicative embedding theorem and showed the smoothness under the assumption $u_3\in L^t(0,T;L^s(\mathbb{R}^3))$ (actually, they work for the space periodic boundary conditions, but the proof for the Cauchy problem is exactly the same), $\frac{2}{t}+\frac{3}{s}<\frac{2}{3}+\frac{2}{3s}$, $s>\frac{7}{2}$. Note that this result is stronger than the result in [9], even though the criterion does not correspond the natural scaling of the Navier–Stokes equations.

Although the result of CAO and TITI is the strongest one among all the criteria for one velocity variable, it seems that the authors did not use all the possibilities of their method. In this short note we want to extend their result in several aspects. First, we show that for the regularity of the weak solutions it is enough to assume

$$u_3\in L^t(0,T;L^s(\mathbb{R}^3)),\quad \frac{2}{t}+\frac{3}{s}\leq\frac{3}{4}+\frac{1}{2s},\quad s>\frac{10}{3}$$

(or the norm in $L^\infty(0,T;L^{\frac{10}{3}}(\mathbb{R}^3)$ is sufficiently small), which is an improvement of the result by Cao and Titi. Next, using a very similar method, we will show that it is enough to have

$$\nabla u_3\in L^t(0,T;L^s(\mathbb{R}^3)),\quad \frac{2}{t}+\frac{3}{s}\leq\begin{cases}\dfrac{19}{12}+\dfrac{1}{2s}, & s\in\left(\dfrac{30}{19},3\right]\\[2mm] \dfrac{3}{2}+\dfrac{3}{4s}, & s\in(3,\infty],\end{cases}$$

(or again the norm in $L^\infty(0,T;L^{\frac{30}{19}}(\mathbb{R}^3))$ sufficiently small). Note that for $s<3$ it precisely corresponds to the result for u_3, using the Sobolev embedding theorem. The result for u_3 will be further applied to improve the regularity criteria for $\frac{\partial p}{\partial x_3}$ from [4] and for $\frac{\partial u_3}{\partial x_3}$ from [14]. (There, $\frac{\partial u_3}{\partial x_3}\in L^\infty((0,T)\times\mathbb{R}^3)$ was required.) Note finally that for $s\to\frac{30}{19}^+$, $\frac{19}{12}+\frac{1}{2s}\to\frac{19}{10}^-$, i.e. we are not so far away from $\frac{2}{t}+\frac{3}{s}=2$ which ensures the regularity if $\nabla\boldsymbol{u}\in L^t(0,T;L^s(\mathbb{R}^3))$ (but also if only either $\nabla_3\boldsymbol{u}$ belongs to this space — see [10] — or only $\frac{\partial u_2}{\partial x_2}$ and $\frac{\partial u_3}{\partial x_3}$, see [14]).

In the whole paper, we will use the standard notation for Lebesgue spaces $L^p(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_p$ and for Sobolev spaces $W^{k,p}(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_{k,p}$. We do not distinguish between the spaces X and their vector analogues X^N, however, all vector- and tensor-valued functions are printed boldfaced.

2 Main results

The aim of this paper is to show the following

t1 **Theorem 1** *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to $\boldsymbol{u}_0 \in W^{1,2}_{div}(\mathbb{R}^3)$ which satisfies the energy inequality. Let additionally*

$$u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{s}+\frac{3}{t} \le \frac{3}{4}+\frac{1}{2s}, \quad s > \frac{10}{3}.$$

Then $\boldsymbol{u}$ is as smooth as the data allow, thus in our case $\boldsymbol{u} \in C^\infty((0,T)\times\mathbb{R}^3)$ and $\boldsymbol{u}$ is unique in the class of all weak solutions satisfying the energy inequality.

t2 **Theorem 2** *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to $\boldsymbol{u}_0 \in W^{1,2}_{div}(\mathbb{R}^3)$ which satisfies the energy inequality. Let additionally*

$$\nabla u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{s}+\frac{3}{t} \le \begin{cases} \dfrac{19}{12}+\dfrac{1}{2s}, & s \in \Big(\dfrac{30}{19},3\Big], \\ \dfrac{3}{2}+\dfrac{3}{4s}, & s\in(3,\infty]. \end{cases}$$

Then $\boldsymbol{u}$ is as smooth as the data allow, thus in our case $\boldsymbol{u} \in C^\infty((0,T)\times\mathbb{R}^3)$ and $\boldsymbol{u}$ is unique in the class of all weak solutions satisfying the energy inequality.

