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On the Relation of One-Dimensional Diffusions on Natural Scale and Their Speed Measures

David Criens[1]

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Abstract
It is well known that the law of a one-dimensional diffusion on natural scale is fully characterized by its speed measure. Stone proved a continuous dependence of such diffusions on their speed measures. In this paper we establish the converse direction, i.e., we prove a continuous dependence of the speed measures on their diffusions. Furthermore, we take a topological point of view on the relation. More precisely, for suitable topologies, we establish a homeomorphic relation between the set of regular diffusions on natural scale without absorbing boundaries and the set of locally finite speed measures.

Keywords Diffusion · Speed measure · Homeomorphism · Convergence of diffusions · Sufficient and necessary conditions · Limit theorem · Vague convergence · Weak convergence · Feller–Dynkin property · Itô diffusion

Mathematics Subject Classification (2020) 60J60 · 60G07 · 60F17

1 Introduction

It is well known (see, e.g., [3, 11]) that the law of a one-dimensional regular continuous strong Markov process on natural scale (called *diffusion* in this short section) is fully characterized by its speed measure. Among other things, Stone [25] proved that diffusions depend continuously on their speed measures and Brooks and Chacon [4] established the converse direction for real-valued diffusions, i.e., they proved a continuous dependence of the speed measures on the diffusions.

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✉ David Criens
david.criens@stochastik.uni-freiburg.de

1 Albert-Ludwigs University of Freiburg, Ernst-Zermelo-Str. 1, 79104 Freiburg, Germany

In this paper we establish a continuous dependence of the speed measures for general diffusions. The real-valued and the general case distinguish in two important points: for real-valued diffusions there is no issue with the boundary behavior and the corresponding speed measures are locally finite, which in particular means that they can be endowed with the vague topology. To treat the general case we use a new method of proof, which is quite different to those of Brooks and Chacon. In Sect. 3.4 we comment in more detail on the methods and compare them to each other.

As a second contribution, we investigate the relation of certain diffusions and their speed measures from a topological perspective. Namely, for suitable topologies, we deduce a homeomorphic relation between the set of regular diffusions on natural scale without absorbing boundaries and the set of locally finite speed measures. As an application of the homeomorphic relation, we discuss properties of certain subsets of the set of diffusions without absorbing boundaries, namely those with the Feller–Dynkin property and Itô diffusions with open state space. More precisely, we show that both of these subsets are dense Borel sets which are neither closed nor open.

The remainder of this paper is structured as follows. In Sect. 2 we introduce some notation and we recall the canonical diffusion framework used in this paper. Thereafter, in Sect. 3, we present and prove our main result, we discuss some of its consequences and we comment on the relation to the work [4]. Finally, in Sect. 4, we present our results on the topological relation of diffusions without absorbing boundaries and their speed measures.

2 Foundations

This section is split into three parts. In the first we recall some notation, in the second we introduce our probabilistic framework and in the third part we introduce the notion of speed measure convergence, which is crucial for the formulation of our main result.

2.1 Notation for Function and Measure Spaces

In this section we introduce our notation for function and measure spaces.

2.1.1 Function Spaces

Let G and F be topological spaces. We denote the set of functions from G into F by $M(G, F)$ and its subspace continuous functions by $C(G, F)$. We write $C(G)$ for the space $C(G, \mathbb{R})$ and we write $C_b(G)$ for its subspace of real-valued bounded continuous functions. Further, we write $C_c(G)$ for the set of continuous functions $G \to \mathbb{R}$ with compact support. In case G is a locally compact topological space, $C_0(G)$ denotes the set of continuous functions $G \to \mathbb{R}$ which are vanishing at infinity. Finally, if G is an open subset of $\mathbb{R}$, then $L^1_{\text{loc}}(G)$ is defined to be the set of all locally integrable Borel functions $G \to \mathbb{R}$, where functions are identified when they agree Lebesgue almost everywhere. We endow $L^1_{\text{loc}}(G)$ with the local L^1-topology, which is generated by the metric

$$d_{L^1_{\text{loc}}(G)}(f,g) \triangleq \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left(\int_{G_k} |f(x) - g(x)| dx, 1 \right), \quad f, g \in L^1_{\text{loc}}(G),$$

where $(G_k)_{k=1}^{\infty} \subset G$ is a sequence of compact subsets of $\mathbb{R}$ such that $G_1 \subset G_2 \subset \cdots$ and $G = \bigcup_{k=1}^{\infty} G_k$. Notice that $L^1_{\text{loc}}(G)$ is a Polish space with this topology.[1]

2.1.2 Measure Spaces

Let G be a Polish space and denote its Borel σ-field by $\mathcal{B}(G)$. The set of probability measures on $(G, \mathcal{B}(G))$ is denoted by $\mathcal{M}_1(G)$. We endow $\mathcal{M}_1(G)$ with the *weak topology*, i.e., with the coarsest topology on $\mathcal{M}_1(G)$ with respect to which all mappings $\mu \mapsto \int f d\mu$, $f \in C_b(G)$, are continuous. It is well known that $\mathcal{M}_1(G)$ is a Polish space.

Suppose now that G is a locally compact Polish space. A measure μ on $(G, \mathcal{B}(G))$ is said to be *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset G$. The set of all locally finite measures on $(G, \mathcal{B}(G))$ is denoted by $\mathcal{M}(G)$. We endow $\mathcal{M}(G)$ with the *vague topology*, i.e., with the coarsest topology on $\mathcal{M}(G)$ with respect to which all mappings $\mu \mapsto \int f d\mu$, $f \in C_c(G)$, are continuous. The space $\mathcal{M}(G)$ is Polish ([2, Theorem 31.5]).

2.2 Canonical Framework for Diffusions

We work with the canonical setting for diffusions as introduced in [21, Sect. V.25]. A quite complete treatment of the theory is given in the monograph of Itô and McKean [11]. Shorter introductions can be found in the monographs [3, 14, 20, 21].

Let $J \subset \mathbb{R}$ be a finite or infinite, closed, open or half-open interval, denote its interior by J° and by $\partial J \triangleq J \backslash J^\circ$ the boundary points in J. We define Ω to be the space of continuous functions from $\mathbb{R}_+ \triangleq [0, \infty)$ into J endowed with the local uniform topology. The coordinate process on Ω is denoted by X, i.e., $\mathsf{X}_t(\omega) = \omega(t)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$. It is well known that the Borel σ-field on Ω is given by $\mathcal{F} \triangleq \sigma(\mathsf{X}_s, s \geq 0)$. For any time $t \in \mathbb{R}_+$, we also set $\mathcal{F}_t \triangleq \sigma(\mathsf{X}_s, s \leq t)$ and we define the shift operator $\theta_t \colon \Omega \to \Omega$ by $(\theta_t \omega)(s) = \omega(t+s)$ for all $s, t \in \mathbb{R}_+$.

We call $(J \ni x \mapsto P_x \in \mathcal{M}_1(\Omega))$ a *(canonical) diffusion*, if $x \mapsto P_x(A)$ is measurable for all $A \in \mathcal{F}$, $P_x(\mathsf{X}_0 = x) = 1$ for all $x \in J$, and, for any $(\mathcal{F}_{t+})_{t \geq 0}$-stopping time τ and any $x \in J$, P_{X_τ} is the regular conditional P_x-distribution of $\theta_\tau \mathsf{X}$ given $\mathcal{F}_{\tau+}$ on $\{\tau < \infty\}$, i.e., P_x-a.s. on $\{\tau < \infty\}$

$$P_x\left(\theta_\tau^{-1} G | \mathcal{F}_{\tau+}\right) = P_{\mathsf{X}_\tau}(G), \quad G \in \mathcal{F}.$$

[1] The space $(L^1_{\text{loc}}(G), d_{L^1_{\text{loc}}(G)})$ is a complete metric space by [16, Lemma I.5.17], and it is separable as $C(G)$ is a dense subset ([9, Lemma 7.2]) and $C(G)$ is separable for the local uniform topology ([14, Lemma A.5.1]).

