On the relationship between logarithmic sensitivity integrals and limiting optimal control problems *

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Abstract

Two seemingly independent streams of control systems research have examined logarithmic sensitivity integrals and limiting linear quadratic optimal control problems. These apparently diverse problems yield some results with an identical right hand side. The main contribution of this paper is to directly explain the commonality between these streams. This explanation involves the use of Parseval's theorem to derive tight inequality bounds between frequency domain logarithmic sensitivity integrals, and the achievable quadratic performance of a linear time invariant system.

1 Introduction

Consider a single input single output linear time invariant plant, with input, output, state and input disturbance denoted by u, y, x and d respectively, described by the state space model

$$\begin{aligned} \dot{x} &= Ax + B(u+d) \\ y &= Cx \end{aligned} \tag{1}$$

We further denote the plant transfer function, and input-to-state transfer function, respectively by

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B \\ G_X(s) &= (sI-A)^{-1}B, \end{aligned} \tag{2}$$

and the reference trajectory for the output by r, and an error signal by e, defined by:

$$e = r - y$$

There has been a large literature of study for systems of the form (1). One aspect of this study has been the examination of performance limitations, from the perspective of frequency domain logarithmic sensitivity integrals.

1.1 Logarithmic Sensitivity Integrals

In this line of research, linear time invariant feedback of the error signal, e, via a controller with transfer function, $K(s)$ is examined:

$$u(t) = -K(s) * e(t). \tag{3}$$

Define the (open) loop transfer function as $L(s) = K(s)G(s)$. It turns out that in this case, important sensitivity, robustness and performance features of the feedback controller system, are described by the sensitivity, $S(s)$, and complementary sensitivity function, $T(s)$,

$$\begin{aligned} S(s) &:= \frac{1}{1+L(s)}, \\ T(s) &:= \frac{L(s)}{1+L(s)} = 1 - S(s). \end{aligned} \tag{4}$$

The Bode Sensitivity Integral, initiated in [2] and generalized in [5], [6], [12], states that for any causal, internally stabilizing controller, the sensitivity function (4) must satisfy the logarithmic integral constraint

$$\frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\,d\omega + \frac{1}{2}k_h = \sum_{p_i \in CRHP} p_i, \tag{5}$$

where $p_i \in CRHP$ denote the closed right half plane (CRHP) open loop poles, that is, the unstable poles of $L(s)$; and k_h is the loop high frequency gain constant defined by:

$$k_h = \lim_{s\to\infty} sL(s)$$

A dual result has been obtained in [9], [12] for the complementary sensitivity function, namely, for any internally stabilizing controller with integral action[1] the complementary sensitivity function must satisfy the logarithmic integral constraint

$$\frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\,\frac{d\omega}{\omega^2} + \frac{k_v}{2} = \sum_{z_i \in CRHP} \frac{1}{z_i}, \tag{6}$$

where $z_i \in CRHP$ denote the CRHP open loop zeros; and k_v is the system velocity constant defined by

$$\begin{aligned} k_v &:= \lim_{s\to 0} \frac{1}{sL(s)} = -T'(0) \\ &= \int_0^\infty e(t)dt. \end{aligned}$$

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[1] Note that the assumption of integral action can be weakened to that of having non-zero steady state gain, at the cost of additional complexity in the equations.

Note that the results above, although motivated by the pure error feedback case, (3), actually apply to any stable, causal feedback structure provided only that (4) hold.

1.2 Limiting Linear Quadratic Optimal Control

We now shift attention to a seemingly unrelated area of research in linear control systems, where we perform state feedback control of (1) in order to optimize a quadratic cost, e.g., [1], [8]. Typically, the cost would involve quadratically weighted terms in both the state and the control signal. Limiting cases arise when we permit the weighting of one of the terms to approach zero. The first of these, where weighting on the states is allowed to approach zero, is described below.