In the proof we will follow the idea from [4]. First, we know that there exists $T^* > 0$ such that on $(0,T^*)$ there is a strong solution to the Navier–Stokes equations, i.e. $\boldsymbol{u} \in L^\infty(0,T^*;W^{1,2}(\mathbb{R}^3)) \cap L^2(0,T^*;W^{2,2}(\mathbb{R}^3))$ with $\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T^*;L^2(\mathbb{R}^3))$; actually as $\boldsymbol{f} \equiv \boldsymbol{0}$, $\boldsymbol{u} \in C^\infty((0,T^*]\times\mathbb{R}^3)$. Let $\mathcal{T}^* < T$ be the first time of the blow up, i.e. necessarily $\limsup_{t\to\mathcal{T}^*}\|\nabla\boldsymbol{u}(t)\|_2 = +\infty$. We will show that for any $\tau < \mathcal{T}^*$ we have $\|\nabla\boldsymbol{u}(\tau)\|_2 \le C < \infty$ with C independent of $\mathcal{T}^*$. This contradicts to the definition of $\mathcal{T}^*$ and thus $\mathcal{T}^* = T$.

3 Proof of Theorem 1

Denote

2.1 (3.1)
$$J^2(t) = \sup_{\tau\in(0,t)} \|\nabla_h\boldsymbol{u}(\tau)\|_2^2 + \int_0^t \|\nabla\nabla_h\boldsymbol{u}(\tau)\|_2^2 d\tau,$$

with

2.2 (3.2)
$$\nabla_h\boldsymbol{u} = \Big(\frac{\partial\boldsymbol{u}}{\partial x_1},\frac{\partial\boldsymbol{u}}{\partial x_2}\Big)$$

and

$$\boxed{2.3}(3.3) \qquad V_s(t) = \int_0^t \|u_3(\tau)\|_s^{\frac{8s}{3s-10}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau, \quad s \in \Big(\frac{10}{3}, \infty\Big],$$

(i.e. $V_\infty = \int_0^t \|u_3(\tau)\|_\infty^{\frac{8}{3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau$). Then, testing ($\overset{1.1}{1.1}$)$_1$ (recall $\nu = 1$ and $\boldsymbol{f} = \boldsymbol{0}$) by $\Delta_2 \boldsymbol{u} = \sum_{i=1}^2 \frac{\partial^2 \boldsymbol{u}}{\partial x_i^2}$ leads to

$$\boxed{2.4}(3.4) \qquad \frac{1}{2}\frac{d}{dt}\|\nabla_h \boldsymbol{u}(t)\|_2^2 + \|\nabla\nabla_h \boldsymbol{u}(t)\|_2^2 = \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\Delta_2 \boldsymbol{u} dx.$$

Then

$$\boxed{2.5}(3.5) \qquad \begin{aligned} & \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\Delta_2 \boldsymbol{u} dx \\ = \; & \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} u_i \frac{\partial u_j}{\partial x_i} \Delta_2 u_j dx + \sum_{i=1}^{3} \int_{\mathbb{R}^3} u_i \frac{\partial u_3}{\partial x_i} \Delta_2 u_3 dx + \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_3 \frac{\partial u_j}{\partial x_3} \Delta_2 u_j dx \\ = \; & J_1 + J_2 + J_3. \end{aligned}$$

We have

$$\begin{aligned} J_1 = \; & \frac{1}{2}\sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dx - \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} dx + \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} dx \\ = \; & -\int_{\mathbb{R}^3} u_3 \frac{\partial^2 u_i}{\partial x_3 \partial x_j}\frac{\partial u_i}{\partial x_j} dx + \int_{\mathbb{R}^3} u_3 \Big(\frac{\partial^2 u_1}{\partial x_3 \partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial^2 u_2}{\partial x_3 \partial x_2}\frac{\partial u_1}{\partial x_1}\Big) dx \\ & - \int_{\mathbb{R}^3} u_3 \Big(\frac{\partial^2 u_1}{\partial x_3 \partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_3 \partial x_1}\frac{\partial u_1}{\partial x_2}\Big) dx, \end{aligned}$$

see e.g. [9]. It expresses the well-known fact that for $\boldsymbol{u} \in W^{1,2}_{div}(\mathbb{R}^2) \cap W^{2,2}(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} (\boldsymbol{u} \cdot \nabla\boldsymbol{u}) \cdot \Delta_2 \boldsymbol{u} dx = 0$. Further

