

On the relationship between radiance and irradiance: determining the illumination from images of a convex Lambertian object

Ravi Ramamoorthi and Pat Hanrahan

Department of Computer Science, Stanford University, Gates Wing 3B-386, Stanford, California 94305

Received October 30, 2000; revised manuscript received March 8, 2001; accepted March 12, 2001

We present a theoretical analysis of the relationship between incoming radiance and irradiance. Specifically, we address the question of whether it is possible to compute the incident radiance from knowledge of the irradiance at all surface orientations. This is a fundamental question in computer vision and inverse radiative transfer. We show that the irradiance can be viewed as a simple convolution of the incident illumination, i.e., radiance and a clamped cosine transfer function. Estimating the radiance can then be seen as a deconvolution operation. We derive a simple closed-form formula for the irradiance in terms of spherical harmonic coefficients of the incident illumination and demonstrate that the odd-order modes of the lighting with order greater than 1 are completely annihilated. Therefore these components cannot be estimated from the irradiance, contradicting a theorem that is due to Preisendorfer. A practical realization of the radiance-from-irradiance problem is the estimation of the lighting from images of a homogeneous convex curved Lambertian surface of known geometry under distant illumination, since a Lambertian object reflects light equally in all directions proportional to the irradiance. We briefly discuss practical and physical considerations and describe a simple experimental test to verify our theoretical results. © 2001 Optical Society of America
OCIS codes: 150.0150, 150.2950.

1. INTRODUCTION

This paper presents a theoretical analysis of the relationship between incoming radiance and irradiance. Radiance and irradiance are basic optical quantities, and their relationship is of fundamental interest to many fields, including computer vision, radiative transfer, and computer graphics. Physically, we are interested in analyzing the properties of the light field generated when a homogeneous convex curved Lambertian surface of known geometry reflects a distant illumination field. A Lambertian surface reflects light proportional to the incoming irradiance, so analysis of this physical system is equivalent to a mathematical analysis of the relationship between incoming radiance and irradiance.

The specific question of interest to us in this paper is the estimation of the incident radiance from the irradiance, i.e., estimation of the lighting from observations of a Lambertian surface. We are able to derive a closed-form formula in terms of the spherical harmonic coefficients for the irradiance and thereby to show that odd modes of the lighting with order greater than 1 cannot be estimated.

The rest of this paper is organized as follows. In Section 2, we briefly discuss some previous work. In Section 3, we introduce the mathematical and physical preliminaries. In Section 4, we obtain the equation for the irradiance in terms of spherical harmonic coefficients. Section 5 applies this result to the main problem of this paper, recovering the input radiance from the irradiance, and derives the main results of this paper. Section 6 briefly discusses some practical considerations and describes a simple experimental verification. We discuss

some of the broader implications of our results and conclude the paper in Section 7.

2. PREVIOUS WORK

The radiance-from-irradiance problem as discussed in this paper is addressed by Preisendorfer¹ in his treatise on hydrologic optics. He considers the recovery of radiance, given irradiance at all surface orientations. Preisendorfer's conclusion is that irradiance and radiance are equivalent and that irradiance can be inverted to give the input radiance. By deriving a simple closed-form formula, we will show that this assertion is not true. More recently, Marschner and Greenberg² have considered the inverse lighting problem, assuming curved Lambertian surfaces, and have proposed a practical solution method. However, they have noted the problem to be ill conditioned and have made extensive use of regularization. By deriving a closed-form formula for the case of convex objects, we can explain the ill conditioning observed by Marschner and Greenberg² and propose alternative algorithms.

Sato *et al.*³ use shadows instead of curvature to recover the lighting in a scene, and they also estimate the reflectance parameters of a planar surface. Inverse problems in transport theory have also been studied in other areas such as radiative transfer and neutron scattering. See McCormick⁴ for a review. However, to the best of our knowledge, these researchers have not addressed the specific radiance-from-irradiance problem treated here,

wherein we use the varying reflected radiance over a curved Lambertian surface to estimate the input lighting.

The work presented here is related to efforts in many different areas of computer graphics and vision and is likely to be of interest to these communities. Within the context of rendering surfaces under distant illumination (referred to as environment mapping within computer graphics), many previous authors such as Miller and Hoffman⁵ and Cabral *et al.*⁶ have qualitatively described reflection as a convolution and have empirically demonstrated that a Lambertian bidirectional reflectance distribution function (BRDF) behaves like a low-pass filter. In computer vision, similar observations have been made by many researchers such as Haddon and Forsyth⁷ and Jacobs *et al.*⁸ Our main contribution is in formalizing these previous qualitative results by deriving analytic quantitative formulas relating the incoming radiance to the irradiance.

Although our goals are different, our work is also related to efforts in the object recognition community to characterize the appearance of a Lambertian surface under all possible illumination conditions, and it is likely that our results will be applicable to this problem. Belhumeur and Kriegman⁹ have theoretically described this set of images in terms of an illumination cone, while empirical results have been obtained by Epstein *et al.*¹⁰ In independent work simultaneous with our own, Basri and Jacobs¹¹ have described Lambertian reflection as a convolution and have applied the results to face recognition.

To derive an analytic formula for the input illumination, we must analyze the properties of the reflected light field from a homogeneous Lambertian surface. The light field¹² is a fundamental quantity in light transport and therefore has wide applicability for both forward and inverse problems in a number of fields. A good introduction to the various radiometric quantities derived from light fields that we will use in this paper is provided by McCluney.¹³ Light fields have been used directly for rendering images from photographs in computer graphics, without considering the underlying geometry,^{14,15} or by parameterizing the light field on the object surface.¹⁶ In previous work, we¹⁷ have performed a theoretical analysis of two-dimensional or flatland, light fields, which is similar in spirit to the analysis in this paper for three-dimensional Lambertian surfaces.

To derive our results, we will represent quantities by using spherical harmonics.^{18–20} In previous work, D’Zmura²¹ has qualitatively analyzed reflection as a linear operator in terms of spherical harmonics. Basis functions have also been used in representing BRDFs for computer graphics. A number of authors^{22–24} have used spherical harmonics, while Koenderink and van Doorn²⁵ have described BRDFs by using Zernike polynomials.

3. PRELIMINARIES

A. Assumptions

Mathematically, we are simply considering the relationship between the irradiance, expressed as a function of surface orientation, and the incoming radiance, expressed as a function of incident angle. The corresponding physical system is a curved convex homogeneous Lambertian

surface reflecting a distant illumination field. For the physical system, we will assume that the surfaces under consideration are convex, so they may be parameterized by the surface orientation, as described by the surface normal, and so that interreflection and shadowing can be ignored. Also, surfaces will be assumed to be Lambertian and homogeneous, so the reflectivity can be characterized by a constant albedo. We will further assume here that the illuminating light sources are distant, so the illumination or the incoming radiance can be represented as a function of direction only. This also means that the incident illumination does not depend directly on surface position but depends only on surface orientation. Finally, for the purposes of experimental measurement, we will assume that the geometry of the object and its location with respect to the camera are known, so that we can relate each image pixel to a particular location on the object surface.

The notation used in the paper is listed in Appendix A. A diagram of the local geometry of the situation is shown in Fig. 1. We will use two types of coordinates. Unprimed global coordinates denote angles with respect to a global reference frame. On the other hand, primed local coordinates denote angles with respect to the local reference frame, defined by the local surface normal and an arbitrarily chosen tangent vector. These two coordinate systems are related simply by a rotation, and this relationship will be detailed shortly.

B. Reflection Equation

In local coordinates, we can relate the irradiance to the incoming radiance by

$$E(\mathbf{x}) = \int_{\Omega'} L(\mathbf{x}, \theta'_i, \phi'_i) \cos \theta'_i d\Omega', \quad (1)$$

where E is the irradiance, as a function of position \mathbf{x} on the object surface, and L is the radiance of the incident light field. As noted in Subsection 3.A, primes denote quantities in local coordinates. The integral is over the upper hemisphere with respect to the local surface normal.