1.2.1 Minimum Energy Control: Suppose that at $t = 0$ the system is initially at $x(0) = B$. We then seek the control which stabilizes the system with minimum control energy. This control can be found by optimizing the cost functional

$$J_u(\varepsilon) := \frac{1}{2}\int_0^\infty u^2(t) + \varepsilon^2 x^T(t)x(t)dt. \tag{7}$$

The minimum energy control is found by taking the limit as $\varepsilon \to 0$. Results in [8] may be used to derive (see [13]) that

$$\lim_{\varepsilon\to 0} J_u(\varepsilon) = \sum_{p_i \in CRHP} p_i \tag{8}$$

where p_i are the unstable poles of the plant. Equation (8) shows that the minimum control energy required is proportional to the unstable poles of the system. Perhaps surprisingly, the RHS of (8) is precisely the RHS of (5). A dual result is known in the cheap control case, described below.

1.2.2 Cheap Control: The cheap control problem [11] is a dual problem to the minimum energy problem. Here we suppose that we initially start at rest, $x(0) = 0$ and, with a unit reference, seek the control which minimizes the L_2 norm of the output error. If we denote by $u(\infty)$ the final control, which gives a steady state with zero error, then we seek to minimize the cost functional

$$J_e(\varepsilon) := \frac{1}{2}\int_0^\infty e^2(t) + \varepsilon^2 v^2(t)dt \tag{9}$$

where $v = u - u(\infty)$ is the control deviation. If we then take the limiting case, $\varepsilon \to 0$ (termed the "Cheap Control" case) it can be shown [10], [13] that

$$\lim_{\varepsilon\to 0} J_e(\varepsilon) = \sum_{z_i \in CRHP} \frac{1}{z_i} \tag{10}$$

where z_i are the CRHP zeros of $G(s)$. Now, the lowest[2] achievable tracking error depends on the non-minimum phase zeros of the plant. Again, perhaps unexpectedly, the RHSs of (10) and of (6) are identical.

[2] In terms of L_2 norm.

The main aim of this paper is to explore and explain the connection between the two apparently dissimilar areas of limiting optimal control, and logarithmic sensitivity integrals.

2 Sensitivity Integral and Minimum Energy Control

We first consider the case where we wish to relate the minimum energy control problem and the sensitivity integral. To do this, consider the case where the plant is initially at rest, the input disturbance is an impulse, and the reference signal is zero,

$$\begin{aligned} x(0^-) &= 0 \\ d(t) &= \delta(t) \\ r(t) &\equiv 0 \end{aligned} \tag{11}$$

Note that in this case it is easy to show that $x(0^+) = B$, which corresponds to the Minimum Energy problem of Section 1.2.1. It therefore follows from Section 1.2.1 that for any control algorithm, the infimal control energy required to stabilize the plant is given by (8). We denote the plant input to state transfer function by $G_X(s) = (sI - A)^{-1}B$ and consider an arbitrary linear time invariant state feedback with transfer function, $K_X(s)$. Then, breaking the loop at the input, we consider the loop transfer function, $L(s) = K_X(s)G_X(s)$, with sensitivity function given by (4). We have the following result.

Theorem 1 *Consider the plant (1), (2) under the conditions (11). Then for any stabilizing dynamic feedback controller,* $K_X(s)$

$$\frac{1}{2}\int_0^\infty u^2(t)dt \geq \frac{k_h}{2} + \frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\,d\omega \tag{12}$$

Proof: Since the input disturbance, $d(t)$, is an impulse function, and therefore has unit Laplace transform, we can show that

$$U(s) = -T(s)D(s) = -T(s) \tag{13}$$

Now by Parseval's theorem, for any bounded energy control signal,

$$\begin{aligned} \frac{1}{2}\int_0^\infty u^2(t)dt &= \frac{1}{2\pi}\int_0^\infty |U(j\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_0^\infty |T(j\omega)|^2 d\omega \end{aligned} \tag{14}$$

We then add and subtract $\frac{1}{\pi}\int_0^\infty \mathrm{Re}\{T(j\omega)\}d\omega$ from (14), noting that $2\int_0^\infty \mathrm{Re}\{T(j\omega)\}d\omega = \int_{-\infty}^\infty T(j\omega)d\omega$ to obtain

$$\frac{1}{2}\int_0^\infty u^2(t)dt = \int_{-\infty}^{\infty} T(j\omega)\frac{d\omega}{2\pi} + \frac{1}{2\pi}\int_0^\infty |T(j\omega)|^2 - 2\,\mathrm{Re}\{T(j\omega)\}\frac{d\omega}{2\pi} \quad (15)$$