The final part is the *strong Markov property*. A diffusion $(x \mapsto P_x)$ is called *regular* if, for all $x \in J^\circ$ and $y \in J$,

$$P_x(\gamma_y < \infty) > 0, \tag{2.1}$$

where

$$\gamma_y \triangleq \inf(s \geq 0 \colon \mathsf{X}_s = y)$$

with the convention that $\inf(\varnothing) \triangleq \infty$. Further, $(x \mapsto P_x)$ is called *completely regular*[2] if (2.1) holds for all $x, y \in J$. Notice that regularity and complete regularity are equivalent for open J. We say that a regular diffusion $(x \mapsto P_x)$ is on *natural scale* if, for all $a, b, x \in J$ with $a < x < b$, we have

$$P_x(\gamma_b < \gamma_a) = \frac{x-a}{b-a}.$$

Any regular diffusion can be brought to natural scale via a homeomorphic space transformation ([3, Proposition 16.34]). Let $(x \mapsto P_x)$ be a regular diffusion on natural scale. According to [3, Theorem 16.36], there exists a unique locally finite measure $\mathfrak{m}$ on $(J^\circ, \mathcal{B}(J^\circ))$ such that, for any $a < b$ with $[a, b] \subset J^\circ$, we have

$$E_x\big[\gamma_a \wedge \gamma_b\big] = \int G_{(a,b)}(x, y)\mathfrak{m}(dy), \quad x \in (a, b),$$

where $G_{(a,b)}$ is the Green function defined by

$$G_{(a,b)}(x, y) \triangleq \begin{cases} \dfrac{2(x \wedge y - a)(b - x \vee y)}{b-a}, & a \leq x, y \leq b, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Next, we extend $\mathfrak{m}$ from the interior J° to the whole space J. Suppose that J is closed at the left side with boundary point $l \in \partial J$. We now define a symmetrized Green function for all intervals of the form $I \triangleq [l, c)$ with $c \in J^\circ$. Let I^* be the reflection of I around l, i.e., $I^* = (l - (c - l), l]$ and, for $y \in I$, let y^* be the reflection of y around l, i.e., $y^* = l - (y - l)$. Then, define

$$G_I(x, y) \triangleq G_{I \cup I^*}(x, y) + G_{I \cup I^*}(x, y^*), \quad x, y \in J, \tag{2.3}$$

where $G_{I \cup I^*}$ is given by (2.2). By [3, Theorem 16.47], $\mathfrak{m}(\{l\})$ can be defined such that, for any $c \in J^\circ$,

$$E_x\big[\gamma_b\big] = \int G_{[l,c)}(x, y)\mathfrak{m}(dy), \quad x \in [l, c).$$

[2] This terminology is new in the sense that it does not appear in [3, 11, 14, 20, 21].

In particular, in case $\mathfrak{m}((l, c)) = \infty$ for some $c \in J^\circ$, we set $\mathfrak{m}(\{l\}) \triangleq \infty$. In the same way, $\mathfrak{m}(\{r\})$ can be defined for a right boundary point r which is an element of J. In this manner, we get a measure $\mathfrak{m}$ on $(J, \mathcal{B}(J))$ which is called the *speed measure* associated to the regular diffusion $(x \mapsto P_x)$. The speed measure is an element of $\mathcal{M}(J)$, i.e., locally finite, if and only if the corresponding diffusion is completely regular. Within the class of regular diffusions on natural scale, the speed measure determines a diffusion uniquely ([3, Corollary 16.73]).

We end this section with some boundary terminology. We say that a boundary point $b \in \partial J$ is *absorbing* if $\mathfrak{m}(\{b\}) = \infty$, and otherwise we call it *reflecting*. This terminology coincides with those from [14, 21] but it differs slightly from those in [3, 11, 20], where finer allocations are given.

2.3 Speed Measure Convergence

We now introduce the nonstandard concept of speed measure convergence. Define $l \triangleq \inf J$ and $r \triangleq \sup J$.

Definition 2.1 We say that the sequence $(\mathfrak{m}^n)_{n=1}^\infty$ of speed measures on J converges in the *speed measure sense* to a speed measure $\mathfrak{m}^0$ on J, which we denote by $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$, if the following hold:

(a) $\mathfrak{m}^n|_{J^\circ} \to \mathfrak{m}^0|_{J^\circ}$ vaguely.
(b) If $l \in J$, then $\int f d\mathfrak{m}^n \to \int f d\mathfrak{m}^0$ for all $0 \le f \in C(J)$ such that $f(l) > 0$ and $f = 0$ off $[l, y)$ for some $y \in J^\circ$.
(c) If $r \in J$, then $\int f d\mathfrak{m}^n \to \int f d\mathfrak{m}^0$ for all $0 \le f \in C(J)$ such that $f(r) > 0$ and $f = 0$ off $(y, r]$ for some $y \in J^\circ$.

Remark 2.2 If $(\mathfrak{m}^n)_{n=0}^\infty \subset \mathcal{M}(J)$, then $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$ if and only if $\mathfrak{m}^n \to \mathfrak{m}^0$ vaguely.

Example 2.3 Consider $J = \mathbb{R}_+$ and $\mathfrak{m}^n(dx) \triangleq dx + n\delta_0(dx)$ for $n \in \mathbb{N}$. Then, $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0(dx) \triangleq dx + \infty\delta_0(dx)$. Broadly speaking, the speed measure of a sticky Brownian motion converges in the speed measure sense to those of an absorbed Brownian motion if the stickiness parameter increases to infinity.

3 Stone's Theorem and Its Converse

In this section, we fix a sequence $(\mathfrak{m}^n)_{n=0}^\infty$ of arbitrary speed measures on J. For each $n \in \mathbb{Z}_+ \triangleq \mathbb{Z} \cap [0, \infty)$, let $(J \ni x \mapsto P_x^n)$ be the regular diffusion on natural scale with speed measure $\mathfrak{m}^n$.

3.1 Stone's Theorem

In his seminal paper [25], Stone investigated limit theorems for random walks, birth and death processes and diffusions. Boiled down to the class of regular diffusions on natural scale, Stone proved the following theorem, which is implied by part (5) of his Corollary 1.

Theorem 3.1 (Stone's Theorem) *If* $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$, *then* $P^n_{x^n} \to P^0_{x^0}$ *weakly for all sequences* $(x^n)_{n=0}^{\infty} \subset J$ *such that* $x^n \to x^0$.

Stone's theorem provides a sufficient condition for the convergence of regular diffusions (on natural scale) in terms of their speed measures. Further, it shows that a regular diffusion $(x \mapsto P_x)$ on natural scale can be seen as a continuous map from J into $\mathcal{M}_1(\Omega)$. This observation deserves a formal statement.

Corollary 3.2 *If* $(x \mapsto P_x)$ *is a regular diffusion on natural scale, then* $x \mapsto P_x$ *is a continuous function from* J *into* $\mathcal{M}_1(\Omega)$, *i.e.,* $(x \mapsto P_x) \in C(J, \mathcal{M}_1(\Omega))$.

As a consequence of Corollary 3.2, $(x \mapsto P^n_x)_{n=1}^{\infty}$ is a sequence in the space $C(J, \mathcal{M}_1(\Omega))$ and hence, we can ask whether it converges in the local uniform topology.

Corollary 3.3 *If* $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$, *then* $(x \mapsto P^n_x) \to (x \mapsto P^0_x)$ *locally uniformly in* $C(J, \mathcal{M}_1(\Omega))$.

Proof of Corollary 3.3 Thanks to Theorem 3.1, for every sequence $(x^n)_{n=0}^{\infty} \subset J$ with $x^n \to x^0$, we have $P^n_{x^n} \to P^0_{x^0}$ weakly. In other words, the sequence $(x \mapsto P^n_x)_{n=1}^{\infty}$ converges continuously to $(x \mapsto P^0_x)$. A theorem by Carathéodory ([18, Theorem on pp. 98–99]) shows that continuous convergence is equivalent to local uniform convergence. □

3.2 A Converse to Theorem 3.1

The following theorem, which can be seen as a converse to Theorem 3.1, is our main result. We present its proof in Sect. 3.3.

