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ON THE RELATION BETWEEN PSEUDO-CONFORMAL AND KAHLER GEOMETRY

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Dedicated to Professor Shigeru Ishihara on his sixtieth birthday.

In [8, 9] S. Webster showed a relation between the geometry of a Kähler manifold and the pseudo-conformal geometry of a real hypersurface in $\boldsymbol{C}^{n+1}$. In particular the Bochner tensor of the Kähler manifold may be identified with the fourth order pseudo-conformal invariant of Chern and Moser [3] of a real hypersurface constructed to be a circle bundle over a neighbourhood of the Kähler manifold. It is this relation between the two geometries that we wish to study further here and we prove two results. The first is that if an infinitesimal pseudo-conformal transformation on the circle bundle of the Webster construction is projectable, the projected vector field is an infinitesimal isometry of the Kähler metric. The second theorem is that if a holomorphic transformation of a Kähler manifold preserves the Bochner tensor, its covariant derivatives and the tensor $D_{\alpha\beta}$ (see below), it is a homothety.

It is known that the Bochner tensor is a conformal invariant [7], but of course a conformal, non-homothetic change of a Kähler metric destroys the Kähler property. Though conceivable that there may be holomorphic transformations preserving the Bochner tensor other than homotheties we conjecture not and Theorem 2 is a result in this direction.

1. Pseudo-conformal geometry.

The problem of pseudo-conformal geometry is, given two real hypersurfaces of $\boldsymbol{C}^{n+1}$, can one find local differential invariants on them whose agreement is equivalent to the hypersurfaces being (locally) biholomorphically equivalent? This problem was solved by Chern and Moser [3] and the invariants are called pseudo-conformal or Chern-Moser invariants.

Consider a hypersurface M in $\boldsymbol{C}^{n+1}$ given by $r(z^1, \cdots, z^{n+1}, \bar{z}^1, \cdots, \bar{z}^{n+1})=0$ and such that for the real form $\theta=i\partial r$, the Levi form $d\theta$ is non-degenerate. In particular θ annihilates the holomorphic tangent space of M. We set $\mathcal{D}_p=\{X\in T_pM \mid \theta(X)=0\}$ and $\mathcal{H}_p=\{X-iJX \mid X\in\mathcal{D}_p\}$ where J is the almost complex structure on $\boldsymbol{C}^{n+1}$. Since J is integrable $[\mathcal{H}, \mathcal{H}]\subset\mathcal{H}$ and hence

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M is a CR-manifold [4]. Let $\mathcal{J}X=JX$ for $X\in\mathcal{D}$; $\mathcal{J}$ is then an almost complex structure on the distribution $\mathcal{D}$ in the sense of Ishihara [5]. If $f: M\to M'$ is a diffeomorphism such that $F_*X\in\mathcal{D}'$ for $X\in\mathcal{D}$ and $f_*\mathcal{J}=\mathcal{J}'f_*$ we say that f is a *pseudo-conformal transformation;* in terms of hypersurfaces this is equivalent to f locally being the restriction of a local biholomorphic mapping [6, 10]. A vector field V on a real hypersurface is called a *pseudo-conformal vector field* if its 1-parameter group is a group of pseudo-conformal transformations. In extrinsic terms V is pseudo-conformal if and only if V is the restriction of a holomorphic vector field X on $\boldsymbol{C}^{n+1}$, i.e. $X|_M=V-iJV$, again see [6, 10].

Let $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$, $\alpha=1, \cdots, n$ be a local coframe on M with θ^α holomorphic and $\theta^{\bar\alpha}=\overline{\theta^\alpha}$. To this coframe we can associate connection forms $\phi, \phi^\alpha_\beta, \phi^\alpha, \psi$ with ϕ and ψ real, such that

$$\begin{aligned} d\theta &= i g_{\alpha\bar\beta}\theta^\alpha\wedge\theta^{\bar\beta}+\theta\wedge\phi\,,\\ d\theta^\alpha &= \theta^\beta\wedge\phi^\alpha_\beta+\theta\wedge\phi^\alpha\,,\\ dg_{\alpha\bar\beta}+g_{\alpha\bar\beta}\phi &= \phi^\gamma_\alpha g_{\gamma\bar\beta}+g_{\alpha\bar\gamma}\phi_{\bar\beta}^{\bar\gamma}\,,\\ d\phi &= i\phi^\alpha\wedge\theta_\alpha+i\theta^\alpha\wedge\phi_\alpha+\theta\wedge\psi\,, \end{aligned} \tag{1.1}$$