Since $T(s)$ is analytic in the CRHP, and for s large, $T(s) \approx \frac{k_h}{s}$ it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} T(j\omega)d\omega = \frac{1}{j2\pi}\oint_{\mathscr{C}} T(s)ds - \lim_{R\to\infty}\frac{1}{j2\pi}\int_{\pi/2}^{-\pi/2} T(e^{j\theta}R)je^{j\theta}Rd\theta = \frac{k_h}{2}. \quad (16)$$

If we now substitute (16) in (15), and note that for any real $x > -1$, the inequality $x \geq \log(1+x)$ holds, then

$$\begin{aligned}\frac{1}{2}\int_0^\infty u^2(t)dt &= \frac{k_h}{2} + \frac{1}{2\pi}\int_0^\infty -2\,\mathrm{Re}\{T(j\omega)\} + |T(j\omega)|^2 d\omega \\ &\geq \frac{k_h}{2} + \frac{1}{2\pi}\int_0^\infty \log\left(1 - 2\,\mathrm{Re}\{T(j\omega)\} + |T(j\omega)|^2\right) d\omega \\ &= \frac{k_h}{2} + \frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\,d\omega,\end{aligned}$$

from which (12) follows. ∎

Theorem 1 presents an inequality linking logarithmic sensitivity and the control energy required to stabilize the plant. Naturally, the *lowest* control energy required is zero if the plant is stable, in which case, the inequality (12) becomes the trivial $\int_0^\infty u^2(t)dt \geq 0$, which follows on using (5).

The inequality (12) is one-way. The following results, based on a dual to a result in [3] for the filtering case, establishes that in a certain sense, the inequality in (12) is tight. We first show that the optimal cost can be expressed in terms of the high frequency gain.

Lemma 2 *For any $\varepsilon > 0$, consider the problem of optimizing $J_u(\varepsilon)$ as defined in (7) with initial state $x(0^+) = B$. The optimal cost in this case is*

$$\min_u J_u(\varepsilon) = \frac{k_h}{2}.$$

Proof: The optimal control u_ε for this case is the time invariant state feedback

$$K_X(s) = B^T P_\varepsilon, \quad (17)$$

where P_ε is the unique symmetric positive definite solution to the Riccati equation

$$P_\varepsilon A + A^T P_\varepsilon - P_\varepsilon BB^T P_\varepsilon + \varepsilon^2 I = 0.$$

Now the loop transfer function corresponding to the control $K_x(s)$ from (17) is

$$\begin{aligned}L(s) &= K_X(s)G_X(s) \\ &= B^T P_\varepsilon (sI - A)^{-1}B\end{aligned}$$

It therefore follows that the high frequency gain is $k_h = B^T P_\varepsilon B$. Furthermore, the optimal cost is [1]

$$\begin{aligned}\min_u J_u(\varepsilon) &= \frac{1}{2}x_0^T P_\varepsilon x_0 \\ &= \frac{1}{2}B^T P_\varepsilon B,\end{aligned}$$

and the result follows. ∎

We now note that for the optimal control problem posed, the sensitivity function and the plant transfer function are related.

Lemma 3 *For any $\varepsilon > 0$, consider the problem of optimizing $J_u(\varepsilon)$ as defined in (7). Denote the sensitivity function achieved in this case by $S_\varepsilon(s)$. Then for all ω:*

$$|S_\varepsilon(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 \|P_X(j\omega)\|^2} \quad (18)$$

Proof: This result is a direct consequence of the Return Difference Identity [1]. ∎

The final result we need to establish the “tightness” of (12) is that the integral of the log of (18) can be made arbitrarily small.