Theorem 3.4 *Suppose that there exists a point* $y \in J^{\circ}$ *such that* $P^n_x \to P^0_x$ *weakly for all* $x \in \partial J \cup \{y\}$. *Then,* $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$.

Remark 3.5 The statement of Theorem 3.4 *cannot* be weakened to the following: if $P^n_{x_n} \to P^0_{x^0}$ weakly for *some* sequence $(x^n)_{n=0}^{\infty} \subset J$, then $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$. Indeed, if J is non-open with $b \in \partial J$ and all diffusions $(x \mapsto P^n_x)_{n=0}^{\infty}$ are absorbed in b, then $P^n_b \to P^0_b$ is trivially true independently of the speed measures.

Corollary 3.6 *The following are equivalent:*

(1) $\mathfrak{m}^n \Rightarrow \mathfrak{m}^0$.
(2) $P^n_{x^n} \to P^0_{x^0}$ *weakly for every sequence* $(x^n)_{n=0}^{\infty} \subset J$ *such that* $x^n \to x^0$.
(3) $(x \mapsto P^n_x) \to (x \mapsto P^0_x)$ *locally uniformly in* $C(J, \mathcal{M}_1(\Omega))$.
(4) *There exists a point* $y \in J^{\circ}$ *such that* $P^n_x \to P^0_x$ *weakly for all* $x \in \partial J \cup \{y\}$.

Proof The implication (1) ⇒ (2) follows from Theorem 3.1, the implication (1) ⇒ (3) is given by Corollary 3.3, that either (2) or (3) implies (4) is trivial and, finally, the implication (4) ⇒ (1) follows from Theorem 3.4. □

Remark 3.7 (1) Corollary 3.6 shows that on the set of regular diffusions on natural scale the sequential topologies of pointwise and local uniform convergence coincide.

(2) It is interesting to compare Corollary 3.6 to a variant of the Trotter–Kato theorem as given by [14, Theorem 17.25], which states that convergence (in a certain sense) of infinitesimal generators is equivalent to weak convergence of the associated Feller–Dynkin processes for arbitrary weakly convergent initial laws. For regular diffusions that are Feller–Dynkin processes, Rosenkrantz and Dorea [23] deduced a version of Stone's theorem from an early variant of [14, Theorem 17.25] which is due to Kurtz [15]. Furthermore, Rosenkrantz [22] emphasized the importance of Kurtz' work, who established a necessary and sufficient condition in his result. In the same spirit, Corollary 3.6 provides a necessary and sufficient condition for weak convergence of regular diffusions on natural scale, which need, in general, not to be Feller–Dynkin processes, cf. [7] for a characterization of the class of regular Feller–Dynkin diffusions (on natural scale) in terms of their speed measures.

(3) It is also interesting to notice the role of the initial values in Corollary 3.6. In this regard, we highlight the implication (4) $\Rightarrow$ (2), which tells us that the weak convergence $P^n_{x^n} \to P^0_{x^0}$ holds for *all sequences* $(x^n)_{n=0}^\infty \subset J$ such that $x^n \to x^0$ once it holds for (at most) *three constant sequences* (one taken from the interior and (at most) two for the attainable boundary points). In particular, in case the state space J is open, weak convergence of regular diffusions for an arbitrary fixed initial value already implies weak convergence of the diffusions for all convergent sequences of initial values. The proof of the implication (4) $\Rightarrow$ (2) fully relies on Theorem 3.4, which shows that at most three constant test sequences of initial values suffice to understand the convergence of the speed measures, which then, by Theorem 3.1, implies weak convergence of the corresponding diffusions for all convergent sequences of initial values.

3.3 Proof of Theorem 3.4

This section is split into four parts. In Sect. 3.3.1 we establish some preliminary technical results. Thereafter, in Sect. 3.3.2, we establish part (a) of Definition 2.1, i.e., we prove vague convergence of the speed measures on the interior of the state space. In Sect. 3.3.3 we consider part (b) of Definition 2.1, i.e., we establish convergence of the speed measures at closed left boundaries. Here, we distinguish between the cases where the boundary point is absorbing or reflecting. Part (c) of Definition 2.1, which deals with closed right boundary points, can be proved similar to part (b) and we omit a detailed proof for brevity. Finally, in Sect. 3.3.4, we connect the pieces and deduce Theorem 3.4.

3.3.1 Preparations

In this section, we prepare the proof of Theorem 3.4 with some technical results. For $y \in J$, we set

$$\tau^+_y \triangleq \inf(s \geq 0\colon \mathsf{X}_s \geq y), \quad \tau^-_y \triangleq \inf(s \geq 0\colon \mathsf{X}_s \leq y),$$

$$\sigma_y^+ \triangleq \inf(s \geq 0\colon \mathsf{X}_s > y), \quad \sigma_y^- \triangleq \inf(s \geq 0\colon \mathsf{X}_s < y),$$

which are functions from Ω into $[0, \infty]$ with the convention that $\inf(\varnothing) = \infty$.

Recall that a function f from a topological space T into $[-\infty, \infty]$ is said to be *upper semicontinuous* if $\{f < c\} = \{t \in T : f(t) < c\}$ is open for every $c \in \mathbb{R}$, and that f is called *lower semicontinuous* if $\{f \leq c\}$ is closed for every $c \in \mathbb{R}$. Equivalently, upper and lower semicontinuity can be defined via nets (or sequences if T is first countable). To be more precise ([1, Lemma 2.42]), assuming that T is first countable, f is upper semicontinuous if and only if $t^n \to t$ implies that $\limsup_{n\to\infty} f(t^n) \leq f(t)$, and f is lower semicontinuous if and only if $t^n \to t$ implies that $\liminf_{n\to\infty} f(t^n) \geq f(t)$. Further, recall that the path space Ω is endowed with the local uniform topology.

Lemma 3.8 *For any $y \in J$, the functions $\Omega \ni \omega \mapsto \sigma_y^{\pm}(\omega) \in [0, \infty]$ are upper semicontinuous and the functions $\Omega \ni \omega \mapsto \tau_y^{\pm}(\omega) \in [0, \infty]$ are lower semicontinuous.*

Proof of Lemma 3.8 The claim is implied by [17, Exercise 2.1 on p. 75]. For completeness, we provide a proof for σ_y^+ and τ_y^+. The arguments for σ_y^- and τ_y^- work the same way. Fix an arbitrary $t > 0$. We have

$$\left\{\sigma_y^+ < t\right\} = \bigcup_{\substack{s\in\mathbb{Q}\\ s<t}} \{\mathsf{X}_s > y\}.$$

As $\omega \mapsto \omega(s)$ is continuous for every $s \in \mathbb{R}_+$, the set $\{\mathsf{X}_s > y\}$ is open, as the inverse image of an open set under a continuous map is open (by definition). As unions of open sets are open (by definition), $\{\sigma_y^+ < t\}$ is also open. Consequently, σ_y^+ is upper semicontinuous.