$$J_2 = -\sum_{i=1}^{3}\sum_{k=1}^{2} \int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_k}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_k} dx = \sum_{i=1}^{3}\sum_{k=1}^{2} \int_{\mathbb{R}^3} u_3 \frac{\partial^2 u_3}{\partial x_i \partial x_k}\frac{\partial u_i}{\partial x_k} dx$$

and thus

$$\begin{aligned} |J_1 + J_2| &\leq C \int_{\mathbb{R}^3} |u_3||\nabla_h \boldsymbol{u}||\nabla\nabla_h \boldsymbol{u}| dx \leq C\|u_3\|_s \|\nabla\nabla_h \boldsymbol{u}\|_2 \|\nabla_h \boldsymbol{u}\|_{\frac{2s}{s-2}} \\ &\leq C\|u_3\|_s \|\nabla\nabla_h \boldsymbol{u}\|_2^{1+\frac{3}{s}} \|\nabla_h \boldsymbol{u}\|_2^{1-\frac{3}{s}} \leq \frac{1}{4}\|\nabla\nabla_h \boldsymbol{u}\|_2^2 + \|u_3\|_s^{\frac{2s}{s-3}} \|\nabla_h \boldsymbol{u}\|_2^2. \end{aligned}$$

Further

$$|J_3| \leq C \int_{\mathbb{R}^3} |u_3||\nabla \boldsymbol{u}||\nabla\nabla_h \boldsymbol{u}| dx \leq C\|u_3\|_s \|\nabla\nabla_h \boldsymbol{u}\|_2 \|\nabla \boldsymbol{u}\|_2^{1-\frac{3}{s}} \|\nabla \boldsymbol{u}\|_6^{\frac{3}{s}}.$$

Exactly as in [4] we apply the multiplicative Gagliardo–Nirenberg inequality

$$\|\nabla \boldsymbol{u}\|_6 \leq C \Big\|\frac{\partial \nabla \boldsymbol{u}}{\partial x_1}\Big\|_2^{\frac{1}{3}} \Big\|\frac{\partial \nabla \boldsymbol{u}}{\partial x_2}\Big\|_2^{\frac{1}{3}} \Big\|\frac{\partial \nabla \boldsymbol{u}}{\partial x_3}\Big\|_2^{\frac{1}{3}}$$

which leads to

$$
\begin{aligned}
J_3 &\le C\|u_3\|_s \|\nabla\nabla_h \boldsymbol{u}\|_2^{1+\frac{2}{s}} \|\nabla \boldsymbol{u}\|_2^{1-\frac{3}{s}} \|\nabla^2 \boldsymbol{u}\|_2^{\frac{1}{s}} \\
&\le C\|u_3\|_s^{\frac{2s}{s-2}} \|\nabla \boldsymbol{u}\|_2^{\frac{2(s-3)}{s-2}} \|\nabla^2 \boldsymbol{u}\|_2^{\frac{2}{s-2}} + \frac{1}{4}\|\nabla\nabla_h \boldsymbol{u}\|_2^2.
\end{aligned}
$$

Thus, integrating (2.5 3.5) over time interval $(0,t)$, $t < \mathcal{T}^*$ and using estimates above yields

$$
\begin{aligned}
&\frac{1}{2}\|\nabla_h \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla\nabla_h \boldsymbol{u}(\tau)\|_2^2 d\tau \le \|\nabla_h \boldsymbol{u}(0)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla\nabla_h \boldsymbol{u}(\tau)\|_2^2 d\tau \\
&+C\int_0^t \|u_3(\tau)\|_s^{\frac{2s}{s-3}} \|\nabla_h \boldsymbol{u}(\tau)\|_2^2 d\tau + C\int_0^t \|u_3(\tau)\|_s^{\frac{2s}{s-2}} \|\nabla \boldsymbol{u}(\tau)\|_2^{\frac{2(s-3)}{s-2}} \|\nabla^2 \boldsymbol{u}(\tau)\|_2^{\frac{2}{s-2}} d\tau,
\end{aligned}
$$

i.e., taking the supremum over time interval in the first term and using Hölder's inequality

2.6
$$
\begin{aligned}
J^2(t) &\le K_0 + C\int_0^t \|u_3(\tau)\|_s^{\frac{2s}{s-3}} \|\nabla_h \boldsymbol{u}(\tau)\|_2^2 d\tau \\
&+C\Big(\int_0^t \|u_3(\tau)\|_s^{\frac{2s}{s-3}} \|\nabla_h \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{s-3}{s-2}} \Big(\int_{\mathbb{R}^3} \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{s-2}},
\end{aligned}
\tag{3.6}
$$

where $K_0 = K_0(\|\boldsymbol{u}_0\|_{1,2})$. Up to now, the proof just copied the proof from [4], with possibly different notation and slightly different arguments. Next we use, as in [4], as test function $-\Delta \boldsymbol{u}$. However, we estimate the convective term more carefully. We have

$$
\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|_2^2 + \|\nabla^2 \boldsymbol{u}(t)\|_2^2 = \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla \boldsymbol{u})\cdot\Delta \boldsymbol{u}.
$$