Practically, we may observe the irradiance by considering the brightness of a homogeneous Lambertian reflector. For a Lambertian surface with constant reflectance, we can relate the reflected radiance to the irradiance by

$$B(\mathbf{x}) = \rho E(\mathbf{x}), \quad (2)$$

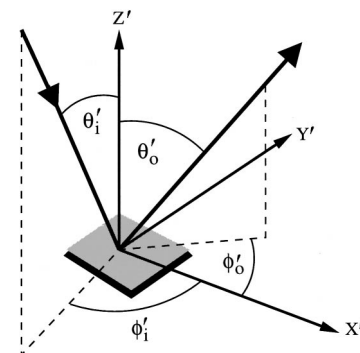


Fig. 1. Local geometry. Quantities are primed because they are all in local coordinates.

where B is the radiant exitance (i.e., brightness) as a function of position \mathbf{x} on the object surface, ρ is the surface reflectance, which lies between 0 and 1, and E is the irradiance defined in Eq. (1). In computer graphics, the radiant exitance B is usually referred to by the older term radiosity.

Note that the radiant exitance B is simply a scaled version of the irradiance E and is exactly equal to E if the Lambertian surface has reflectance $\rho = 1$. Furthermore, for a Lambertian object, the reflected radiance is the same in all directions and is therefore directly proportional to the radiant exitance, being given by B/π . In the rest of this paper, we will analyze Eq. (1), relating the irradiance E to the incident radiance L . It should be understood that the practically measurable quantities for a Lambertian surface—the radiant exitance B or the corresponding reflected radiance—are simply scaled versions of the irradiance and can be derived trivially from E and the surface reflectance ρ by using Eq. (2).

We now manipulate Eq. (1) by performing a number of substitutions. First, the assumption of distant illumination means that the illumination field is homogeneous over the surface, i.e., independent of surface position \mathbf{x} , and depends only on the *global* incident angle (θ_i, ϕ_i) . This allows us to make the substitution $L(\mathbf{x}, \theta'_i, \phi'_i) \rightarrow L(\theta_i, \phi_i)$. Second, consider the assumption of a curved convex surface. This ensures that there is no shadowing or interreflection, so the irradiance is due only to the distant illumination field L . This fact is implicitly assumed in Eq. (1). Furthermore, since the illumination is distant, we may reparameterize the surface simply by the surface normal \mathbf{n} . Equation (1) now becomes

$$E(\mathbf{n}) = \int_{\Omega'} L(\theta_i, \phi_i) \cos \theta'_i d\Omega'. \quad (3)$$

The goal of this paper is to determine what we can learn about the incident radiance L from measuring the functional dependence of the irradiance E on the surface normal \mathbf{n} .

To proceed further, we will parameterize \mathbf{n} by its spherical angular coordinates (α, β, γ) . Here (α, β) define the angular coordinates of the local normal vector, i.e.,

$$\mathbf{n} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha); \quad (4)$$

γ defines the local tangent frame, i.e., rotation of the coordinate axes about the normal vector. For isotropic surfaces—those where there is no preferred tangential direction, i.e., where rotation of the tangent frame about the surface normal has no physical effect—the parameter γ has no physical significance, and we have therefore not explicitly considered it in Eq. (3). We will include γ for completeness in the ensuing discussion on rotations but will eventually eliminate it from our equations after showing mathematically that it does in fact have no effect on the final results. Finally, for convenience, we will define a transfer function $A(\theta'_i) = \cos \theta'_i$. With these modifications, Eq. (3) becomes

$$E(\alpha, \beta, \gamma) = \int_{\Omega'} L(\theta_i, \phi_i) A(\theta'_i) d\Omega'. \quad (5)$$

Note that local and global coordinates are mixed. The lighting is expressed in global coordinates, since it is constant over the object surface when viewed with respect to a global reference frame, while the transfer function $A = \cos \theta'_i$ is expressed naturally in local coordinates. To analyze this equation further, we will need to apply a rotation corresponding to the surface orientation (α, β, γ) in order to convert the lighting into local coordinates.

C. Rotating the Lighting

We have assumed that the incoming light field remains constant (in a global frame) over the object. To convert to the local coordinates in which (θ'_i, ϕ'_i) are expressed, we must perform the appropriate rotation on the lighting. Let $L(\theta_i, \phi_i)$ be the global incoming radiance and (α, β, γ) the parameters corresponding to the local surface orientation. We define $R^{\alpha, \beta, \gamma}$ to be a rotation operator that rotates (θ'_i, ϕ'_i) into global coordinates (θ_i, ϕ_i) . $R^{\alpha, \beta, \gamma}$ can be expressed in terms of the standard Euler-angle representation and is given by $R^{\alpha, \beta, \gamma} = R_z(\beta)R_y(\alpha)R_z(\gamma)$, where R_z is a rotation about the Z axis and R_y is a rotation about the Y axis. Refer to Fig. 2 for an illustration. It is easy to verify that this rotation correctly transforms the local coordinates $(0', 0')$ —corresponding to the local representation of the Z axis, i.e., the surface normal—to the global coordinates (α, β) —corresponding to the global representation of the surface normal. The relevant transformations are given below:

$$\begin{aligned} R^{\alpha, \beta, \gamma} &= R_z(\beta)R_y(\alpha)R_z(\gamma), \\ (\theta_i, \phi_i) &= R^{\alpha, \beta, \gamma}(\theta'_i, \phi'_i), \\ L(\theta_i, \phi_i) &= L(R^{\alpha, \beta, \gamma}(\theta'_i, \phi'_i)). \end{aligned} \quad (6)$$

Note that the angular parameters are rotated as if they were a unit vector pointing in the appropriate direction. It should also be noted that this rotation of parameters is equivalent to an inverse rotation of the function, with R^{-1} being given by $R_z(-\gamma)R_y(-\alpha)R_z(-\beta)$.

Finally, we can plug Eqs. (6) into Eq. (5) to derive

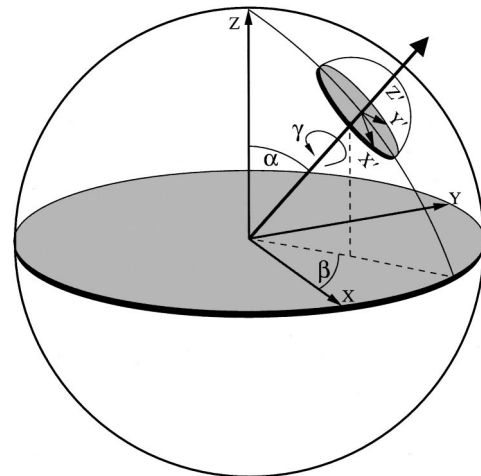


Fig. 2. Diagram showing how the rotation corresponding to (α, β, γ) transforms between local (primed) and global (unprimed) coordinates.

$$E(\alpha, \beta, \gamma) = \int_{\Omega'} L(R^{\alpha, \beta, \gamma}(\theta'_i, \phi'_i)) A(\theta'_i) d\Omega'. \quad (7)$$

Note that this equation is essentially a convolution, although we have a rotation operator rather than a translation. The irradiance can be viewed as a convolution of the incident illumination L and the transfer function $A = \cos \theta'_i$. Different observations of the irradiance E , at points on the object surface with different orientations, correspond to different rotations of the transfer function (since the local upper hemisphere is rotated), which can also be thought of as different rotations of the incident light field. In Section 4, we will see that this integral becomes a simple product when transformed to spherical harmonics, further stressing the analogy with convolution. Our goal will then be to deconvolve the irradiance in order to recover the incident illumination.