where $g_{\alpha\bar\beta}$ is used to raise and lower indices. Strictly speaking these are connection forms on a particular subbundle Y of the frame bundle of the line bundle E over M determined by θ[2, 3]. The curvature forms Φ^α_β, Φ^α are expressed by

$$\Phi^\alpha_\beta=S^\alpha_{\beta\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma}+V^\alpha_{\beta\rho}\theta^\rho\wedge\theta-V^\alpha_{\beta\bar\sigma}\theta^{\bar\sigma}\wedge\theta\,,$$

$$\Phi^\alpha=V^\alpha_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma}+P^\alpha_\rho\theta^\rho\wedge\theta+Q^\alpha_{\bar\sigma}\theta^{\bar\sigma}\wedge\theta\,.$$

Given θ, θ^α, ψ the forms ϕ^α_β, ϕ^α are uniquely determined by (1.1) and the trace condition $\Phi^\alpha_\alpha=0$. This connection is called the *pseudo-conformal connection* and for $n>1$ the pseudo-conformal invariants are given by S and its covariant derivatives with respect to this connection [3, p. 270] [9, p. 35].

The relation of pseudo-conformal to Kähler geometry was pointed out by Webster [8, 9] and is now easily seen. Let N^{2n} be a Kähler manifold with Kähler metric g and almost complex structure J. Then locally g is given by $g_{\alpha\bar\beta}=\dfrac{\partial^2 h}{\partial z^\alpha\partial\bar z^\beta}$ where $h=h(z, \bar z)$, $z=(z^1, \cdots, z^n)$ is a positive function defined on a coordinate neighbourhood U on N^{2n}. Now on $U\times\boldsymbol{C}$ let $r=h(z, \bar z)w\bar w-1$ where w is the coordinate on $\boldsymbol{C}$. Then the hypersurface M given by $r=0$ is a trivial circle bundle over U and the Levi form $d\theta$ is just the fundamental 2-form of the Kähler structure $\Omega(X, Y)=g(X, JY)$ lifted to M. θ itself in terms of local coordinates $z^\alpha=x^\alpha+iy^\alpha$, $w=u+iv$ is given by

$$\theta=i\partial r=\frac{1}{2}\Big(\frac{\partial\log h}{\partial y^\alpha}dx^\alpha-\frac{\partial\log h}{\partial x^\alpha}dy^\alpha\Big)+\frac{vdu-udv}{u^2+v^2}\,. \tag{1.2}$$

Note that if we set $t=\tan^{-1}(u/v)$ the last term is just dt. θ is a contact structure and $\frac{\partial}{\partial t}=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v}$ its characteristic vector field (see e.g. [1] p. 6).

Webster showed that the curvature tensor S may be identified with the Bochner tensor B of a Kähler manifold. If θ^α is a local basis of holomorphic 1-forms on N^{2n}, then the $g_{\alpha\bar\beta}$'s are given by $ds^2=g_{\alpha\bar\beta}\theta^\alpha\otimes\theta^{\bar\beta}$ and in turn on M by (1.1) with $\phi=0$. Let ω_β^α be the Riemannian connection of g and $R_{\beta\bar\alpha\rho\bar\sigma}$ its curvature tensor. We then have the following lemma of Webster [8] relating the connections and giving S as the Bochner tensor.

LEMMA. *Relative to the above coframe we have*

$$\phi_\beta^\alpha=\omega_\beta^\alpha+D_\beta^\alpha\theta\,,$$

$$\phi^\alpha=D_\beta^\alpha\theta^\beta+E^\alpha\theta\,,$$

$$\phi=i(E_\gamma\theta^\gamma-E_{\bar\gamma}\theta^{\bar\gamma})+G\theta\,,$$

where

$$D_{\alpha\beta}=i\left(\frac{R_{\alpha\bar\beta}}{n+2}-\frac{Rg_{\alpha\bar\beta}}{4(n+1)(n+2)}\right), \tag{1.3}$$

$$E_\alpha=\frac{R_{,\alpha}}{2(n+1)(n+2)}\,,$$

and G is a real function which can be uniquely determined. The curvature tensors are related by

$$S_{\beta\bar\alpha\rho\bar\sigma}=B_{\beta\bar\alpha\rho\bar\sigma}\,;$$

$$2(n+1)(n+2)Q_{\alpha\beta}=R_{,\alpha\beta}\,.$$

The comma denotes covariant differentiation with respect to the Riemannian connection ω on N^{2n}.