Now

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla \boldsymbol{u})\cdot\Delta \boldsymbol{u} = \\
&= \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \frac{\partial u_j}{\partial x_3}\Delta_2 u_j dx + \sum_{i=1}^2\sum_{j=1}^3 \int_{\mathbb{R}^3} u_i \frac{\partial u_j}{\partial x_i}\Delta u_j dx + \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \frac{\partial u_j}{\partial x_3}\frac{\partial^2 u_j}{\partial x_3^2} dx \\
&= K_1 + K_2 + K_3.
\end{aligned}
$$

We have

$$
\begin{aligned}
K_1 &= -\sum_{j=1}^3\sum_{k=1}^2 \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k} dx + \frac{1}{2}\sum_{j=1}^3\sum_{k=1}^2 \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k} dx, \\
K_2 &= -\sum_{i=1}^2\sum_{j=1}^3\sum_{k=1}^3 \int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k} dx + \frac{1}{2}\sum_{i=1}^2\sum_{j=1}^3\sum_{k=1}^3 \int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k} dx, \\
K_3 &= -\frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_3} dx = \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^3} \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_3} dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{2}\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau &\le C\int_0^t\int_{\mathbb{R}^3} |\nabla_h \boldsymbol{u}||\nabla \boldsymbol{u}|^2 dx d\tau + \frac{1}{2}\|\nabla \boldsymbol{u}_0\|_2^2 \\
&\le C\int_0^t \|\nabla_h \boldsymbol{u}(\tau)\|_2 \|\nabla \boldsymbol{u}(\tau)\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{u}(\tau)\|_6^{\frac{3}{2}} d\tau + \frac{1}{2}\|\nabla \boldsymbol{u}_0\|_2^2.
\end{aligned}
$$

Using again the multiplicative embedding theorem yields

$$\frac{1}{2}\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau$$
$$\leq K_0 + C\|\nabla_h \boldsymbol{u}\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}\|\nabla\nabla_h \boldsymbol{u}\|_{L^2(0,t;L^2(\mathbb{R}^3))}\|\nabla \boldsymbol{u}\|^{\frac{1}{2}}_{L^2(0,t;L^2(\mathbb{R}^3))}\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}$$
$$\leq K_0 + CJ^2(t)\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}.$$

Now, we can use the estimate of $J(t)$ from (3.6) and get (the rest of the proof follows again the approach from [4])

$$J^2(t)\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}} \leq C\Big[K_0 + \int_0^t \|u_3(\tau)\|_s^{\frac{2s}{s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau$$
$$+\Big(\int_0^t \|u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{s-3}{s-2}}\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{s-2}}\Big]\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}} = A.$$

Now, let $\beta \geq 1$. Using Hölder's and Young's inequality we get

$$A \leq CK_0\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}} + C\Big(\int_0^t \|u_3(\tau)\|_s^{\beta\frac{2s}{s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{\beta}} \times$$
$$\times\Big(\int_0^t \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{\beta-1}{\beta}}\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}} + C\Big(\int_0^t \|u_3(\tau)\|_s^{\beta\frac{2s}{s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{s-3}{s-2}\frac{1}{\beta}} \times$$
$$\times\Big(\int_0^t \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{\beta-1}{\beta}\frac{s-3}{s-2}}\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}+\frac{1}{s-2}}$$
$$\leq CK_0 + C\Big(\int_0^t \|u_3(\tau)\|_s^{\beta\frac{2s}{s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{4}{3\beta}}$$
$$+C\Big(\int_0^t \|u_3(\tau)\|_s^{\beta\frac{2s}{s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{s-3}{s-2}\frac{1}{\beta}\frac{4(s-2)}{3s-10}} + \frac{1}{2}\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau,$$

as $\frac{s+2}{4(s-2)} < 1$ for $s > \frac{10}{3}$.