4. SPHERICAL HARMONIC REPRESENTATION

We now proceed to construct a closed-form description of the irradiance in terms of spherical harmonic coefficients. Spherical harmonics are the analog on the sphere to the Fourier basis on the line or the circle. The spherical harmonic $Y_{l,m}$ is given by

$$Y_{l,m}(\theta, \phi) = N_{l,m} P_{l,m}(\cos \theta) \exp(Im\phi), \quad (8)$$

where $N_{l,m}$ is a normalization factor. In the above equation, the azimuthal dependence is expanded in terms of Fourier basis functions. The θ dependence is expanded in terms of the associated Legendre functions $P_{l,m}$. The indices obey $l \geq 0$ and $-l \leq m \leq l$. Thus there are $2l + 1$ basis functions for given order l . Inui *et al.*¹⁸ is a good reference for background on spherical harmonics and their relationship to rotations.

We begin by expanding the lighting in global coordinates in terms of spherical harmonics:

$$L(\theta_i, \phi_i) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} L_{l,m} Y_{l,m}(\theta_i, \phi_i). \quad (9)$$

We will then transform to local coordinates by applying the appropriate rotations.

A. Rotation of Spherical Harmonics

Let us now build up the rotation operator on the spherical harmonics. We will use notation of the form $R(\cdot)\{Y_{l,m}(\cdot)\}$ to stand for a rotation by the rotation operator R of the parameters of the spherical harmonic $Y_{l,m}$. For instance,

$$R_z(\beta)\{Y_{l,m}(\theta'_i, \phi'_i)\} = Y_{l,m}(R_z(\beta)\{\theta'_i, \phi'_i\}). \quad (10)$$

First, from the form of the spherical harmonics, rotation about z is simple. Specifically,

$$\begin{aligned} R_z(\beta)\{Y_{l,m}(\theta'_i, \phi'_i)\} &= Y_{l,m}(\theta'_i, \phi'_i + \beta) \\ &= \exp(Im\beta) Y_{l,m}(\theta'_i, \phi'_i). \end{aligned} \quad (11)$$

Rotation about y is more complicated and is given by

$$R_y(\alpha)\{Y_{l,m}(\theta'_i, \phi'_i)\} = \sum_{m'=-l}^l D_{m,m'}^l(\alpha) Y_{l,m'}(\theta'_i, \phi'_i), \quad (12)$$

where D^l is a $(2l + 1) \times (2l + 1)$ matrix that tells us how a spherical harmonic transforms under rotation about the Y axis, i.e., how to rewrite a rotated spherical harmonic as a linear combination of all the spherical harmonics of the same order. The important thing to note here is that the m indices are *mixed*—a spherical harmonic after rotation must be expressed as a combination of other spherical harmonics with different m indices. However, the l indices are not mixed; rotations of spherical harmonics with order l are composed entirely of other spherical harmonics with order l .

Finally, we can combine the above two equations, with a similar result for the z rotation by γ , to derive the required rotation formula:

$$\begin{aligned} R^{\alpha, \beta, \gamma}\{Y_{l,m}(\theta'_i, \phi'_i)\} &= R_z(\beta) R_y(\alpha) R_z(\gamma)\{Y_{l,m}(\theta'_i, \phi'_i)\} \\ &= \sum_{m'=-l}^l \tilde{D}_{m,m'}^l(\alpha, \beta, \gamma) Y_{l,m'}(\theta'_i, \phi'_i), \\ \tilde{D}_{m,m'}^l(\alpha, \beta, \gamma) &= D_{m,m'}^l(\alpha) \exp(Im\beta) \exp(Im'\gamma). \end{aligned} \quad (13)$$

In terms of group theory, the matrix \tilde{D} can be viewed as the $2l + 1$ -dimensional representation of the rotation group $SO(3)$, with the interesting α dependence being encapsulated by D . Equation (13) is simply the standard rotation formula for spherical harmonics.

Since the transfer function $A = \cos \theta'_i$ has no azimuthal dependence, terms with $m' \neq 0$ will vanish when we perform the integration in Eq. (7). Therefore we will be most interested in the coefficient of the term with $m' = 0$, i.e., $\tilde{D}_{m,0}^l = D_{m,0}^l(\alpha) \exp(Im\beta)$. It can be shown that this is simply equal to $\sqrt{4\pi/(2l+1)} Y_{l,m}(\alpha, \beta)$. This result will be assumed in Eq. (23).

Another way to derive the result just stated without appealing to the properties of the matrix D is to realize that we simply want the coefficient of the term with no azimuthal dependence. At $\theta'_i = 0$, the rotated function is determined only by the term with $m' = 0$. In fact, it can be shown (Jackson,¹⁹ Eq. 3.59) that, with $\tilde{D}_{m,0}^l$ equal to the desired coefficient, we have

$$R^{\alpha, \beta, \gamma}\{Y_{l,m}(0', \phi'_i)\} = \left(\frac{2l+1}{4\pi}\right)^{1/2} \tilde{D}_{m,0}^l(\alpha, \beta, \gamma). \quad (14)$$

Noting that $R^{\alpha, \beta, \gamma}\{Y_{l,m}(0', \phi'_i)\}$ corresponds to evaluating $Y_{l,m}$ at the local Z axis or normal, i.e., global coordinates of (α, β) , we see that $\tilde{D}_{m,0}^l = [4\pi/(2l+1)]^{1/2} \times Y_{l,m}(\alpha, \beta)$.

B. Representing the Transfer Function

We switch our attention now to representing the transfer function $A(\theta'_i) = \cos \theta'_i$. Since an object reflects only the upper hemisphere, $A(\theta'_i)$ is nonzero only when $\cos \theta'_i > 0$. The transfer function $A(\theta'_i) = 0$ over the

lower hemisphere where $\cos \theta'_i < 0$. We may refer to the transfer function as the clamped-cosine function, since it is equal to the cosine function over the upper hemisphere but is clamped to 0 when $\cos \theta'_i < 0$ over the lower hemisphere. We will need to use many formulas for representing integrals of spherical harmonics, for which a reference²⁰ will be useful. First, we expand the transfer function in terms of spherical harmonics without azimuthal dependence:

$$A(\theta'_i) = \cos \theta'_i = \sum_{n=0}^{\infty} A_n Y_{n,0}(\theta'_i). \quad (15)$$

The coefficients are given by

$$A_n = 2\pi \int_0^{\pi/2} Y_{n,0}(\theta'_i) \cos \theta'_i \sin \theta'_i d\theta'_i. \quad (16)$$

The factor of 2π comes from integrating 1 over the azimuthal dependence. It is important to note that the limits of the integral range from 0 to $\pi/2$ and not π , because we are considering only the upper hemisphere. The expression above may be simplified by writing in terms of Legendre polynomials $P(\cos \theta'_i)$. Putting $u = \cos \theta'_i$ in the above integral and noting that $P_1(u) = u$ and that $Y_{n,0}(\theta'_i) = [(2n + 1)/(4\pi)]^{1/2} P_n(\cos \theta'_i)$, we obtain

$$A_n = 2\pi \left(\frac{2n + 1}{4\pi} \right)^{1/2} \int_0^1 P_n(u) P_1(u) du. \quad (17)$$