2. The main results.

We first consider the question of a pseudo-conformal vector field V on the hypersurface $M\subset U\times \boldsymbol{C}$ which is projectable to U. We have already noted that V is the restriction of a holomorphic vector field X, i.e. $X|_M=V-iJV$. In terms of the local coordinates (z^ε, w) $\varepsilon=1,\cdots,n$, $X=\alpha^\varepsilon\frac{\partial}{\partial z^\varepsilon}+\beta\frac{\partial}{\partial w}$ where α^ε and β are holomorphic functions of (z^ε, w). Writing $\alpha^\varepsilon=a^\varepsilon+ib^\varepsilon$ and $\beta=c+ie$, V is then given by

$$V=\frac{1}{2}\left(a^\varepsilon\frac{\partial}{\partial x^\varepsilon}+b^\varepsilon\frac{\partial}{\partial y^\varepsilon}+c\frac{\partial}{\partial u}+e\frac{\partial}{\partial v}\right)$$

and the condition that V is tangent to M is $Vr=0$, i.e.

$$a^{\varepsilon}\frac{\partial \log h}{\partial x^{\varepsilon}}+b^{\varepsilon}\frac{\partial \log h}{\partial y^{\varepsilon}}+2chu+2ehv=0\,. \tag{2.1}$$

Tanaka proved that if $V\in\mathcal{D}$, i.e. $\theta(V)=0$ then $V=0$. Here we suppose that V is projectable and prove the following result.

THEOREM 1. *Let V be a pseudo-conformal vector field on $M\subset U\times \boldsymbol{C}$ which is projectable to U. Let $\underline{V}$ be its projection on the Kähler manifold U. Then $\underline{V}$ is Killing.*

Proof. We first show that $\underline{V}h=0$. $\xi=v\dfrac{\partial}{\partial u}-u\dfrac{\partial}{\partial v}$ is the tangent field to the fibres of M as a circle bundle over U. Then the projectability of V means that

$$2[\xi,\ V]=(\xi a^{\varepsilon})\frac{\partial}{\partial x^{\varepsilon}}+(\xi b^{\varepsilon})\frac{\partial}{\partial y^{\varepsilon}}+(\xi c-e)\frac{\partial}{\partial u}+(\xi e+c)\frac{\partial}{\partial v}$$

is collinear with ξ. Thus

$$v\frac{\partial a^{\varepsilon}}{\partial u}-u\frac{\partial a^{\varepsilon}}{\partial v}=0\,,\qquad v\frac{\partial b^{\varepsilon}}{\partial u}-u\frac{\partial b^{\varepsilon}}{\partial v}=0 \tag{2.2}$$

and

$$v\frac{\partial c}{\partial u}-u\frac{\partial c}{\partial v}=\rho v+e\,,\qquad v\frac{\partial e}{\partial u}-u\frac{\partial e}{\partial v}=-\rho u-c\,. \tag{2.3}$$

As an immediate consequence of projectability or from (2.2) and the Cauchy-Riemann equations, a^{ε} and b^{ε} are independent of u and v. Similarly (2.3) and the Cauchy-Riemann equations give

$$\frac{\partial c}{\partial u}=\rho+\frac{ev+cu}{u^2+v^2}\,,\qquad \frac{\partial c}{\partial v}=\frac{cv-eu}{u^2+v^2}\,. \tag{2.4}$$

Now since a^{ε} and b^{ε} are independent of u and v, (2.1) implies that $cu+ev$ is independent of u and v. Differentiating $cu+ev$ with respect to u and v, using (2.4) and $hw\bar{w}=1$ we have

$$c+\rho u+ch(u^2-v^2)+2ehuv=0\,,$$

$$e+\rho v+2chuv-eh(u^2-v^2)=0\,.$$

Multiplying the first of these by u, the second by v and adding we obtain

$$\rho=-2h(cu+ev)\,.$$

Using this and (2.4), we have

$$0=\frac{\partial^2 c}{\partial u^2}+\frac{\partial^2 c}{\partial v^2}=2hc$$

and hence that $c=0$ and in turn $e=0$. Equation (2.1) now gives $\underline{V}h=0$.