Take $t \in \mathbb{R}_+$ and let $d_y(x) \triangleq \inf_{z\geq y} |z - x|$ for $x \in J$. We have

$$\left\{\tau_y^+ \leq t\right\} = \left\{\inf_{s\in\mathbb{Q}\cap[0,t]} d_y(\mathsf{X}_s) = 0\right\}.$$

For every $s \in \mathbb{Q} \cap [0, t]$ and $\omega, \omega' \in \Omega$, we get

$$\inf_{r\in\mathbb{Q}\cap[0,t]} d_y(\omega(r)) \leq d_y(\omega(s)) \leq \sup_{r\leq t} |\omega(r) - \omega'(r)| + d_y(\omega'(s)).$$

Taking the infimum over s and using symmetry yields that

$$\left|\inf_{r\in\mathbb{Q}\cap[0,t]} d_y(\omega(r)) - \inf_{r\in\mathbb{Q}\cap[0,t]} d_y(\omega'(r))\right| \leq \sup_{r\leq t} |\omega(r) - \omega'(r)|.$$

Consequently, $\omega \mapsto \inf_{s\in\mathbb{Q}\cap[0,t]} d_y(\omega(s))$ is continuous (in the local uniform topology) and $\{\tau_y^+ \leq t\}$ is closed, as the inverse image of a closed set under a continuous map is closed (by definition). Finally, we conclude that τ_y^+ is lower semicontinuous.

□

Lemma 3.9 *Let $x_0 \in J$ and let $(J \ni x \mapsto P_x)$ be a regular diffusion on natural scale. For every $a \in (l, x_0)$ and $b \in (x_0, r)$, we have P_{x_0}-a.s.*

$$\tau_a^- = \sigma_a^- = \gamma_a \text{ and } \tau_b^+ = \sigma_b^+ = \gamma_b.$$

Proof of Lemma 3.9 We only show that P_{x_0}-a.s. $\tau_a^- = \sigma_a^- = \gamma_a$. The proof for the other claim is similar. As P_{x_0}-a.s. $\mathsf{X}_0 = x_0$ and $a < x_0$, P_{x_0}-a.s. $\tau_a^- = \gamma_a$ is clear. Thanks to [8, Lemma 2.12], as $a \in J^\circ$, it holds that P_a-a.s. $\sigma_a^- = 0$. Now, using the strong Markov property, we get

$$\begin{aligned} P_{x_0}(\tau_a^- = \sigma_a^-, \tau_a^- < \infty) &= P_{x_0}(\sigma_a^-(\theta_{\tau_a^-}\mathsf{X}) = 0, \tau_a^- < \infty) \\ &= E_{x_0}\Big[P_{\mathsf{X}_{\tau_a^-}}(\sigma_a^- = 0)\mathbb{1}_{\{\tau_a^- < \infty\}}\Big] \\ &= P_a(\sigma_a^- = 0)P_{x_0}(\tau_a^- < \infty) \\ &= P_{x_0}(\tau_a^- < \infty). \end{aligned}$$

Since $\tau_a^- \leq \sigma_a^-$, we clearly have $\tau_a^- = \sigma_a^-$ on $\{\tau_a^- = \infty\}$. This completes the proof. □

For $P \in \mathcal{M}_1(\Omega)$, we say that a Borel function $f: \Omega \to [-\infty, \infty]$ is P-a.s. continuous if

$$P(\{\omega \in \Omega : f \text{ is discontinuous at } \omega\}) = 0.$$

Equivalently, f is P-a.s. continuous if there exists a set $G \in \mathcal{F}$ such that f is continuous at every $\omega \in G$ and $P(G) = 1$.

Lemma 3.10 *Let $x_0 \in J$ and let $(J \ni x \mapsto P_x)$ be a regular diffusion on natural scale. For every $a \in J^\circ \backslash \{x_0\}$, the function $\Omega \ni \omega \mapsto \gamma_a(\omega) \in [0, \infty]$ is P_{x_0}-a.s. continuous.*

Proof of Lemma 3.10 We suppose that $a < x_0$. The case $a > x_0$ works the same way. Set

$$G \triangleq \{\omega \in \Omega : \omega(0) = x_0, \tau_a^-(\omega) = \sigma_a^-(\omega)\}.$$

By Lemma 3.9, we have $P_{x_0}(G) = 1$. Take $\omega^0 \in G$ and let $(\omega^n)_{n=1}^\infty \subset \Omega$ be such that $\omega^n \to \omega^0$ locally uniformly. W.l.o.g. we may assume that $\omega^n(0) > a$ for all $n = 1, 2, \ldots$. Since, by Lemma 3.8, $\omega \mapsto \tau_a^-(\omega)$ is lower semicontinuous and $\omega \mapsto \sigma_a^-(\omega)$ is upper semicontinuous, and $\tau_a^- \leq \sigma_a^-$, we get that

$$\tau_a^-(\omega^0) \leq \liminf_{n\to\infty} \tau_a^-(\omega^n) \leq \limsup_{n\to\infty} \tau_a^-(\omega^n) \leq \limsup_{n\to\infty} \sigma_a^-(\omega^n) \leq \sigma_a^-(\omega^0). \quad (3.1)$$

By definition of G and because $\omega^0 \in G$, we have $\sigma_a^-(\omega^0) = \tau_a^-(\omega^0) = \gamma_a(\omega^0)$. Thus, since $\tau_a^-(\omega^n) = \gamma_a(\omega^n)$ for all $n = 1, 2, \ldots$, we conclude from (3.1) that

$$\gamma_a(\omega^0) \leq \liminf_{n\to\infty} \gamma_a(\omega^n) \leq \limsup_{n\to\infty} \gamma_a(\omega^n) \leq \gamma_a(\omega^0).$$

This shows that $\omega \mapsto \gamma_a(\omega)$ is continuous at ω^0. □

Lemma 3.11 *Let $(P^n)_{n=0}^{\infty} \subset \mathcal{M}_1(\Omega)$ be a sequence such that $P^n \to P^0$ weakly. For every $b \in J$ and $t > 0$, there are numbers $N_b^+ \in \mathbb{Z}_+$ and $N_b^- \in \mathbb{Z}_+$ such that*

$$\sup_{n \in \mathbb{Z}_+} P^n(\sigma_b^+ \geq t) = P^{N_b^+}(\sigma_b^+ \geq t), \qquad \sup_{n \in \mathbb{Z}_+} P^n(\sigma_b^- \geq t) = P^{N_b^-}(\sigma_b^- \geq t).$$

Proof of Lemma 3.11 Take $b \in J$ and $t > 0$. By Lemma 3.8, the maps $\omega \mapsto \sigma_b^{\pm}(\omega)$ are upper semicontinuous. Hence, by [1, Theorem 15.5], the maps $\mathcal{M}_1(\Omega) \ni P \mapsto P(\sigma_b^{\pm} \geq t) \in [0, 1]$ are also upper semicontinuous. Since $P^n \to P^0$ weakly, the set $\{P^n : n \in \mathbb{Z}_+\}$ is compact in $\mathcal{M}_1(\Omega)$. Now, the claim follows from the fact that real-valued upper semicontinuous functions attain a maximum value on a compact set ([1, Theorem 2.43]). □

Lemma 3.12 *Let $[a, b] \subset J$ be a proper interval and let $(J \ni x \mapsto P_x)$ be a regular diffusion on natural scale. Furthermore, take $x, x_0 \in [a, b]$. If $x \leq x_0$, then, for every $t > 0$,*

$$P_x(\gamma_a \geq t) \leq P_{x_0}(\gamma_a \geq t), \tag{3.2}$$

and if $x_0 \leq x$, then, for every $t > 0$,

$$P_x(\gamma_b \geq t) \leq P_{x_0}(\gamma_b \geq t). \tag{3.3}$$

In particular, for all $t > 0$, we have

$$P_x(\gamma_{a,b} \geq t) \leq \max\big(P_{x_0}(\gamma_a \geq t), P_{x_0}(\gamma_b \geq t)\big). \tag{3.4}$$

Proof of Lemma 3.12 We argue as in the proof of [3, Lemma 16.25]. If $x \leq x_0$, the strong Markov property yields that

$$\begin{aligned} P_{x_0}(\gamma_a < t) &\leq P_{x_0}(\gamma_x < \infty, \gamma_a(\theta_{\gamma_x}\mathsf{X}) < t) \\ &= E_{x_0}\big[\mathbb{1}_{\{\gamma_x<\infty\}} P_{\mathsf{X}_{\gamma_x}}(\gamma_a < t)\big] \\ &= P_{x_0}(\gamma_x < \infty) P_x(\gamma_a < t) \\ &\leq P_x(\gamma_a < t). \end{aligned}$$

This shows the inequality (3.2). If $x_0 \leq x$, the same computation yields the inequality (3.3). The final inequality (3.4) is an immediate consequence of (3.2) and (3.3). □

3.3.2 Proof for Convergence in the Interior: Part (a) from Definition 2.1

Let $(J \ni x \mapsto P_x^n)_{n=0}^{\infty}$ be a sequence of regular diffusions on natural scale. Recall that $l = \inf J$ and $r = \sup J$. In particular, this notation means that $J^\circ = (l, r)$.