To gain further insight, we need some facts regarding the Legendre polynomials. P_n is odd if n is odd and even if n is even. The Legendre polynomials are orthogonal over the domain $[-1, 1]$, with the orthogonality relationship being given by

$$\int_{-1}^1 P_a(u) P_b(u) du = \frac{2}{2a + 1} \delta_{a,b}. \quad (18)$$

From this, we can establish some results about Eq. (17). When n is equal to 1, the integral evaluates to half the norm above, i.e., $1/3$. When n is odd but greater than 1, the integral in Eq. (17) vanishes. This is because for $a = n$ and $b = 1$, we can break the left-hand side of Eq. (18), by using the oddness of a and b , into two equal integrals over $[-1, 0]$ and $[0, 1]$. Therefore both of these integrals must vanish, and the latter integral is the right-hand integral in Eq. (17). When n is even, the required formula is given by manipulating Eq. (20) in Chap. 5 of MacRobert.²⁰ Putting it all together, we have

$$\begin{aligned} n = 1: & \quad A_n = \sqrt{\pi/3}, \\ n > 1, \text{ odd:} & \quad A_n = 0, \\ n \text{ even:} & \quad A_n = 2\pi \left(\frac{2n + 1}{4\pi} \right)^{1/2} \frac{(-1)^{n/2-1}}{(n + 2)(n - 1)} \\ & \quad \times \left[\frac{n!}{2^n (n!/2)^2} \right]. \end{aligned} \quad (19)$$

We can determine the asymptotic behavior of A_n for large even n by using Stirling's formula. The bracketed term goes as $n^{-1/2}$, which cancels the term in the square root. Therefore the asymptotic behavior for even terms is

$A_n \sim n^{-2}$. A plot of A_n for the first few terms is shown in Fig. 3, and approximation of the clamped cosine by spherical harmonic terms as n increases is shown in Fig. 4.

C. Spherical Harmonic Version of the Reflection Equation

We now have the necessary tools to write Eq. (7) in terms of spherical harmonics. Substituting Eqs. (13) and (15), we obtain

$$\begin{aligned} E(\alpha, \beta, \gamma) &= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{m'=-l}^l L_{l,m} A_n D_{m,m'}^l(\alpha) \\ &\quad \times \exp(Im\beta) \exp(Im'\gamma) T_{n,l,m'}, \\ T_{n,l,m'} &= \int_{\phi'_i=0}^{2\pi} \int_{\theta'_i=0}^{\pi} Y_{l,m'}(\theta'_i, \phi'_i) Y_{n,0}(\theta'_i, \phi'_i) \\ &\quad \times \sin \theta'_i d\theta'_i d\phi'_i. \end{aligned} \quad (20)$$

Note that we have summed over all indices and have removed the restriction on the integral to the upper hemisphere, because that restriction has now already been folded into the coefficients A_n . By orthonormality of the spherical harmonics,

$$\begin{aligned} \int_{\phi'_i=0}^{2\pi} \int_{\theta'_i=0}^{\pi} Y_{l,m'}(\theta'_i, \phi'_i) Y_{n,0}(\theta'_i, \phi'_i) \sin \theta'_i d\theta'_i d\phi'_i \\ = \delta_{l,n} \delta_{m',0}. \end{aligned} \quad (21)$$

Therefore terms that do not satisfy $n = l, m' = 0$ will vanish. Making these substitutions in Eq. (20), we obtain

$$E(\alpha, \beta, \gamma) = \sum_{l=0}^{\infty} \sum_{m=-1}^l L_{l,m} A_l D_{m,0}^l(\alpha) \exp(Im\beta). \quad (22)$$

As noted in Subsection 4.B, it can be shown that

$$D_{m,0}^l(\alpha) \exp(Im\beta) = \left(\frac{4\pi}{2l + 1} \right)^{1/2} Y_{l,m}(\alpha, \beta). \quad (23)$$

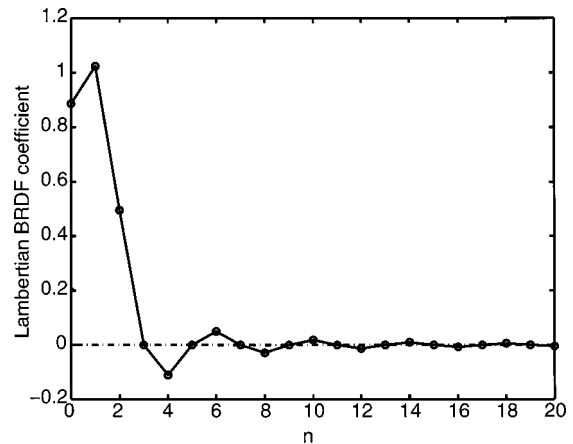


Fig. 3. The solid curve is a plot of A_n versus n . It can be seen that odd terms with $n > 1$ have $A_n = 0$. Also, as n increases, the coefficients rapidly decay.

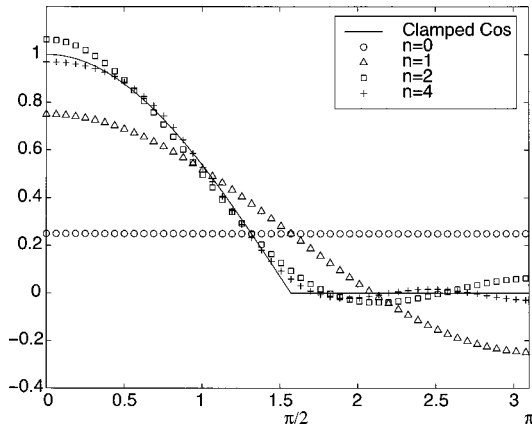


Fig. 4. Successive approximations to the clamped-cosine function by adding more spherical harmonic terms. For $n = 2$, we already get a very good approximation.

With this relation, we can write

$$E(\alpha, \beta, \gamma) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\frac{4\pi}{2l+1} \right)^{1/2} A_l L_{l,m} Y_{l,m}(\alpha, \beta). \quad (24)$$

To complete the expansion in terms of spherical harmonics, we first drop the γ dependence of E . We can see that the right-hand side of the equation above does not depend on γ , as required by physical considerations, since γ has no physical significance. Then we complete the expansion in terms of spherical harmonics by also expanding the irradiance $E(\alpha, \beta)$, i.e.,

$$E(\alpha, \beta) = \sum_{l=0}^{\infty} \sum_{m=-l}^l E_{l,m} Y_{l,m}(\alpha, \beta). \quad (25)$$

We can now equate coefficients to obtain the reflection equation in terms of spherical harmonic coefficients:

$$E_{l,m} = \left(\frac{4\pi}{2l+1} \right)^{1/2} A_l L_{l,m}. \quad (26)$$

D. Discussion

Equation (26) states that the standard direct illumination integral in Eq. (7) can be viewed as a simple product in terms of spherical harmonic coefficients. This is not really surprising, considering that Eq. (7) can be interpreted as showing that the irradiance is a *convolution* of the incident illumination and the transfer function, with different observations E corresponding to different rotations of the incident light field. Since Eq. (26) is in terms of spherical harmonic coefficients, it can be viewed in signal processing terms as a filtering operation; the output light field can be obtained by filtering the input lighting by using the clamped-cosine transfer function.

Since A_l vanishes for odd values of $l > 1$, as seen in Eqs. (19), the irradiance has zero projection onto odd-order modes, i.e., $E_{l,m} = 0$ when $l > 1$ and odd. In terms of the filtering analogy, since the filter corresponding to $A = \cos \theta'_i$ destroys high-frequency odd terms in the spherical harmonic expansion of the lighting, the corresponding terms are not found in the irradiance. Further, for large even l , the asymptotic behavior of

$E_{l,m} \sim l^{-5/2}$, since $A_l \sim l^{-2}$. The transfer function A acts as a low-pass filter, causing the rapid decay of high-frequency terms in the lighting.

We are now ready to answer the question motivating this paper: To what extent can we estimate the incoming radiance or illumination distribution given the irradiance at all orientations or surface normals. This can be viewed as a problem of deconvolution. Equation (26) makes it trivial to formulate a closed-form solution.