Now as $\underline{V}-iJ\underline{V}$ is holomorphic, $\mathcal{L}_{\underline{V}}J=0$ and hence to show that $\mathcal{L}_{\underline{V}}g=0$, which can be done directly but is more lengthy, it suffices to show that $\mathcal{L}_{\underline{V}}\Omega=0$. But the lift of Ω to M is just $d\theta$ and the proof is completed simply by showing that $\mathcal{L}_V\theta=0$. We compute this with respect to the coordinates $(x^\alpha, y^\alpha, u, v)$. Recall that θ is given by (1.2). For x^α we have

$$4(\mathcal{L}_V\theta)_\alpha = a^\varepsilon\frac{\partial^2\log h}{\partial x^\varepsilon\partial y^\alpha}+b^\varepsilon\frac{\partial^2\log h}{\partial y^\varepsilon\partial y^\alpha}+\frac{\partial a^\varepsilon}{\partial x^\alpha}\frac{\partial\log h}{\partial y^\varepsilon}-\frac{\partial b^\varepsilon}{\partial x^\alpha}\frac{\partial\log h}{\partial x^\varepsilon}$$

since $c=e=0$ as shown above. Using the Cauchy-Riemann equations we see that the right hand side is $\dfrac{\partial}{\partial y^\alpha}(\underline{V}\log h)$ and hence vanishes. The computation of the other components is similar.

We now turn to our second question, namely what holomorphic transformations of a Kähler manifold preserve the Bochner tensor ? We conjecture that only homotheties do and prove the following weaker result.

THEOREM 2. *Let f be a holomorphic transformation of a Kähler manifold N^{2n} which preserves the Bochner tensor B, its covariant derivatives and the tensor $D_{\alpha\bar\beta}$ (1.3). Then f is a homothety.*

Proof. Let U and U' be coordinate neighborhoods on N^{2n} such that $f: U\to U'$. In the lemma of Webster we saw that the pseudo-conformal connection forms ϕ^α_β on the hypersurface $M\subset U\times \boldsymbol{C}$ are related to the connection forms of the Kähler metric on U by $\phi^\alpha_\beta=\omega^\alpha_\beta+D^\alpha_\beta\theta$. Consequently since f preserves B, its covariant derivatives and D, the map $\hat{f}: M\to M'$ given by $\hat{f}(z,t)=(f(z),t)$ preserves S and its covariant derivatives with respect to the pseudo-conformal connection and hence is a pseudo-conformal transformation. Here as we mentioned in section 1, $t=\tan^{-1}(u/v)$. If (x^α, y^α, t) are the local coordinates on U and if we let (u^α, v^α, t') denote the coordinates on U', the matrix of $\hat{f}_*$ is

$$\begin{pmatrix} \dfrac{\partial u^\alpha}{\partial x^\beta} & \dfrac{\partial u^\alpha}{\partial y^\beta} & 0 \\ \dfrac{\partial v^\alpha}{\partial x^\beta} & \dfrac{\partial v^\alpha}{\partial y^\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now from (1.2) the distribution $\mathcal{D}$ is spanned by

$$X_\alpha=\frac{\partial}{\partial x^\alpha}-\frac{1}{2}\frac{\partial\log h}{\partial y^\alpha}\frac{\partial}{\partial t}, \qquad X_{\bar\alpha}=\mathcal{J}X_\alpha=\frac{\partial}{\partial y^\alpha}+\frac{\partial\log h}{\partial x^\alpha}\frac{\partial}{\partial t}.$$

Thus in terms of these coordinates J is given by the matrix

$$\begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ \dfrac{1}{2}\dfrac{\partial \log h}{\partial x^{\alpha}} & \dfrac{1}{2}\dfrac{\partial \log h}{\partial y^{\alpha}} & 0 \end{pmatrix}.$$

Similarly $\mathcal{J}'$ is given by

$$\begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ \dfrac{1}{2}\dfrac{\partial \log h'}{\partial u^{\alpha}} & \dfrac{1}{2}\dfrac{\partial \log h'}{\partial v^{\alpha}} & 0 \end{pmatrix}.$$

$\mathcal{J}'\hat{f}_*=\hat{f}_*\mathcal{J}$ now gives

$$\frac{\partial \log h'}{\partial x^{\beta}}=\frac{\partial \log h}{\partial x^{\beta}}, \qquad \frac{\partial \log h'}{\partial y^{\beta}}=\frac{\partial \log h}{\partial y^{\beta}}$$

and hence h' is a constant times h giving that g' is homothetic to g.

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