Lemma 3.13 *Assume that $P^n_{x_0} \to P^0_{x_0}$ weakly for some $x_0 \in J^\circ$. Then, for all $a \in (l, x_0)$, $b \in (x_0, r)$ and $f \in C_c(J^\circ)$,*

$$E^n_{x_0}\Big[\int_0^{\gamma_a \wedge \gamma_b} f(X_s)ds\Big] \to E^0_{x_0}\Big[\int_0^{\gamma_a \wedge \gamma_b} f(X_s)ds\Big] \tag{3.5}$$

as $n \to \infty$.

Before we prove this lemma, we deduce part (a) of Definition 2.1.

Corollary 3.14 *If $P^n_{x_0} \to P^0_{x_0}$ weakly for some $x_0 \in J^\circ$, then $\mathfrak{m}^n|_{J^\circ} \to \mathfrak{m}^0|_{J^\circ}$ vaguely.*

Proof of Corollary 3.14 Let $f \in C_c(J^\circ)$ and let $[a, b] \subset J^\circ$ be a proper interval such that $f = 0$ off $[a, b]$. Take $c, d \in J^\circ$ such that $c < a$ and $d > b$, and such that $x_0 \in (c, d)$. Recall that $G_{(c,d)}$ denotes the Green function as defined in (2.2). Then, $(x \mapsto f(x)/G_{(c,d)}(x_0, x)) \in C_c(J^\circ)$, and [20, Corollary VII.3.8] yields that

$$E^n_{x_0}\Big[\int_0^{\gamma_c \wedge \gamma_d} \frac{f(\mathsf{X}_s)ds}{G_{(c,d)}(x_0, \mathsf{X}_s)}\Big] = \int G_{(c,d)}(x_0, y)\,\frac{f(y)}{G_{(c,d)}(x_0, y)}\,\mathfrak{m}^n(dy) = \int f d\mathfrak{m}^n$$

for all $n \in \mathbb{Z}_+$. Finally, thanks to Lemma 3.13, we obtain that

$$\begin{aligned}\int f d\mathfrak{m}^n &= E^n_{x_0}\Big[\int_0^{\gamma_c \wedge \gamma_d} \frac{f(\mathsf{X}_s)ds}{G_{(c,d)}(x_0, \mathsf{X}_s)}\Big] \\ &\to E^0_{x_0}\Big[\int_0^{\gamma_c \wedge \gamma_d} \frac{f(\mathsf{X}_s)ds}{G_{(c,d)}(x_0, \mathsf{X}_s)}\Big] = \int f d\mathfrak{m}^0.\end{aligned}$$

This proves that $\mathfrak{m}^n|_{J^\circ} \to \mathfrak{m}^0|_{J^\circ}$ vaguely. □

In the remainder of this section, we prove Lemma 3.13. The two main ingredients of the proof are the continuous mapping theorem and a uniform second moment bound for the stopping time $\gamma_a \wedge \gamma_b$, which is established by the following lemma.

Lemma 3.15 *Assume that $P^n_{x_0} \to P^0_{x_0}$ weakly for some $x_0 \in J^\circ$ and let $[a, b] \subset J^\circ$ be a proper interval such that $x_0 \in (a, b)$. Then,*

$$\sup_{n \in \mathbb{Z}_+} E^n_{x_0}\big[\gamma^2_{a,b}\big] < \infty \quad \textit{with} \quad \gamma_{a,b} \triangleq \gamma_a \wedge \gamma_b.$$

Proof of Lemma 3.15 We fix $t > 0$ and define

$$\alpha \triangleq \sup\big(P^n_x(\gamma_{a,b} \geq t) : n \in \mathbb{Z}_+, a \leq x \leq b\big).$$

Thanks to Lemmata 3.9, 3.11 and 3.12, there are numbers $N^-_a, N^+_b \in \mathbb{Z}_+$ such that

$$\alpha \leq \sup_{n \in \mathbb{Z}_+}\big(\max(P^n_{x_0}(\gamma_a \geq t), P^n_{x_0}(\gamma_b \geq t))\big)$$

$$= \sup_{n\in\mathbb{Z}_+} \big(\max(P^n_{x_0}(\sigma_a^- \geq t), P^n_{x_0}(\sigma_b^+ \geq t)) \big)$$
$$\leq \max\big(P^{N_a^-}_{x_0}(\sigma_a^- \geq t), P^{N_b^+}_{x_0}(\sigma_b^+ \geq t)\big).$$

Thanks to [5, Theorem 1.1], regular diffusions hit (attainable) points arbitrarily fast with positive probability, which implies that

$$\max\left(P^{N_a^-}_{x_0}(\sigma_a^- \geq t), P^{N_b^+}_{x_0}(\sigma_b^+ \geq t)\right) < 1.$$

We conclude that $\alpha < 1$. Let $n \in \mathbb{Z}_+$ be arbitrary. Using the Markov property, for every $m \in \mathbb{Z}_+$, we get

$$\begin{aligned} P^n_{x_0}(\gamma_{a,b} \geq mt + t) &= P^n_{x_0}(\gamma_{a,b} \geq mt, \gamma_{a,b}(\theta_{mt}\mathsf{X}) \geq t) \\ &= E^n_{x_0}\big[\mathbb{1}_{\{\gamma_{a,b}\geq mt\}} P^n_{\mathsf{X}_{mt}}(\gamma_{a,b} \geq t)\big] \\ &\leq P^n_{x_0}(\gamma_{a,b} \geq mt)\,\alpha. \end{aligned}$$

Thus, by induction, we obtain, for every $m \in \mathbb{Z}_+$, that

$$P^n_{x_0}(\gamma_{a,b} \geq mt) \leq \alpha^m.$$

Finally, we estimate

$$E^n_{x_0}\big[\gamma^2_{a,b}\big] = \sum_{m=0}^{\infty} \int_{mt}^{(m+1)t} 2s P^n_{x_0}(\gamma_{a,b} \geq s)ds \leq \sum_{m=0}^{\infty}(m+1)^2 t^2 \alpha^m < \infty.$$

The proof is complete. □

Proof of Lemma 3.13 Take $f \in C_c(J^\circ)$, $a \in (l, x_0)$ and $b \in (x_0, r)$. By Lemma 3.15, we have

$$\sup_{n\in\mathbb{N}} E^n_{x_0}\left[\left(\int_0^{\gamma_{a,b}} f(\mathsf{X}_s)ds\right)^2\right] \leq \sup_{y\in[a,b]} |f(y)|^2 \sup_{n\in\mathbb{N}} E^n_{x_0}\left[\gamma^2_{a,b}\right] < \infty.$$

Thus, the family

$$\left\{P^n_{x_0} \circ \left(\int_0^{\gamma_{a,b}} f(\mathsf{X}_s)ds\right)^{-1} : n \in \mathbb{N}\right\}$$

is uniformly integrable. Furthermore, by Lemma 3.10, the function

$$\omega \mapsto \int_0^{\gamma_{a,b}(\omega)} f(\omega(s))ds$$

is $P^0_{x_0}$-a.s. continuous (and real-valued, as $P^0_{x_0}$-a.s. $\gamma_{a,b} < \infty$ by Lemma 3.15). Consequently, the continuous mapping theorem ([14, Theorem 5.27]) yields the convergence in (3.5). □

3.3.3 Proof of Convergence up to the Boundaries: Part (b) of Definition 2.1

We now prove property (b) from Definition 2.1. Let $(J \ni x \mapsto P^n_x)_{n=0}^\infty$ be a sequence of regular diffusions on natural scale. In the following we distinguish between absorbing or reflecting boundaries.
The absorbing case.