Thus

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq K_0 + C\big(V_s(t)^{\frac{4}{3\beta}} + V_s^{\frac{4(s-3)}{\beta(3s-10)}}\big),$$

which allows to use the Gronwall lemma provided $\beta = \frac{4(s-3)}{\beta(3s-10)}$. Note that the other condition, $\beta \geq \frac{4}{3}$, is less restrictive. As $\beta\frac{2s}{s-3} = \frac{8s}{3s-10}$, we get that $\|\nabla \boldsymbol{u}(t)\|_2$ is bounded independently of $\mathcal{T}^*$, provided $u_3 \in L^{\frac{8s}{3s-10}}(0,T;L^s(\mathbb{R}^3))$, $s > \frac{10}{3}$. Finally, if $\|u_3\|_{L^\infty(0,T;L^{\frac{10}{3}}(\mathbb{R}^3))} << 1$, we can transfer the corresponding term to the left-hand side. The proof of Theorem 1 is finished.

4 Proof of Theorem 2

The proof is similar to the proof of Theorem 1. We replace (3.3) by

2.7 (4.1)
$$V_s(t) = \begin{cases} \displaystyle\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{24s}{19s-30}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau, & s \in \big(\frac{30}{19}, 3\big], \\ \displaystyle\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{8s}{6s-9}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau, & s \in (3, \infty] \end{cases}$$

with the standard convention $V_\infty(t) = \int_0^t \|\nabla u_3(\tau)\|_\infty^{\frac{4}{3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau$. As before, first we multiply (1.1)$_1$ by $-\Delta_2 \boldsymbol{u}$ and integrate over Ω. We get as before

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h \boldsymbol{u}(t)\|_2^2 + \|\nabla\nabla_h \boldsymbol{u}(t)\|_2^2 = \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\Delta_2\boldsymbol{u}dx.$$

But

$$\begin{aligned}&\int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\Delta_2\boldsymbol{u}dx\\ =\ & \frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dx - \int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}dx + \int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}dx\\ -\ & \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\frac{\partial u_i}{\partial x_k}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_k}dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} u_3\frac{\partial u_j}{\partial x_3}\Delta_2 u_j dx.\end{aligned}$$

The first four terms can be estimated by

$$\int_{\mathbb{R}^3} |\nabla u_3||\nabla_h \boldsymbol{u}|^2 dx \le \|\nabla u_3\|_s \|\nabla_h \boldsymbol{u}\|_{\frac{2s}{s-1}} \le C\|\nabla u_3\|_s^{\frac{2s}{2s-3}}\|\nabla_h\boldsymbol{u}\|_2^2 + \frac{1}{4}\|\nabla\nabla_h\boldsymbol{u}\|_2^2,$$

while the last term

$$\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_3\frac{\partial u_j}{\partial x_3}\Delta_2 u_j dx = -\sum_{j,k=1}^{2}\Big(\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k}dx + \frac{1}{2}\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}dx\Big).$$

The second term can be estimated as above, and the first term

$$\Big|\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k}dx\Big| \le C\int_{\mathbb{R}^3}|\nabla u_3||\nabla_h\boldsymbol{u}||\nabla\boldsymbol{u}|dx = B.$$

Now we estimate separately B for $s \le 3$ and $s > 3$. We have
a) $s \in [\frac{3}{2}, 3]$:

$$\begin{aligned}B &\le C\|\nabla u_3\|_s\|\nabla_h\boldsymbol{u}\|_6\|\nabla\boldsymbol{u}\|_{\frac{6s}{5s-6}} \le C\|\nabla u_3\|_s\|\nabla\nabla_h\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2^{\frac{2s-3}{s}}\|\nabla\boldsymbol{u}\|_6^{\frac{3-s}{s}}\\ &\le C\|\nabla u_3\|_s\|\nabla\nabla_h\boldsymbol{u}\|_2^{1+\frac{2}{3}\frac{3-s}{s}}\|\nabla\boldsymbol{u}\|_2^{\frac{2s-3}{s}}\|\nabla^2\boldsymbol{u}\|_2^{\frac{1}{3}\frac{3-s}{s}},\end{aligned}$$

where we used the multiplicative embedding theorem. Thus

$$B \le \frac{1}{4}\|\nabla\nabla_h\boldsymbol{u}\|_2^2 + C\|\nabla u_3\|_s^{\frac{6s}{5s-6}}\|\nabla\boldsymbol{u}\|_2^{\frac{6(2s-3)}{5s-6}}\|\nabla^2\boldsymbol{u}\|_2^{\frac{2(3-s)}{5s-6}}.$$