Lemma 4 *For any plant input-to-state transfer function, $P_X(s) = (sI - A)^{-1}B$,*

$$\lim_{\varepsilon\to 0^+}\left\{\int_0^\infty \log\left(1 + \varepsilon^2\|P_X(j\omega)\|^2\right) d\omega\right\} = 0.$$

Proof: In the case where A has no eigenvalues on the imaginary axis, the proof is straightforward, as $0 \leq \log\left(1 + \varepsilon^2\|P_X(j\omega)\|^2\right) \leq \varepsilon^2\|P_X(j\omega)\|^2$. More generally, however, let $\gamma_\varepsilon(s)$ denote a marginally stable, unity high frequency gain, minimum phase spectral factor of $1 + \varepsilon^2\|P_X(j\omega)\|^2$, that is,

$$\begin{aligned}\gamma_\varepsilon(s)\gamma_\varepsilon(-s) &= 1 + \varepsilon^2 P_X(-s)^T P_X(s) \\ &= \prod_{i=1}^n \frac{\left(s - z_{i,\varepsilon}\right)}{\left(s - \lambda_i^+\right)} \prod_{i=1}^n \frac{\left(s + z_{i,\varepsilon}\right)}{\left(s + \lambda_i^+\right)}\end{aligned}$$

where $\lambda_i^+ = -|\sigma_i| + j\omega_i \in CLHP$ denote the marginally stable reflections of the eigenvalues $\lambda_i = \sigma_i + j\omega_i$ of the matrix A. Then $\log\left(1+\varepsilon^2 \|P_X(j\omega)\|^2\right) = 2\log|\gamma_\varepsilon(j\omega)|$ and therefore using a minor variant of (5)

$$\begin{aligned}\int_0^\infty \log\left(1+\varepsilon^2 \|P_X(j\omega)\|^2\right) d\omega &= 2\int_0^\infty \log|\gamma_\varepsilon(j\omega)|\, d\omega \\ &= 2\pi \lim_{s\to\infty} \left(s\left(\gamma_\varepsilon(s)-1\right)\right) \\ &= 2\pi \sum_{i=1}^n \left(z_{i,\varepsilon} - \lambda_i^+\right)\end{aligned}$$

The result then follows on noting that $\lim_{\varepsilon\to 0} z_{i,\varepsilon} = \lambda_i^+$. ■

We are now in a position to prove the following theorem.

Theorem 5 *The inequality of Theorem 1 is tight in the sense that for any $\delta > 0$ there exists a controller which achieves*

$$\frac{1}{2}\int_0^\infty u^2(t)dt \le \delta + \frac{k_h}{2} + \frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\, d\omega. \qquad (19)$$

Proof: For any $\delta > 0$ we pick the control which optimizes (7) for a sufficiently small ε. To see that this suffices, note from Lemma 2 and Lemma 3 that

$$\begin{aligned}\frac{1}{2}\int_0^\infty u^2(t)dt &\le \frac{1}{2}\int_0^\infty u^2(t) + \varepsilon^2 x^T(t)x(t)dt \\ &= J_u(\varepsilon) = \frac{k_h}{2} \\ &= \frac{k_h}{2} + \frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\, d\omega \\ &+ \frac{1}{2\pi}\int_0^\infty \log\left(1+\varepsilon^2 \|P_X(j\omega)\|^2\right) d\omega \qquad (20)\end{aligned}$$

Equation (19) now follows since from Lemma 4 the last term on the RHS of (20) can be made arbitrarily small. ■

We have therefore established that for any controller, the logarithmic sensitivity integral and the high frequency gain yield a lower bound on the achievable control energy for any stabilizing controller. Moreover, by using limiting versions (approximate minimum energy) of linear quadratic optimal control, this lower bound is shown to be tight.