Lemma 3.16 *Assume that* $l = \inf J \in J$ *is an absorbing boundary point of* $(x \mapsto P^0_x)$*, i.e.,* $\mathfrak{m}^0(\{l\}) = \infty$*. Furthermore, assume that* $P^n_l \to P^0_l$ *weakly. Then,* $\int f d\mathfrak{m}^n \to \infty$ *for all* $0 \le f \in C(J)$ *such that* $f(l) > 0$ *and* $f = 0$ *off* $[l, y)$ *for some* $y \in J^\circ$*, i.e., part (b) of Definition 2.1 holds.*

Proof of Lemma 3.16 For every $b \in J^\circ$ and $t > 0$, the set $\{\tau_b^+ > t\}$ is open by Lemma 3.8. Using the Portmanteau theorem, we get

$$\liminf_{n\to\infty} P^n_l(\tau_b^+ > t) \ge P^0_l(\tau_b^+ > t) = 1.$$

Hence, Fatou's lemma yields that

$$\infty = \int_0^\infty \liminf_{n\to\infty} P^n_l(\tau_b^+ > t)dt \le \liminf_{n\to\infty} \int_0^\infty P^n_l(\tau_b^+ > t)dt = \liminf_{n\to\infty} E^n_l\big[\tau_b^+\big].$$

Take $0 \le f \in C(J)$ such that $f(l) > 0$ and $f = 0$ off $[l, y)$ for some $y \in J^\circ$. Let $b \in (l, y)$ be such that $f > 0$ on $[l, b]$ and choose $z \in J^\circ$ such that $y < z$. By [20, Proposition VII.3.10], we have

$$E^n_l\left[\int_0^{\tau_z^+} \frac{f(\mathsf{X}_s)ds}{G_{[l,z)}(l, \mathsf{X}_s)}\right] = \int f d\mathfrak{m}^n, \tag{3.6}$$

where $G_{[l,z)}$ is the symmetrized Green function as defined in (2.3). Now, as $n \to \infty$, we obtain

$$\int f d\mathfrak{m}^n \ge E^n_l\left[\int_0^{\tau_b^+} \frac{f(\mathsf{X}_s)ds}{G_{[l,z)}(l, \mathsf{X}_s)}\right] \ge \min_{x\in[l,b]} \frac{f(x)}{G_{[l,z)}(l, x)} E^n_l\left[\tau_b^+\right] \to \infty.$$

This completes the proof. □

Remark 3.17 Broadly speaking, in case l is an absorbing boundary point for the diffusion $(x \mapsto P^0_x)$, Lemma 3.16 shows that we can deduce some convergence properties of the sequence $(\mathfrak{m}^n)_{n=1}^\infty$ around l from the weak convergence $P^n_l \to P^0_l$. The special case where l is absorbing for all diffusions $(x \mapsto P^n_x)_{n=0}^\infty$ shows that it is in general not possible to deduce convergence properties of $(\mathfrak{m}^n|_{J^\circ})_{n=1}^\infty$ from the weak convergence $P_l^n \to P_l^0$, see also Remark 3.5. We emphasis that Lemma 3.16 also covers the case where neither of the diffusions $(x \mapsto P_x^n)_{n=1}^\infty$ is absorbed in l, while the limiting diffusion $(x \mapsto P_x^0)$ is absorbed in l, see Example 2.3.

The reflecting case.

Lemma 3.18 *Assume that $l = \inf J \in J$ is a reflecting boundary point of $(x \mapsto P_x^0)$, i.e., $\mathfrak{m}^0(\{l\}) < \infty$. Furthermore, assume that $P_l^n \to P_l^0$ weakly. Then, $\int f d\mathfrak{m}^n \to \int f d\mathfrak{m}^0$ for all $0 \le f \in C(J)$ such that $f(l) > 0$ and $f = 0$ off $[l, y)$ for some $y \in J^\circ$, i.e., part (b) of Definition 2.1 holds.*

Proof of Lemma 3.18 Take $z \in J^\circ$ and $t > 0$, and take $0 \le f \in C(J)$ such that $f(l) > 0$ and $f = 0$ off $[l, y)$ for some $y \in J^\circ$. Using Lemma 3.8, the Portmanteau theorem and [5, Theorem 1.1], we get

$$\liminf_{n\to\infty} P_l^n\left(\sigma_z^+ < t\right) \ge P_l^0\left(\sigma_z^+ < t\right) > 0.$$

Recall that a real-valued sequence converges to a limit L if and only if any of its subsequences contains a further subsequence which converges to L. Thus, to get $\int f d\mathfrak{m}^n \to \int f d\mathfrak{m}^0$, we have to prove that any subsequence of $(\int f d\mathfrak{m}^n)_{n=1}^\infty$ contains a further subsequence which converges to $\int f d\mathfrak{m}^0$. Let $(k(n))_{n=1}^\infty \subset \mathbb{N}$ be an arbitrary subsequence of $(n)_{n=1}^\infty$. Then,

$$\liminf_{n\to\infty} P_l^{k(n)}\left(\sigma_z^+ < t\right) \ge \liminf_{n\to\infty} P_l^n\left(\sigma_z^+ < t\right) > 0.$$

Thus, there exists a subsequence $(m(n))_{n=1}^\infty$ of $(k(n))_{n=1}^\infty$ such that

$$P_l^{m(n)}\left(\sigma_z^+ < t\right) > 0, \quad \forall n \in \mathbb{N}.$$

To simplify our notation, we assume that $P_l^n(\sigma_z^+ < t) > 0$ for every $n \in \mathbb{N}$. Consequently, for each $n \in \mathbb{Z}_+$, the point l is a reflecting boundary of $(x \mapsto P_x^n)$ and, recalling again [5, Theorem 1.1], we have

$$P_l^n\left(\sigma_c^+ < t\right) > 0 \text{ for all } n \in \mathbb{Z}_+ \text{ and } c \in J^\circ. \tag{3.7}$$

Take $b \in J^\circ$ such that $y < b$. Here, recall that $y \in J^\circ$ is such that $f = 0$ off $[l, y)$. We claim that

$$E_l^n\left[\int_0^{\gamma_b} \frac{f(\mathsf{X}_s)ds}{G_{[l,b)}(l, \mathsf{X}_s)}\right] \to E_l^0\left[\int_0^{\gamma_b} \frac{f(\mathsf{X}_s)ds}{G_{[l,b)}(l, \mathsf{X}_s)}\right]. \tag{3.8}$$

Recalling (3.6), the convergence in (3.8) implies that

$$\int f d\mathfrak{m}^n \to \int f d\mathfrak{m}^0,$$

which completes the proof. Hence, it remains to prove (3.8), which is done by the same strategy as used in the proof of Lemma 3.13. Our first step is to show that

$$\sup_{n\in\mathbb{Z}_+} E^n_l\big[\gamma_b^2\big] < \infty, \tag{3.9}$$

which implies that the family

$$\left\{ P^n_l \circ \left(\int_0^{\gamma_b} \frac{f(\mathsf{X}_s)ds}{G_{[l,b)}(l,\mathsf{X}_s)} \right)^{-1} : n \in \mathbb{N} \right\}. \tag{3.10}$$

is uniformly integrable. Fix $t > 0$ and set

$$\beta \triangleq \sup\big(P^n_x(\gamma_b \geq t) \colon n \in \mathbb{Z}_+, l \leq x \leq b\big).$$