5. RADIANCE FROM IRRADIANCE

In this section, we discuss the inverse lighting problem. We want to find $L_{l,m}$ given the functional dependence of the irradiance on the surface normal. A practical realization is that we want to find the incident illumination from observations of a homogeneous convex curved Lambertian surface. From Eq. (26), it is trivial to derive a simple closed-form relation:

$$L_{l,m} = \left(\frac{2l+1}{4\pi} \right)^{1/2} \frac{E_{l,m}}{A_l}. \quad (27)$$

We will have difficulty solving for $L_{l,m}$ only if for all m , $E_{l,m}$ and A_l both vanish, in which case the right-hand side cannot be determined. We have already seen in Subsection 4.E that this happens when l is odd and $l > 1$. This brings us to our main result.

Theorem 1. In general, it is not possible to recover the odd-order spherical harmonic modes with order >1 of a distant radiance distribution from information about the irradiance at every surface orientation. In practical terms, observations of a homogeneous convex curved Lambertian surface do not determine the odd-order modes with order >1 of the incoming distant illumination field.

This theorem is simple to understand in terms of signal processing. The filter A has no amplitude along certain modes, and it annihilates the corresponding lighting coefficients when convolved with the incident illumination. Therefore a deconvolution method cannot recover the corresponding original components of the lighting. A stronger version of the theorem is that adding a perturbation to the incident light field consisting only of odd modes with order >1 does not change the irradiance.

Theorem 2. A perturbation of the incident distant illumination field consisting entirely of a linear superposition of odd-order spherical harmonic modes with order >1 has no effect on the irradiance at any surface orientation and hence no effect on the reflected radiance from, or on the appearance of, a convex curved Lambertian object.

As a simple example, consider adding a perturbation $\Delta L = Y_{3,m}(\theta_i, \phi_i)$ for any m to the incident lighting L . We consider the change in irradiance ΔE at a point with surface normal in spherical coordinates (α, β) . To evaluate this, we need to rotate the lighting into the local coordinate frame. We know that for spherical harmonics, rotation does not change the order but merely mixes the indices. Since the cosine in the irradiance integral does not have azimuthal dependence, the only term that can

affect the irradiance is the term without azimuthal dependence, i.e., $Y_{3,0}$ in local coordinates. The precise coefficient of this term is not important, and for similar reasons, we ignore the factor of 2π that comes from integrating over the azimuthal angle. We can simply denote $C(\alpha, \beta)$ as a constant factor and write

$$\Delta E(\alpha, \beta) = C(\alpha, \beta) \int_0^{\pi/2} Y_{3,0}(\theta'_i) \cos \theta'_i \sin \theta'_i d\theta'_i. \quad (28)$$

Substituting for $Y_{3,0}$, ignoring the premultiplying constant that we absorb into C , and then substituting $u = \cos \theta'_i$, we obtain

$$\begin{aligned} \Delta E(\alpha, \beta) &= C(\alpha, \beta) \int_0^{\pi/2} \left(\frac{5}{3} \cos^3 \theta'_i - \cos \theta'_i \right) \\ &\quad \times \cos \theta'_i \sin \theta'_i d\theta'_i \\ &= C(\alpha, \beta) \int_0^1 \left(\frac{5}{3} u^3 - u \right) u du \\ &= C(\alpha, \beta) \int_0^1 \left(\frac{5}{3} u^4 - u^2 \right) du \\ &= C(\alpha, \beta) \left(\frac{5}{3} \frac{1}{5} u^5 \Big|_0^1 - \frac{1}{3} u^3 \Big|_0^1 \right) \\ &= C(\alpha, \beta) \left(\frac{1}{3} - \frac{1}{3} \right) \\ &= 0. \end{aligned} \quad (29)$$

To make matters concrete, we can now claim that the following two lighting functions are equivalent in that they produce the same irradiance at all surface orientations and therefore cannot be distinguished by observations of a convex Lambertian surface. The constants have been chosen to ensure that the lighting remains nonnegative everywhere:

$$\begin{aligned} L_1(\theta_i, \phi_i) &= 1, \\ L_2(\theta_i, \phi_i) &= 1 + \frac{5}{3} \cos^3 \theta_i - \cos \theta_i. \end{aligned} \quad (30)$$

A. Comparison with Preisendorfer

Preisendorfer¹ (Vol. 2, pp. 143–151) concludes that radiance and irradiance are equivalent, with radiance always being recoverable from measurements of irradiance at all surface orientations. His argument is that a positive sum lighting perturbation must result in a positive norm change in the integrated irradiance over all surface orientations. In fact, he shows that the norm of the change in the integral of the irradiance is proportional to the sum of the lighting perturbation. Therefore he concludes that any lighting perturbation must result in a corresponding perturbation of the reflected light field. He neglects to consider that while overall, the lighting must be nonnegative, it is possible to devise a zero-sum perturbation to the lighting, since the perturbation can have both positive and negative components, with the only physical requirement being that the net lighting is nowhere negative. In-

deed, all the odd-order modes, including the example above, have zero sum, since their integral with $Y_{0,0}$, the constant term, must be 0 by orthogonality. Nevertheless, the condition that the lighting must be positive to be physically realizable is important and provides a further constraint on allowable perturbations. We will see that because of the constraint of positivity, there are several important special cases where the irradiance distribution does in fact uniquely determine the radiance or input illumination distribution. However, in general, as evidenced by the example above, the irradiance distribution fails to completely specify the radiance, as summarized in Theorems 1 and 2.

B. Constraining the Lighting to be Positive

We have so far not considered the physical requirement that the lighting be everywhere positive. To be physically realizable, any perturbations of the incident illumination must be small enough that the lighting remains nonnegative everywhere. Here we will show how this constraint somewhat restricts the set of *allowable* perturbations—perturbations that do not affect the irradiance. Our analysis is fairly straightforward, and we leave for future work a more complete characterization. We start by enumerating two important properties that allowable perturbations must satisfy. Here we will use ΔL to denote a perturbation.

1. We know that an allowable perturbation must be constructed of odd-order spherical harmonic modes. These modes have the property of being *odd* over the sphere—the value of a function is negated at the antipodal point on the sphere. This can be written as

$$\Delta L(\theta_i, \phi_i) = -\Delta L(\pi - \theta_i, \pi + \phi_i). \quad (31)$$

2. The remaining condition is that the perturbation's projection onto order-1 modes must be 0 (since the order-1 modes can be recovered). The order-1 modes are simply the linear terms x , y , and z . These modes are linear and odd—their value negates at the antipodal point. Therefore, in conjunction with condition 1, an allowable perturbation must have zero linear moment over any hemisphere—since the antipodal hemisphere is negated for both the function and the order-1 mode. Condition 2 is important in showing that for a directional source, there is no allowable nonzero norm perturbation.

We first consider allowable values of a perturbation, as described by the following lemma.

Lemma 1. If L is the true value of the incident illumination, the values of an allowable perturbation ΔL satisfy

$$-L(\theta_i, \phi_i) \leq \Delta L(\theta_i, \phi_i) \leq L(\pi - \theta_i, \pi + \phi_i). \quad (32)$$

The norm of the maximum allowable perturbation therefore satisfies

$$|\Delta L(\theta_i, \phi_i)| \leq \max[L(\theta_i, \phi_i), L(\pi - \theta_i, \pi + \phi_i)]. \quad (33)$$

The first inequality in relation (32) holds trivially to maintain positivity of L . The second inequality in relation (32) follows from the need to maintain positivity at the antipodal point, since $\Delta L(\pi - \theta_i, \pi + \phi_i) = -\Delta L(\theta_i, \phi_i)$. This simple lemma leads to a number

of corollaries for various special cases, where we show that increasingly complex lighting conditions admit no nonzero norm perturbation.