3 Complementary Sensitivity Integral and Cheap Control

We now consider the case where we wish to relate the cheap control problem and the complementary sensitivity integral. To do this, consider the case where plant is initially at rest, the input disturbance is zero, and the reference signal is a unit step:

$$\begin{aligned} x(0) &= 0 \\ d(t) &= 0 \qquad (21) \\ r(t) &\equiv \mathbf{1}(t) \end{aligned}$$

We also consider the class of linear time invariant dynamic state feedback controllers

$$u(t) = -K_X(s) * x(t) + K_E(s) * e(t).$$

In this case we define the sensitivity function (4) as the transfer function from the reference signal to the error signal

$$e(t) = S(s) * r(t).$$

We assume that the loop transfer function has infinite steady state gain, which ensures zero steady state error, that is, $S(0) = 0$ and $T(0) = 1$. Note that this may be achieved either by making $(A - BK(0))$ singular (i.e. generating integral action via the state feedback) or by including integral action in the error feedback $K_E(s)$. The velocity constant can be shown to satisfy $k_v = (S'(0))$. We have the following result.

Theorem 6 *Consider the plant (1) under the conditions (21). Then for any stabilizing dynamic feedback controller with zero steady state error,*

$$\frac{1}{2}\int_0^\infty e^2(t)dt \ge \frac{k_v}{2} + \frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\frac{d\omega}{\omega^2}. \qquad (22)$$

Proof: The proof follows similar lines to the proof of Theorem 1. Since the reference input is a unit step, (21), the Laplace Transform of the error signal can be expressed as:

$$E(s) = S(s)R(s) = \frac{S(s)}{s} \qquad (23)$$

Now by Parseval's theorem, for any bounded energy error signal[3],

$$\frac{1}{2}\int_0^\infty e^2(t)dt = \frac{1}{2\pi}\int_0^\infty |S(j\omega)|^2 \frac{d\omega}{\omega^2} \qquad (24)$$

We then add and subtract $\frac{1}{\pi}\int_0^\infty \mathrm{Re}\{S(j\omega)\}\frac{d\omega}{\omega^2}$ from (24) to obtain

$$\begin{aligned}\frac{1}{2}\int_0^\infty e^2(t)dt &= \frac{1}{2\pi}\int_{-\infty}^\infty S(j\omega)\frac{d\omega}{\omega^2} \\ &+ \frac{1}{2\pi}\int_0^\infty |S(j\omega)|^2 - 2\,\mathrm{Re}\{S(j\omega)\}\frac{d\omega}{\omega^2} \\ &= \frac{k_v}{2} + \frac{1}{2\pi}\int_0^\infty \left(-2\,\mathrm{Re}\{S(j\omega)\} + |S(j\omega)|^2\right)\frac{d\omega}{\omega^2}\end{aligned}$$

[3] Note that the zero steady state error plus stability assumptions ensure that the error signal is bounded energy.

As in the proof of Theorem 1, using the inequality $x \geq \log(1+x)$, valid for for any real $x > -1$, we obtain

$$\frac{1}{2}\int_0^\infty e^2(t)dt \geq \frac{k_v}{2} + \frac{1}{2\pi}\int_0^\infty \log\left(1 - 2\,\mathrm{Re}\{S(j\omega)\} + |S(j\omega)|^2\right)\frac{d\omega}{\omega^2} = \frac{k_v}{2} + \frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\,\frac{d\omega}{\omega^2}$$

from which (22) follows. ∎

Theorem 6 presents an inequality linking logarithmic complementary sensitivity and the tracking error energy required to stabilize the plant. It was shown in [13] that the least tracking error energy is the minimum energy required to stabilize the unstable zeros of the plant. Hence, in particular, the least tracking error energy becomes zero if the plant is minimum phase, in which case the inequality (22) becomes the trivial $\int_0^\infty e^2(t)dt \geq 0$, which follows on using (6).

Note that (22) is a one-way inequality. The following claim, based on a dual to the results in Section 2, suggests that in a certain sense, the inequality (22) is tight.