Thanks to Lemmata 3.9, 3.11 and 3.12, there exists a number $N^+_b \in \mathbb{Z}_+$ such that

$$\beta \leq \sup_{n\in\mathbb{Z}_+} P^n_l(\sigma^+_b \geq t) = P^{N^+_b}_l(\sigma^+_b \geq t) < 1,$$

where the final (strict) inequality follows from (3.7). As in the proof of Lemma 3.15, using the Markov property and induction, we obtain that

$$P^n_l(\gamma_b \geq mt) \leq \beta^m, \quad n, m \in \mathbb{Z}_+.$$

Now, for every $n \in \mathbb{Z}_+$, we obtain

$$E^n_l\big[\gamma_b^2\big] = \sum_{m=0}^{\infty} \int_{mt}^{(m+1)t} 2s P^n_l(\gamma_b \geq s)ds \leq \sum_{m=0}^{\infty} (m+1)^2 t^2 \beta^m < \infty,$$

which implies (3.9). We are in the position to complete the proof. Namely, by Lemma 3.10, the map

$$\omega \mapsto \int_0^{\gamma_b(\omega)} \frac{f(\omega(s))ds}{G_{[l,b)}(l,\omega(s))}$$

is P^0_l-a.s. continuous (and real-valued, as P^0_l-a.s. $\gamma_b < \infty$ by (3.9)). Hence, (3.8) follows from the continuous mapping theorem and the uniform integrability of (3.10). This completes the proof. □

3.3.4 Conclusion: Proof of Theorem 3.4

Under the hypothesis of Theorem 3.4, part (a) from Definition 2.1 follows from Corollary 3.14 and part (b) follows from Lemmata 3.16 and 3.18. Part (c) can be proved similar to part (b) and we omit the details for brevity. We conclude that Theorem 3.4 holds. □

3.4 Comments on Related Literature

For the real-valued case, i.e., $J = \mathbb{R}$, Theorem 3.4 follows from the main result of [4]. Theorem 3.4 seems to be new in its generality, as it covers diffusions with arbitrary state spaces and arbitrary boundary behavior. Further, our method of proof seems to be new and quite different to those from [4].

Let us outline the main differences. The set of speed measures of real-valued diffusions is a subset of $\mathcal{M}(\mathbb{R})$, i.e., it consists of locally finite measures. Thus, it can be endowed with the vague topology. This well-understood topological structure is heavily used in the proof from [4], which is mainly split into two parts.

First, the sequence $(\mathfrak{m}^n)_{n=1}^{\infty}$ is proved to be bounded on every compact subset of J uniformly in n. This property is known to be equivalent to relative compactness of the set $\{\mathfrak{m}^n : n \in \mathbb{N}\}$ in the vague topology ([19, Proposition 3.16]). Hence, to conclude $\mathfrak{m}^n \to \mathfrak{m}^0$ vaguely, it suffices to show that any vague accumulation point of $(\mathfrak{m}^n)_{n=1}^{\infty}$ coincides with $\mathfrak{m}^0$.

The speed measure of a diffusion (on natural scale) can be characterized via the Itô–McKean representation of the diffusion as a time change of Brownian motion, see [14, Theorem 33.9]. This representation is called the *canonical form*.

In the second step of the proof from [4], any vague accumulation point of $(\mathfrak{m}^n)_{n=1}^{\infty}$ is shown to be a speed measure. Thanks to this observation, it can be deduced from Stone's theorem [25, Corollary 1], and the canonical form representation, that any vague accumulation point of $(\mathfrak{m}^n)_{n=1}^{\infty}$ coincides with $\mathfrak{m}^0$.

As our general setting contains diffusions without locally finite speed measures, we cannot use arguments based on properties of the vague topology as done in the proof from [4]. Corollary 3.14 provides an alternative strategy for the real-valued case from [4], which is more direct in the sense that vague convergence is established without a relative compactness argument. Moreover, our proof does not rely on Stone's theorem and the canonical form of a diffusion (on natural scale). Instead, we use the continuous mapping theorem and a uniform second moment bound for exit times, which appears to us more elementary.

4 A Topological Point of View on the Relation of Completely Regular Diffusions and Their Speed Measures

In the remainder of this paper we take a look at the relation of completely regular diffusions on natural scale and their speed measures from a topological point of view. Let $\mathcal{S}$ be the set of all locally finite speed measures and let $\mathcal{D}$ be the set of all completely regular diffusions on natural scale. We endow $\mathcal{S}$ with the vague topology, which turns it into a metrizable space. Thanks to Corollary 3.2, we can treat $\mathcal{D}$ as a subspace of $C(J, \mathcal{M}_1(\Omega))$ endowed with the local uniform topology, which renders it into a metrizable space. The goal of this section is to establish a homeomorphic relation between $\mathcal{S}$ and $\mathcal{D}$.

Remark 4.1 It would also be natural to consider regular diffusions as elements of the space $M(J, \mathcal{M}_1(\Omega))$ of functions from J into $\mathcal{M}_1(\Omega)$ endowed with the topology of pointwise weak convergence, i.e., the product weak topology. As the space $M(J, \mathcal{M}_1(\Omega))$ is not first countable ([24, Theorem 7.1.7]), we cannot a priori[3] check continuity via sequential continuity. The space $C(J, \mathcal{M}_1(\Omega))$ (endowed with the local uniform topology) on the other hand is metrizable and therefore also sequential.

Corollary 3.6 gives us the following result, which we call a theorem rather than a corollary, since we think it deserves this name.

Theorem 4.2 *The map $\Phi\colon \mathcal{D} \to \mathcal{S}$ which maps a completely regular diffusion on natural scale to its speed measure is a homeomorphism, i.e., Φ is a continuous bijection with continuous inverse Φ^{-1}.*

Related to Remark 4.1, Corollary 3.6 implies the following.

Corollary 4.3 *In case $\mathcal{D}$ is seen as a subspace of $M(J, \mathcal{M}_1(\Omega))$ endowed with the product weak topology, the map Φ is a sequential homeomorphism, i.e., a sequentially continuous bijection with sequentially continuous inverse.*[4]

In the remainder of this section we use Theorem 4.2 to study properties of certain subsets of $\mathcal{D}$. More precisely, we consider the set of completely regular diffusions (on natural scale) with the Feller–Dynkin property and the set of (driftless) Itô diffusions with open state space.

4.1 On the Set of Diffusions with the Feller–Dynkin Property

We say that a diffusion $(x \mapsto P_x)$ has the *Feller–Dynkin property* if $(x \mapsto E_x[f(\mathsf{X}_t)]) \in C_0(J)$ for all $f \in C_0(J)$ and $t > 0$. Let $\mathcal{O}$ be the set of all completely regular diffusions with the Feller–Dynkin property.

Corollary 4.4 *If J is bounded, then $\mathcal{O} = \mathcal{D}$ and, in particular, $\mathcal{O}$ is clopen in $\mathcal{D}$. Conversely, if J is unbounded, then $\mathcal{O}$ is a dense Borel subset of $\mathcal{D}$ and it is neither closed nor open in $\mathcal{D}$.*

Proof of Corollary 4.4 According to [7, Theorem 1.1], a regular diffusion with speed measure $\mathfrak{m}$ has the Feller–Dynkin property if and only if any infinite boundary point of J is natural, i.e.,

$$\begin{cases} \forall\, c \in J^\circ\colon \displaystyle\int_c^\infty |x|\mathfrak{m}(dx) = \infty, & \text{if } \infty \text{ is a boundary point,} \\ \forall\, c \in J^\circ\colon \displaystyle\int_{-\infty}^c |x|\mathfrak{m}(dx) = \infty, & \text{if } -\infty \text{ is a boundary point.} \end{cases}$$

[3] The space of continuous functions $[0, 1] \to [0, 1]$ is *not* sequential when endowed with the product topology, see [13, Beispiel on p. 102].

[4] Of course, the inverse Φ^{-1} is even continuous, as $\mathcal{S}$ is sequential.