It yields

$$\begin{aligned}\|\nabla_h\boldsymbol{u}(t)\|_2^2 + \int_0^t\|\nabla\nabla_h\boldsymbol{u}(\tau)\|_2^2 d\tau &\le K_0 + C\int_0^t\|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla\boldsymbol{u}(\tau)\|_2^2 d\tau\\ &+C\int_0^t\|\nabla u_3(\tau)\|_s^{\frac{6s}{5s-6}}\|\nabla\boldsymbol{u}(\tau)\|_2^{\frac{6(2s-3)}{5s-6}}\|\nabla^2\boldsymbol{u}(\tau)\|_2^{\frac{2(3-s)}{5s-6}} d\tau.\end{aligned}$$

As $\frac{6(2s-3)}{5s-6} \leq 2$ for $s \leq 3$, we get due to the Hölder inequality after taking the supremum over time on the left-hand side

$$\boxed{\texttt{2.7a}}\,(4.2)\qquad \begin{aligned} J^2(t) &\leq K_0 + C\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau \\ C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{3(2s-3)}{5s-6}} &\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{3-s}{5s-6}}. \end{aligned}$$

b) $s \in (3,\infty]$:
We estimate the convective term differently

$$\begin{aligned} B &\leq \int_{\mathbb{R}^3} |\nabla u_3||\nabla_h \boldsymbol{u}||\nabla \boldsymbol{u}|dx \leq C\|\nabla u_3\|_s \|\nabla \boldsymbol{u}\|_2 \|\nabla_h \boldsymbol{u}\|_{\frac{2s}{s-2}} \\ &\leq C\|\nabla u_3\|_s\|\nabla \boldsymbol{u}\|_2\|\nabla_h \boldsymbol{u}\|_2^{1-\frac{3}{s}}\|\nabla_h \boldsymbol{u}\|_6^{\frac{3}{s}} \leq \frac{1}{4}\|\nabla\nabla_h \boldsymbol{u}\|_2^2 + C\|\nabla u_3\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}\|_2^2. \end{aligned}$$

Thus

$$J^2(t) \leq K_0 + C\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau.$$

Next we use as test function $\Delta \boldsymbol{u}$. We get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|_2^2 + \|\nabla^2 \boldsymbol{u}(t)\|_2^2 = \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla \boldsymbol{u})\cdot\Delta \boldsymbol{u}dx \\ =\;& \sum_{i=1}^2\sum_{j=1}^3\int_{\mathbb{R}^3} u_i\frac{\partial u_j}{\partial x_i}\Delta u_j dx + \sum_{j=1}^3\int_{\mathbb{R}^3} u_3\frac{\partial u_j}{\partial x_3}\Delta u_j dx \\ =\;& -\sum_{i=1}^2\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}dx + \frac{1}{2}\sum_{i=1}^2\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}dx \\ &-\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k}dx + \frac{1}{2}\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}dx \\ \leq\;& C\int_{\mathbb{R}^3}|\nabla_h \boldsymbol{u}||\nabla \boldsymbol{u}|^2 dx + C\int_{\mathbb{R}^3}|\nabla u_3||\nabla \boldsymbol{u}|^2 dx = D_1 + D_2. \end{aligned}$$

We have as before

$$D_2 \leq \frac{1}{2}\|\nabla^2 \boldsymbol{u}\|_2^2 + C\|\nabla u_3\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}\|_2^2,$$

while using the multiplicative embedding theorem

$$\begin{aligned} D_1 &\leq \|\nabla_h \boldsymbol{u}\|_2\|\nabla \boldsymbol{u}\|_4^2 \leq \|\nabla_h \boldsymbol{u}\|_2\|\nabla \boldsymbol{u}\|_2^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_6^{\frac{3}{2}} \\ &\leq C\|\nabla_h \boldsymbol{u}\|_2\|\nabla \boldsymbol{u}\|_2^{\frac{1}{2}}\|\nabla\nabla_h \boldsymbol{u}\|_2\|\nabla^2 \boldsymbol{u}\|_2^{\frac{1}{2}}, \end{aligned}$$

i.e.

$$\begin{aligned} \|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau &\leq K_0 + C\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}}\|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau \\ +C\int_0^t \|\nabla_h \boldsymbol{u}(\tau)\|_2\|\nabla \boldsymbol{u}(\tau)\|_2^{\frac{1}{2}}&\|\nabla\nabla_h \boldsymbol{u}(\tau)\|_2\|\nabla^2 \boldsymbol{u}(\tau)\|_2^{\frac{1}{2}}d\tau. \end{aligned}$$