Corollary 1. Allowable perturbations $\Delta L(\theta_i, \phi_i)$ must be 0 at points where the true lighting value $L(\theta_i, \phi_i)$ and the true lighting value for the antipodal point $L(\pi - \theta_i, \pi + \phi_i)$ are both 0. In particular, if the true lighting L is everywhere 0, there is no allowable perturbation.

This follows trivially from relation (33). Note that if the true lighting is a single directional source, this condition forces the allowable perturbation to be 0 everywhere except at the source and its antipodal point.

Corollary 2. If the true lighting is a single directional source, there is no nonzero norm allowable perturbation, and therefore the true radiance can be recovered from the irradiance distribution at all surface orientations.

From the discussion below Corollary 1, the allowable perturbation can have a nonzero (negative) value only at the source—with a corresponding negation (positive value) at the antipodal point. A perturbation $-\Delta L$ at the source and $+\Delta L$ at the antipodal point yields a nonzero moment with respect to one or both of $\cos \theta_i$ and $\sin \theta_i$ —the order-1 modes. In other words, since we are on a sphere, one of x , y , or z has to be nonzero, so the linear moments cannot vanish. Note that by the second required property, these moments must vanish in an allowable perturbation. Therefore a perturbation consisting of a negative spike—with an equal positive spike at the antipodal point—cannot be constructed by using only odd-order spherical harmonics with order >1 . In particular, this allows Corollary 2 to be extended to the case of two antipodal sources, since an allowable perturbation must have the same form—a negative spike at one of the sources along with an equal positive spike at the other source. A much stronger statement is found below.

Corollary 3. If the true lighting consists of two distinct directional sources or a nondegenerate arrangement of three directional sources, there is no allowable nonzero norm perturbation.

From the above discussions, we know that allowable perturbations can be nonzero only at the sources and their antipodal points. The condition of vanishing moments with respect to the linear order-1 modes leads to a set of three simultaneous equations requiring us to choose perturbation intensities at the sources so that the moments in x , y , and z vanish. This is not possible for nondegenerate configurations of three sources unless all perturbation intensities are 0. It is also not possible for any two distinct sources, since we can reparameterize so that one of the sources is on the $+Z$ axis. To make all linear moments vanish, the other source would need to have $x = y = 0$, i.e., be antipodal at $-Z$, in which case, by the argument below Corollary 2, there is no allowable perturbation.

The corollary does not extend to more than three directional sources. For four directional sources, we may set the (negative) perturbation arbitrarily at one of the

sources to fix the scale. We can then solve the vanishing moment equations for the three remaining perturbations, and the condition for a nonzero norm perturbation is that all three perturbations at the sources are negative, as required by relation (32).

With more than four directional sources, we have much more freedom in choosing the perturbations, and it is likely that there always exist allowable perturbations if the sources are far enough apart, i.e., do not satisfy the conditions of Corollary 4 below. Our primary consideration in this paper has been continuous lighting distributions or area sources. For these sources (which can be thought of as made up of an infinite number of directional sources), Corollary 3 clearly does not apply.

Corollary 4. If there exists a hemispherical region in which the true lighting $L = 0$ everywhere, i.e., all the sources are strictly confined to one hemisphere, there is no allowable nonzero norm perturbation.

We can always reparameterize so that all the sources lie in $+Z$ for instance, and since $\Delta L \leq 0$ over this hemisphere (since L at the antipode is 0), the linear moment over Z must be negative, violating the condition that the linear moments of the perturbation must vanish. This corollary may have implications for natural lighting when the upper hemisphere is the major contributor. Note that the sources must be *strictly* confined to one hemisphere—a collection of sources at the “equator” or any other great circle does not satisfy the requirements of the corollary.

In Section 6, we will consider practical issues with lighting recovery. It will be shown that regardless of the theoretical results derived here, inverse lighting is in practice poorly conditioned, so even in cases where a unique solution exists, a numerical algorithm is unlikely to find it. Nevertheless, the results of this set of corollaries tells us that it will be helpful to try to explicitly ensure positivity of the recovered solutions in any numerical algorithm; this makes possible the solution of some cases—or reduces the norm of the maximum allowable perturbation—that are otherwise ambiguous by Theorems 1 and 2.

6. PRACTICAL CONSIDERATIONS

This section briefly discusses some practical considerations with respect to the radiance-from-irradiance problem and describes a simple experiment verifying the formulas derived. Equation (26) and the discussion in Subsection 4.E considering asymptotic forms show that the coefficients of the irradiance fall off rapidly, i.e., $E_{l,m} \sim l^{-5/2} L_{l,m}$. This indicates that in practice the inverse lighting problem is very poorly conditioned. In fact, we can explicitly write out numerically the first few terms for the irradiance:

$$E_{0,0} = 3.142L_{0,0},$$

$$E_{1,m} = 2.094L_{1,m},$$

$$E_{2,m} = 0.785L_{2,m},$$

$$E_{3,m} = 0,$$

$$\begin{aligned}
 E_{4,m} &= -0.131L_{4,m}, \\
 E_{5,m} &= 0, \\
 E_{6,m} &= 0.049L_{6,m}.
 \end{aligned}
 \tag{34}$$

We see that already for $E_{4,m}$, the coefficient is only approximately 4% of what it is for $E_{0,0}$. Therefore in real applications—where surfaces are only approximately Lambertian and there are errors in measurements—we expect to robustly measure the irradiance only up to order 2, and this is the maximum order at which we can recover the incident illumination. Since there are $2l + 1$ indices (values of m , which range from $-l$ to $+l$) for order l , this corresponds to nine coefficients for $l \leq 2$: one term with order 0, three terms with order 1, and five terms with order 2. Note that the single order-0 mode $Y_{0,0}$ is a constant, the three order-1 modes are linear functions of the Cartesian coordinates—in real form, they are simply x , y , and z —and the five order-2 modes are quadratic functions of the Cartesian coordinates. Therefore the irradiance—or, equivalently, the reflected light field from a convex Lambertian object—can be well approximated as a quadratic polynomial of the Cartesian coordinates of the surface normal vector.

Enforcement of positivity constraints and consideration of error metrics based on higher-order derivatives may improve the results somewhat and make them physically more plausible, but it will still be virtually impossible to recover higher-order coefficients of the lighting.

A. Discussion

Thus, even though in theory we can recover all the even modes of the lighting, in practice we expect to recover only the first nine coefficients of the lighting—modes up to order 2. For practical purposes, the irradiance in general can be characterized by only its first nine coefficients; the others vanish or are too small to be accurately measured. Thus the Lambertian BRDF acts as a very low-pass filter, passing through only the first nine coefficients of the lighting, with the irradiance effectively restricted to being at most quadratic in the Cartesian coordinates of the surface normal vector. These observations help explain the results of Marschner and Greenberg.² In that paper, an attempt was made to solve the inverse lighting problem, treating the surfaces as Lambertian. The authors noted that the problem appeared ill conditioned and not amenable to accurate solution. Therefore they had to rely heavily on a regularizing term that preserved the smoothness of the solution. The results in this paper show why the problem is ill conditioned and suggest a different regularization scheme. We can assume the high-frequency lighting coefficients to be inaccurate, so we do not attempt to recover them, and we merely set them to 0. This indicates that a spherical harmonic basis is ideal for recovering the lighting.

The fact that the irradiance is sufficiently slowly varying across the surface that it can be described by so few parameters has implications for a number of research areas, a fact that is explored in Section 7. Since Lambertian surfaces are a reasonably close approximation to

many real-world objects and are a widely used approximation in computer graphics and vision, we expect this result to have wide applicability.