Theorem 7 *The inequality of Theorem 1 is tight in the sense that for any $\delta > 0$ there exists a controller which achieves*

$$\frac{1}{2}\int_0^\infty e^2(t)dt \leq \delta + \frac{k_v}{2} + \frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\,\frac{d\omega}{\omega^2} \qquad (25)$$

Proof: Firstly, we note that any linear system (1) of relative degree r can be rewritten, after state feedback and change of coordinates, in the "zero dynamics" form (cf. [13]):

$$\frac{d}{dt}\begin{bmatrix} Z \\ \xi \end{bmatrix} = \begin{bmatrix} I^+ & 0 \\ B_0C_0 & A_0 \end{bmatrix}\begin{bmatrix} Z \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} v$$
$$e = \begin{bmatrix} C_0 & 0 \end{bmatrix}\begin{bmatrix} Z \\ \xi \end{bmatrix} \qquad (26)$$

where $I^+ \in \mathbf{R}^{r\times r}$ is a matrix which is all zeros, except for ones on the super-diagonal, $B_1 = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$, $C_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, A_0 and B_0 are matrices which define the zero dynamics with state ξ, $Z = \begin{bmatrix} e & \dot{e} & \dots & e^{(r-1)} \end{bmatrix}^T$ and v is the deviation of the control from the steady state control, together with a state feedback term. For simplicity we take A_0 to be completely anti-stable, that is, it has all eigenvalues strictly in the right half plane.

Note then that the problem of minimizing the integral square tracking error for a unit step reference is equivalent to minimizing the integral square of the "output" $e(t)$ in (26), with initial conditions $Z(0) = -C_0^T$ and $\xi(0) = -A_0^{-1}B_0C_0Z(0) = A_0^{-1}B_0$. This is a singular optimal control problem, that is, the control v which minimizes (9) subject to (26) is unbounded. We can, however, construct controls which approach this singular optimal case as follows. We denote by $e^* = -B_0^TP_0\xi$ the target trajectory that we would like the error signal e to follow, where P_0 is the unique positive definite solution to the Riccati equation

$$A_0^TP_0 + P_0A_0 - P_0B_0B_0^TP_0 = 0. \qquad (27)$$

We then define the target error state vector

$$Z^* = \begin{bmatrix} e^* & \frac{d}{dt}e^* & \dots & \left(\frac{d}{dt}\right)^{(r-1)}e^* \end{bmatrix}^T \qquad (28)$$

and the deviation of the actual error vector from the target as $\eta = Z - Z^*$. It then follows (after some algebraic manipulations) that we can rewrite (26) as

$$\frac{d}{dt}\begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} I^+ & 0 \\ B_0C_0 & A_0 - B_0B_0^TP_0 \end{bmatrix}\begin{bmatrix} \eta \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w$$
$$e = \begin{bmatrix} 0 & -B_0^TP_0 \end{bmatrix}\begin{bmatrix} \eta \\ \xi \end{bmatrix} \qquad (29)$$

where

$$w = v + B_0^TP_0A_0^r\xi + \begin{bmatrix} B_0^TP_0A_0^{(r-1)}B_0 & B_0^TP_0A_0^{(r-2)}B_0 & \dots & B_0^TP_0B_0 \end{bmatrix}^T Z,$$

and $\eta(0) = \left(B_0^TP_0A_0^{-1}B_0 - 1\right)C_0^T$.

We now select any Hurwitz polynomial $\gamma(s)$ of degree r

$$\gamma(s) = s^r + \gamma_{r-1}s^{r-1} + \dots + \gamma_0,$$

and define the vector $\Gamma = \begin{bmatrix}\gamma_0 & \gamma_1 & \dots & \gamma_{r-1}\end{bmatrix}$ and, for any $\varepsilon > 0$, the scaling matrix $D_\varepsilon = \mathrm{diag}(\varepsilon^{-r}, \varepsilon^{-(r-1)}, \dots, \varepsilon^{-1})$. We consider a family of control laws w_ε, parameterized by ε, and defined by

$$w_\varepsilon = -\Gamma D_\varepsilon \eta \qquad (30)$$

For the control law (30) it can be shown, after some algebra, that the response, $\eta_\varepsilon := D_\varepsilon\eta$ satisfies

$$C_0\eta_\varepsilon(t) = C_0e^{\left(I^+ - B_1\Gamma\right)\frac{t}{\varepsilon}}\eta(0) \qquad (31)$$