The last term can be estimated

$$\int_0^t \|\nabla_h \boldsymbol{u}(\tau)\|_2 \|\nabla \boldsymbol{u}(\tau)\|_2^{\frac{1}{2}} \|\nabla\nabla_h \boldsymbol{u}(\tau)\|_2 \|\nabla^2 \boldsymbol{u}(\tau)\|_2^{\frac{1}{2}} d\tau$$
$$\leq \|\nabla_h \boldsymbol{u}\|_{L^\infty(0,t;L^2(\mathbb{R}^3))} \|\nabla\nabla_h \boldsymbol{u}\|_{L^2(0,t;L^2(\mathbb{R}^3))} \|\nabla \boldsymbol{u}\|^{\frac{1}{2}}_{L^2(0,t;L^2(\mathbb{R}^3))} \Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}$$
$$\leq C J^2(t) \Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}.$$

We may therefore employ estimates of $J^2(t)$ and get separately
a) $s \leq 3$:

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq K_0 + C\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau$$
$$+C\Big[K_0 + \int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau + \Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{3(2s-3)}{5s-6}} \times$$
$$\times \Big(\int_0^t \|\nabla^2 \boldsymbol{u}\|_2^2 d\tau\Big)^{\frac{3-s}{5s-6}}\Big] \Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}.$$

Again, for $\beta \geq 1$ we have

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq CK_0 + \frac{1}{2}\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau$$
$$+C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}\beta} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{\beta}} + C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}\beta} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{3}{4\beta}}$$
$$+C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}\beta} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{12(2s-3)}{19s-30}\frac{1}{\beta}}.$$

Thus

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq CK_0 + CV_s^{\frac{1}{\beta}} + CV_s^{\frac{3}{4\beta}} + CV_s^{\frac{12(2s-3)}{19s-30}\frac{1}{\beta}}.$$

As $\frac{12(2s-3)}{19s-30} \geq \frac{4}{3}$ for $s \in (\frac{30}{19}, 3]$, we have for $\frac{12(2s-3)}{19s-30} = \beta$

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq K + CV_s(t),$$

where $K = K(\|\boldsymbol{u}_0\|_{1,2}, \beta)$. Thus the Gronwall inequality implies that $\|\nabla \boldsymbol{u}(t)\|_2 \leq C < \infty$ for any $t < \mathcal{T}^*$. As in Theorem 1, if $s = \frac{30}{19}$, we need $\|\nabla u_3\|_{L^\infty(0,T;L^{\frac{30}{19}}(\mathbb{R}^3))} << 1$.

Similarly,
b) $s > 3$:

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq K_0 + C\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau$$
$$+C\Big(K_0 + \int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)\Big(\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{4}}$$
$$\leq CK_0 + C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{1}{\beta}}$$
$$+C\Big(\int_0^t \|\nabla u_3(\tau)\|_s^{\frac{2s}{2s-3}} \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau\Big)^{\frac{4}{3\beta}} + \frac{1}{2}\int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau.$$

Thus, if $\beta = \frac{4}{3}$, $\frac{2s}{2s-3}\frac{4}{3} = \frac{8s}{6s-9}$, we have

$$\|\nabla \boldsymbol{u}(t)\|_2^2 + \int_0^t \|\nabla^2 \boldsymbol{u}(\tau)\|_2^2 d\tau \leq K + CV_s,$$

with $K = K(\|\boldsymbol{u}_0\|_{1,2}, \beta)$. The Gronwall lemma finishes the proof of Theorem 2.

5 Two additional criteria

Note that we proved that if for a $q > \frac{10}{3}$ we have $\|u_3\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} < \infty$, then the solution to the Navier–Stokes equations is regular. This fact enables us to prove the following two corollaries:

c1 **Corollary 5.1** *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to $\boldsymbol{u}_0 \in W^{1,2}_{div}(\mathbb{R}^3)$ which satisfies the energy inequality. Let additionally*

$$\frac{\partial u_3}{\partial x_3} \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} < \frac{4}{5}, \quad s \in \Big(\frac{15}{4}, \infty\big].$$

Then the solution is regular.

r1 **Remark 5.1** In [14] is has been proved that the regularity is ensured if $\frac{\partial u_3}{\partial x_3} \in L^\infty((0,T)\times \mathbb{R}^3)$. This result is thus quite a big improvement of this result.