B. Experimental Verification

We describe a simple experiment to verify the results of the paper. Using a camera mounted on a spherical gantry, we took a few calibrated gray-scale images of a Teflon sphere with known radius and position from different viewing positions using the same illumination—primarily from a distant ceiling light and an umbrella lamp. The lighting was measured in high dynamic range by using an almost perfectly specular mirror sphere (gazing ball). The spherical harmonic coefficients of the lighting were then computed. We were able to compute $B(\alpha, \beta)$, the radiant exitance as a function of the surface normal, on the Teflon sphere by discarding specularities and averaging measurements of the same surface location on the Teflon sphere as seen from different viewing directions. We then directly used Eq. (27) to determine the first nine spherical harmonic coefficients of the lighting. These could then be compared with those obtained from the lighting measured by using the mirror sphere. Since we did not know the relative reflectances of the Teflon and mirror spheres—or, equivalently, the precise scaling factor relating the radiant exitance B to the irradiance E —there was a scale factor that we did not recover. Therefore we uniformly scaled one set of lighting coefficients to be able to make meaningful comparisons with the other.

Figure 5 shows results from our experimental test. In (A), we show an image of the mirror and Teflon spheres. We see that the image of the Teflon sphere is a low-pass filtered version of the lighting, retaining essentially none of the high-frequency content that is visible in the image of the mirror sphere. Image (B) shows the high-resolution “real” illumination distribution, as recovered from the mirror sphere. Images (C)–(F) compare recovered and real illumination distributions. Note that since we do not explicitly enforce positivity, there are some darker negative regions in images (C)–(F). Image (F) represents a failed attempt; it is included to show the futility of attempting to recover the higher-order modes of the lighting. However, a comparison of images (C) and (D) indicates that the first nine coefficients of the lighting can be well recovered from observation of a curved convex Lambertian surface. Thus we see that we are able to recover the first nine coefficients of the lighting. But, as predicted by our theory, we fail to recover higher-order coefficients.

Table 1 shows a numerical comparison of real and recovered lighting coefficients (in real form) for the first two orders of spherical harmonics. The second column shows the (scaled) irradiance coefficients $E_{l,m}$. These are divided by the third column, as per Eq. (27), to obtain the recovered lighting coefficients $L_{l,m}$, given in the penultimate column. The final column has the real lighting coefficients, as found by using a gazing sphere. We see that the real and recovered values match closely. We also see that the irradiance coefficients are lower for $l = 2$ than for $l = 0$ or $l = 1$, since there is greater attenuation by the BRDF filter (A_l is smaller). Finally, Table 1 shows

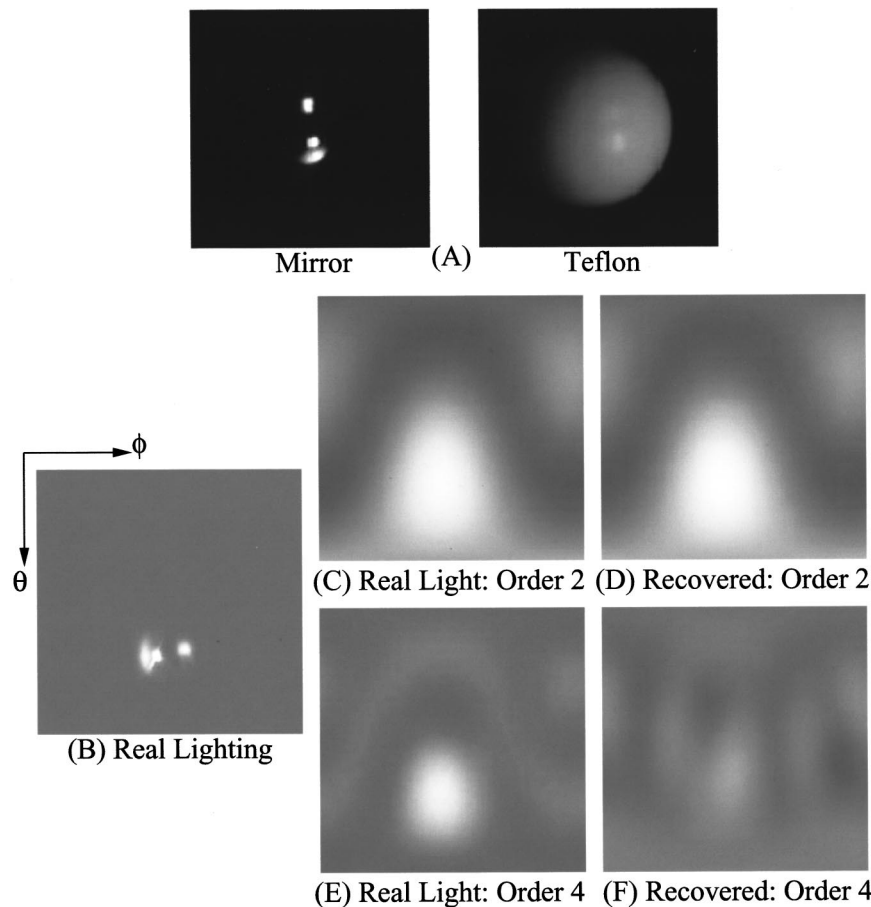


Fig. 5. (A) One of the photographs of the mirror sphere (left) and the Teflon sphere (right), (B) the real lighting as recovered by using the mirrored sphere, (C) the lighting obtained by considering only the first nine coefficients of (B), i.e., up to order 2, (D) the recovered lighting obtained by calculating the first nine coefficients of the light from the radiant exitance of the Teflon sphere, (E) the real lighting up to order 4, (F) an attempt to recover the lighting up to order 4 by also calculating the nine order-4 modes. Images (B)–(F) are visualizations obtained by unwrapping spherical coordinates of the lighting. θ_i ranges over $[0, \pi]$ uniformly from top to bottom, and ϕ_i ranges over $[0, 2\pi]$ uniformly from left to right. The zero of the lighting is the gray color used for the background of image (B).

Table 1. Comparison of Recovered and Real Lighting Coefficients

(l,m)	$E_{l,m}$	$[4\pi/(2l+1)]^{1/2}A_l$	$L_{l,m}$ (Rec.)	$L_{l,m}$ (Real)
(0,0)	3.39	3.14	1.08	0.96
(1,-1)	-0.63	2.09	-0.30	-0.31
(1,0)	-1.99	2.09	-0.95	-0.97
(1,1)	2.55	2.09	-1.22	-1.20
(2,-2)	-0.38	0.79	-0.48	-0.44
(2,-1)	0.37	0.79	0.47	0.42
(2,0)	0.05	0.79	0.06	0.05
(2,1)	-1.21	0.79	-1.54	-1.55
(2,2)	0.50	0.79	0.64	0.74
(3,-3)	-0.02	0		
(3,-2)	0.00	0		
(3,-1)	-0.03	0		
(3,0)	0.02	0		
(3,1)	0.03	0		
(3,2)	0.00	0		
(3,3)	-0.04	0		

the irradiance coefficients $E_{l,m}$ for $l = 3$. According to the theory, these coefficients should be identically 0, since $A_3 = 0$. We see that the experimental values are indeed very close to 0.

7. CONCLUSIONS

We have presented a theoretical analysis of the relationship between radiance and irradiance. We have shown that the operation of reflection is analogous to convolution of the illumination and a clamped-cosine function and have derived a simple closed-form formula in terms of spherical harmonic coefficients. We have further demonstrated that the clamped cosine—or, equivalently, the Lambertian BRDF—acts as a very low-pass filter, making deconvolution to recover the lighting difficult. In fact, we have demonstrated that odd-order modes of the incident illumination with order > 1 cannot be recovered from the irradiance, i.e., from observation of a Lambertian surface. In other words, the radiance-from-irradiance problem is ill posed or ambiguous. We have also presented evidence showing that in practical terms, the reflected light field from a Lambertian surface is characterized only by spherical harmonic modes up to order 2, and we can

therefore reliably estimate only the first nine coefficients of the incident illumination. These results confirm some previous empirical results and also open up the possibility of novel algorithms for problems in many areas of computer graphics and vision.