Now, if J is bounded, it is clear that $\mathcal{O} = \mathcal{D}$. Suppose that J is unbounded. Notice that $\Phi(\mathcal{O})$ is a Borel set, as the above characterization of the Feller–Dynkin property can be reduced to a countable number of measurable operations. We claim that $\Phi(\mathcal{O})$ is neither closed nor open. Let us explain this claim in more detail. Take $\mathfrak{m}^0 \in \mathcal{S}\backslash\Phi(\mathcal{O})$, define $\mathfrak{m}^n(dx) \triangleq \mathfrak{m}^0(dx) + dx/n$ and notice that $\mathfrak{m}^n \in \Phi(\mathcal{O})$ and that $\mathfrak{m}^n \to \mathfrak{m}^0$ vaguely. Consequently, $\Phi(\mathcal{O})$ is not closed in $\mathcal{S}$. Similarly, noting that $e^{-|x|/n}dx \to dx$ vaguely shows that $\mathcal{S}\backslash\Phi(\mathcal{O})$ is not closed and hence, that $\Phi(\mathcal{O})$ is not open. We conclude from Theorem 4.2 that the set $\mathcal{O}$ is Borel but neither closed nor open. Finally, the claim that $\mathcal{O}$ is dense in $\mathcal{D}$ follows from Theorem 4.2 and the fact that any locally finite measure can be approximated in the vague topology by a sequence of discrete measures ([2, Theorem 30.4]), i.e., a sequence whose elements are of the form $\sum_{j=1}^k \alpha_j \delta_{x_j}$, where $k \in \mathbb{N}, \alpha_1, \dots, \alpha_k$ are non-negative real numbers and $\delta_{x_1}, \dots, \delta_{x_k}$ are Dirac measures concentrated on the points $x_1, \dots, x_k$. To be more precise, let $\mathfrak{m}^0 \in \mathcal{S}$, $\mathfrak{m} \in \mathcal{O}$ and let $\mathfrak{n}^1, \mathfrak{n}^2, \dots$ be a sequence of discrete measures such that $\mathfrak{n}^n \to \mathfrak{m}^0$ vaguely. Then, $\mathfrak{m}^n \triangleq \mathfrak{n}^n + \frac{1}{n}\mathfrak{m} \in \Phi(\mathcal{O})$ and $\mathfrak{m}^n \to \mathfrak{m}^0$ vaguely. This shows that $\Phi(\mathcal{O})$ is dense in $\mathcal{S}$ and hence, by Theorem 4.2, $\mathcal{O}$ is dense in $\mathcal{D}$. The proof is complete. □

4.2 On the Set of Itô Diffusions

Let us assume that $J = (l, r)$ is open. We call a (completely) regular diffusion (with state space J) an *Itô diffusion* if its speed measure $\mathfrak{m}$ is absolutely continuous w.r.t. the Lebesgue measure, i.e., $\mathfrak{m}(dx) = f(x)dx$ for some $f \in L^1_{\text{loc}}(J)$. Denote the set of Itô diffusions by $\mathcal{I}$.

Remark 4.5 Let $(x \mapsto P_x) \in M(J, \mathcal{M}_1(\Omega))$. Then, $(x \mapsto P_x) \in \mathcal{I}$ with speed measure $\mathfrak{m}(dx) = f(x)dx$ if and only if, for every $x \in J$, P_x is the (unique) law of a solution process to the stochastic differential equation

$$dY_t = \frac{dW_t}{\sqrt{f(Y_t)}}, \quad Y_0 = x,$$

where W is a one-dimensional standard Brownian motion, see [14, Chapter 33] for more details.

Corollary 4.6 *$\mathcal{I}$ is a dense Borel subset of $\mathcal{D}$ and it is neither closed nor open in $\mathcal{D}$.*

The non-closedness of the set of real-valued Itô diffusions with drift was already observed in [22]. Corollary 4.6 provides a refined picture for the set of Itô diffusions without drift.

Proof of Corollary 4.6 Let $\mathcal{A}$ be the set of speed measures which are absolutely continuous w.r.t. the Lebesgue measure, and let $S_+(J)$ be the set of all $f \in L^1_{\text{loc}}(J)$ such that $\int_a^b f(x)dx > 0$ for all $a, b \in J$ with $a < b$ and

$$\begin{cases} \forall c \in J^\circ\colon \int_c^r |r - x| f(x)dx = \infty, & \text{if } r < \infty, \\ \forall c \in J^\circ\colon \int_l^c |l - x| f(x)dx = \infty, & \text{if } l > -\infty. \end{cases}$$

Notice that $S_+(J) \in \mathcal{B}(L^1_{\text{loc}}(J))$, as the above characterization can be reduced to a countable number of measurable operations. Consider the map $\psi\colon S_+(J) \to \mathcal{M}(J)$ defined by $\psi(f)(G) = \int_G f(x)dx$ for $G \in \mathcal{B}(J)$. As ψ is a continuous injection from a Borel subset of a Polish space into a Polish space, [6, Theorem 8.2.7] yields that $\psi(S_+(J)) \in \mathcal{B}(\mathcal{M}(J))$. As $\psi(S_+(J)) = \mathcal{A}$ by [3, Proposition 16.43, Theorem 16.56], $\mathcal{A}$ is a Borel subset of $\mathcal{S}$. We claim that $\mathcal{A}$ is neither closed nor open. Let us elaborate this claim in more detail. Fix some $x_0 \in J^\circ$ and $\mathfrak{m} \in \mathcal{A}$. For $n \in \mathbb{N}$, set $\mathfrak{m}^n(dx) \triangleq \mathfrak{m}(dx) + ne^{-n(x-x_0)}\mathbb{1}_{\{x \geq x_0\}}dx$ and notice that $\mathfrak{m}^n \in \mathcal{A}$ but $\mathfrak{m}^n \to \mathfrak{m} + \delta_{x_0} \in \mathcal{S}\backslash\mathcal{A}$ vaguely. This shows that $\mathcal{A}$ is not closed. Similarly, noting that $\mathfrak{m} + \frac{1}{n}\delta_{x_0} \to \mathfrak{m}$ vaguely shows that $\mathcal{S}\backslash\mathcal{A}$ is not closed, which means that $\mathcal{A}$ is not open. We conclude from Theorem 4.2 that $\mathcal{I}$ has the same properties, i.e., it is a Borel set but it is neither closed nor open. Finally, let us explain that $\mathcal{I}$ is dense in $\mathcal{D}$. By Theorem 4.2, it suffices to show that $\mathcal{A}$ is dense in $\mathcal{S}$. We provide some details. Take $\mathfrak{m} \in \mathcal{A}$ and $\mathfrak{m}^0 \in \mathcal{S}$. Then, as the set of discrete measures is dense in $\mathcal{M}(J)$, there exists a sequence of discrete measures $(\mathfrak{n}^k)_{k=1}^\infty \subset \mathcal{M}(J)$ such that $\mathfrak{n}^k \to \mathfrak{m}^0$ vaguely. Notice that any discrete measure can be approximated in the vague topology by a sequence of absolutely continuous measures. To see this, recall that $N(\mu, \sigma^2) \to \delta_\mu$ vaguely for $\sigma^2 \to 0$, where $N(\mu, \sigma^2)$ denotes the normal distribution with expectation μ and variance σ^2. Hence, for every $k \in \mathbb{N}$, there exists a sequence $(\mathfrak{n}^{n,k})_{n=1}^\infty$ of absolutely continuous measures in $\mathcal{M}(J)$ such that $\mathfrak{n}^{n,k} \to \mathfrak{n}^k$ vaguely. Consequently, there exists a sequence $(n(k))_{k=1}^\infty \subset \mathbb{N}$ such that $n(k) \to \infty$ and $\mathfrak{n}^{n(k),k} \to \mathfrak{m}^0$ vaguely as $k \to \infty$. Finally, we have $\mathcal{A} \ni \mathfrak{n}^{n(k),k} + \frac{1}{k}\mathfrak{m} \to \mathfrak{m}^0$ vaguely as $k \to \infty$. The proof is complete. $\square$

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