Proof. We proceed as before, i.e. we work on the time interval where the solution is smooth. We test the equation for u_3 by $|u_3|^{q-2}u_3$, $q > \frac{10}{3}$. Then

$$\frac{1}{q}\frac{d}{dt}\|u_3\|_q^q + C(q)\int_{\mathbb{R}^3} \big|\nabla |u_3|^{\frac{q}{2}}\big|^2 dx = (q-1)\int_{\mathbb{R}^3} p\frac{\partial u_3}{\partial x_3}|u_3|^{q-2}dx.$$

We need to estimate the right-hand side. We have

$$\begin{aligned}\int_{\mathbb{R}^3} p\frac{\partial u_3}{\partial x_3}|u_3|^{q-2}dx &\leq \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s \|u_3\|_q^{q-2}\|p\|_{\frac{qs}{2s-q}} \leq C\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s \|u_3\|_q^{q-2}\|\boldsymbol{u}\|^2_{\frac{2qs}{2s-q}} \\ \leq C\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s \|u_3\|_q^{q-2}\|\boldsymbol{u}\|_2^{\frac{6s-3q-qs}{qs}}\|\boldsymbol{u}\|_6^{\frac{3qs-6s+3q}{qs}} &\leq C\|u_3\|_q^{q-2}\Big(\|\boldsymbol{u}\|_6^2 + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s^{\frac{2qs}{6s-3q-qs}}\Big),\end{aligned}$$

where $\frac{3qs-6s+3q}{qs} \leq 2$, i.e. $s \geq \frac{3q}{6-q}$. Thus, passing with $q \to \frac{10}{3}^+$, $\frac{2qs}{6s-3q-qs} \to \frac{10s}{4s-15}^+$, $\frac{3q}{6-q} \to \frac{15}{4}^+$. The corollary is proved. $\square$

r2 **Remark 5.2** Indeed, in the proof of Corollary 5.1, instead of the estimate of u_3 in $L^q(\mathbb{R}^3)$, we could consider the estimate in $L^{3q}(\mathbb{R}^3)$, or in between. It would lead to the same result.

We could also use part of the information from the first energy estimate to increase the range of s. However, this would lead to worse (i.e. more regular) scale for $\frac{\partial u_3}{\partial x_3}$, thus we omit it.

Next result concerns the criterion on $\frac{\partial p}{\partial x_3}$. We get slightly better result than in [4], due to the fact that we improved the criterion on u_3. We have c2 **Corollary 5.2** *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to $\boldsymbol{u}_0 \in W^{1,2}_{div}(\mathbb{R}^3)$ which satisfies the energy inequality. Let additionally*

$$\frac{\partial p}{\partial x_3} \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s} < \frac{29}{10}, \quad s \in \Big(\frac{30}{23}, \frac{10}{3}\Big].$$

Then the solution is regular.

Proof. We proceed as above, only in the term on the right-hand side we do not integrate by parts. Thus we have

$$\frac{1}{q}\frac{d}{dt}\|u_3\|_q^q + C(q)\int_{\mathbb{R}^3} \big|\nabla |u_3|^{\frac{q}{2}}\big|^2 dx = -\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_3}|u_3|^{q-2}u_3 dx.$$

Now (note that $q \leq \frac{(q-1)s}{s-1} \leq 3q$ implies $\frac{3q}{2q+1} \leq s \leq q$)

$$\Big|\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_3}|u_3|^{q-2}u_3 dx\Big| \leq \Big\|\frac{\partial p}{\partial x_3}\Big\|_s \|u_3\|^{q-1}_{\frac{(q-1)s}{s-1}}$$

$$\leq \Big\|\frac{\partial p}{\partial x_3}\Big\|_s \|u_3\|_q^{\frac{2qs+s-3q}{2s}} \|u_3\|_{3q}^{\frac{3(q-s)}{2s}} \leq \varepsilon\|u_3\|^q_{3q} + C(\varepsilon)\Big\|\frac{\partial p}{\partial x_3}\Big\|_s^{\frac{2qs}{2qs+3s-3q}} \|u_3\|_q^{q\frac{2qs+s-3q}{2qs+3s-3q}}.$$

Thus

$$\frac{1}{q}\frac{d}{dt}\|u_3\|_q^q \leq C\Big\|\frac{\partial p}{\partial x_3}\Big\|_s^{\frac{2qs}{2qs+3s-3q}} \|u_3\|_q^{q\frac{2qs+s-3q}{2qs+3s-3q}}.$$

The Gronwall inequality implies the result, provided $\frac{\partial p}{\partial x_3} \in L^t(0,T;L^s(\mathbb{R}^3))$ with $\frac{2}{t}+\frac{3}{s} = 2+\frac{3}{q}$, $\frac{3q}{2q+1} \leq s \leq q$. Passing with $q \to \frac{10}{3}^+$ we get the result. □

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