Since $\gamma(s)$ is Hurwitz, it follows that $\left(I^+ - B_1\Gamma\right)$ has all its eigenvalues strictly in the left half plane. It then follows that there exists a constant k_η such that

$$\int_0^\infty \left(C_0\eta_\varepsilon(t)\right)^2 dt = k_\eta\varepsilon^2. \qquad (32)$$

We now turn to the behavior of the ξ dynamics in (29) under these conditions. Note that with the control (30) the response can be split into two parts:

$$\xi(t) = \xi_0(t) + \xi_{\eta_\varepsilon}(t)$$
$$e(t) = e_0(t) + e_{\eta_\varepsilon}(t), \qquad (33)$$

where the subscript zero denotes the initial condition response, i.e., the response with $C_0\eta = 0$, and the subscript η_ε denotes the response with $\xi(0) = 0$, but with $\eta_\varepsilon(t)$ as a driving term. Because the ξ sub-system (29) is asymptotically stable, it is also finite L_2 gain stable, which, together with (32), guarantees that there exists a constant $k_1 > 0$ such that

$$\int_0^\infty e_{\eta_\varepsilon}^2(t)\,dt \le k_1\varepsilon^2 \tag{34}$$

On the other hand, simple observability Grammian results give that

$$\int_0^\infty e_0^2(t)\,dt = \xi_0^T P_0 \xi_0 = B_0^T A_0^{-T} P_0 A_0^{-1} B_0. \tag{35}$$

Thus, using the Cauchy-Schwarz inequality and equations (34) and (35), we have that for any $\varepsilon > 0$ the control (30) gives

$$\sqrt{\int_0^\infty e^2(t)\,dt} \le \sqrt{B_0^T A_0^{-T} P_0 A_0^{-1} B_0} + \sqrt{k_1}\varepsilon \tag{36}$$

We now turn to examining the remaining terms on the RHS of (25). From (6), the remaining terms can be found as

$$\frac{k_v}{2} + \frac{1}{\pi}\int_0^\infty \log|T_\varepsilon(j\omega)|\frac{d\omega}{\omega^2} = \sum_{z_i\in CRHP}\frac{1}{z_{\varepsilon_i}}$$

where $T_\varepsilon(s)$ denotes the complementary sensitivity function achieved with the control (30) and z_{ε_i} are the zeros of $T_\varepsilon(s)$. From (23) and (33) it follows that

$$T_\varepsilon(s) = 1 - sE_0(s) - sE_{\eta_\varepsilon}(s)$$

From (31) it follows that $\lim_{\varepsilon\to 0}\{sE_{\eta_\varepsilon}(s)\} = 0$ uniformly for all s in the CRHP. Hence,

$$\lim_{\varepsilon\to 0}\{z_{\varepsilon_i}\} = z_{0_i}$$

where z_{0_i} are the zeros of $T_0(s) = 1 - sE_0(s)$. After some lengthy algebra, it can be shown that

$$T_0(s) = \frac{1 - B_0^T P_0 A_0^{-1} B_0}{1 + B_0^T P_0 (sI - A_0)^{-1} B_0},$$

and therefore the zeros of $T_0(s)$ are precisely the plant zeros, that is, the eigenvalues of A_0. We therefore have

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\{\frac{k_v}{2} + \frac{1}{\pi}\int_0^\infty \log|T_\varepsilon(j\omega)|\frac{d\omega}{\omega^2}\right\} &= \sum_{z_i\in CRHP}\frac{1}{z_i} \\ &= \text{trace}\left[A_0^{-1}\right] \\ &= \frac{1}{2}B_0^T A_0^{-T} P_0 A_0^{-1} B_0 \end{aligned} \tag{37}$$

where the last equality follows after some algebra on the Riccati equation (27). The result then follows from (37) and (36). ■

4 Conclusions

In this paper, we have briefly reviewed two seemingly disparate areas of logarithmic sensitivity integrals and limiting linear quadratic optimal control problems. The results of this paper link these two areas, and in particular provide a direct link between minimum energy LQ control, and the Bode Sensitivity Integral. Dual results establish a direct link between cheap control problems, and the complementary sensitivity integral.

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