For instance, in the general context of inverse rendering to recover lighting and BRDFs, our results suggest the use of a spherical harmonic basis as a suitable representation, with regularization obtained by limiting the number of modes used. In rendering images in computer graphics with complex illumination represented by environment maps, our results indicate that for largely diffuse surfaces, an accurate lighting description is not necessary. Efficient algorithms might result from rendering in frequency space, considering only the first few spherical harmonic coefficients of the lighting. For object recognition under varying illumination, our results indicate that the space of all possible images of an object can be easily described by a small basis set of images, corresponding to the lowest-order modes of the lighting. Our work may also have applications in visual perception. Since our results indicate that lighting cannot lead to rapid variation of intensity over a Lambertian surface, such variation must be because of secularity or texture, and this result may be useful in explaining how one can perceive these quantities independently of the illumination.

Further work must be done on developing this theory for non-Lambertian surfaces and in considering other effects such as shadows and interreflections. Further practical work is required on the problems just mentioned. We believe that the techniques developed in this paper are of fundamental interest and may provide a firm theoretical foundation for novel algorithms in many different research areas.

APPENDIX A: NOTATION

L	Incoming radiance
$L_{l,m}$	Coefficients of spherical harmonic expansion of L
E	Irradiance
$E_{l,m}$	Coefficients of spherical harmonic expansion of E
B	Radiant exitance
ρ	Surface reflectance
A	Cosine of local incident angle $A = \cos \theta'_i$
A_l	Coefficients of spherical harmonic expansion of A
$\theta'_i (\theta_i)$	Incident elevation angle in local (global) coordinates
$\phi'_i (\phi_i)$	Incident azimuthal angle in local (global) coordinates
$\theta'_o (\theta_o)$	Outgoing elevation angle in local (global) coordinates
$\phi'_o (\phi_o)$	Outgoing azimuthal angle in local (global) coordinates
$\Omega' (\Omega)$	Hemisphere of integration in local (global) coordinates
\mathbf{x}	Surface position
\mathbf{n}	Surface normal

α	Surface normal parameterization—elevation angle
β	Surface normal parameterization—azimuthal angle
γ	Tangent frame angle
$\bar{D}_{m,m'}^l$	Representation matrices of SO(3)
$Y_{l,m}$	Spherical harmonic
I	$\sqrt{-1}$

ACKNOWLEDGMENTS

We thank the anonymous reviewers, especially reviewer 1, for their suggestions, which have led to a significant improvement in the exposition of the paper. Steve Marschner and Szymon Rusinkiewicz read early drafts and provided many insightful comments. This work was supported in part by a Hodgson-Reed Stanford graduate fellowship.

Address correspondence to Ravi Ramamoorthi at the location on the title page. The authors may also be contacted by e-mail at {ravr, hanrahan}@graphics.stanford.edu or through the website <http://graphics.stanford.edu/papers/invlamb/>.

REFERENCES

1. R. W. Preisendorfer, *Hydrologic Optics* (U.S. Department of Commerce, Washington, D.C., 1976).
2. S. R. Marschner and D. P. Greenberg, "Inverse lighting for photography," in *Fifth Color Imaging Conference: Color Science, Systems and Applications* (Society for Imaging Science and Technology, Springfield, Va., 1997), pp. 262–265.
3. I. Sato, Y. Sato, and K. Ikeuchi, "Illumination distribution from brightness in shadows: adaptive estimation of illumination distribution with unknown reflectance properties in shadow regions," in *Seventh IEEE International Conference on Computer Vision* (IEEE Computer Society, Los Alamitos, Calif., 1999), pp. 875–882.
4. N. McCormick, "Inverse radiative transfer problems: a review," *Nucl. Sci. Eng.* **112**, 185–198 (1992).
5. G. Miller and C. Hoffman, "Illumination and reflection maps: simulated objects in simulated and real environments," in *SIGGRAPH 84 Advanced Computer Graphics Animation Seminar Notes* (Association for Computing Machinery, New York, 1984).
6. B. Cabral, M. Olano, and P. Nemeč, "Reflection space image based rendering," in *SIGGRAPH 99, Computer Graphics Proceedings, Annual Conference Series*, A. Rockwood, ed. (Association for Computing Machinery, New York, 1999), pp. 165–170.
7. J. Haddon and D. Forsyth, "Shape representations from shading primitives," in *Fifth European Conference on Computer Vision*, H. Burkhardt and B. Neumann, eds. (Springer-Verlag, Berlin, 1998), pp. 415–431.
8. D. Jacobs, P. Belhumeur, and R. Basri, "Comparing images under variable illumination," in *IEEE Conference on Computer Vision and Pattern Recognition* (IEEE Computer Society, Los Alamitos, Calif., 1998), pp. 610–617.
9. P. Belhumeur and D. Kriegman, "What is the set of images of an object under all possible illumination conditions?" *Int. J. Comput. Vision* **28**, 245–260 (1998).
10. R. Epstein, P. W. Hallinan, and A. Yuille, "5 plus or minus 2 eigenimages suffice: an empirical investigation of low-dimensional lighting models," in *IEEE Workshop on Physics-Based Modeling in Computer Vision* (IEEE Computer Society, Los Alamitos, Calif., 1995), pp. 108–116.
11. R. Basri and D. Jacobs, "Lambertian reflectance and linear subspaces," *Tech. Rep. 2000-172R* (NEC Research Institute, Princeton, N.J., 2000).

12. A. Gershun, "The light field," *J. Math. Phys.* **18**, 51–151 (1939) (translated by P. Moon and G. Timoshenko).
13. R. McCluney, *Introduction to Radiometry and Photometry* (Artech House, Norwood, Mass., 1994).
14. S. Gortler, R. Grzeszczuk, R. Szeliski, and M. Cohen, "The lumigraph," in *SIGGRAPH 96, Computer Graphics Proceedings, Annual Conference Series*, H. Rushmeier, ed. (Association for Computing Machinery, New York, 1996), pp. 43–54.
15. M. Levoy and P. Hanrahan, "Light field rendering," in *SIGGRAPH 96, Computer Graphics Proceedings, Annual Conference Series*, H. Rushmeier, ed. (Association for Computing Machinery, New York, 1996), pp. 31–42.
16. D. Wood, D. Azuma, K. Aldinger, B. Curless, T. Duchamp, D. Salesin, and W. Stuetzle, "Surface light fields for 3D photography," in *SIGGRAPH 2000, Computer Graphics Proceedings, Annual Conference Series*, K. Akeley, ed. (Association for Computing Machinery, New York, 2000), pp. 287–296.
17. R. Ramamoorthi and P. Hanrahan, "Analysis of planar light fields from homogeneous convex curved surfaces under distant illumination," <http://graphics.stanford.edu/papers/planarlf>.
18. T. Inui, Y. Tanabe, and Y. Onodera, *Group Theory and its Applications in Physics* (Springer Verlag, New York, 1990).
19. J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
20. T. M. MacRobert, *Spherical Harmonics; An Elementary Treatise on Harmonic Functions, with Applications* (Dover, New York, 1948).
21. M. D'Zmura, "Shading ambiguity: reflectance and illumination," in *Computational Models of Visual Processing*, M. S. Landy and J. A. Movshon, eds. (MIT Press, Cambridge, Mass. 1991), pp. 187–207.
22. B. Cabral, N. Max, and R. Springmeyer, "Bidirectional reflection functions from surface bump maps," *Comput. Graph.* **21**(4), 273–281 (1987).
23. S. Westin, J. Arvo, and K. Torrance, "Predicting reflectance functions from complex surfaces," *Comput. Graph.* **26**(2), 255–264 (1992).
24. F. Sillion, J. Arvo, S. Westin, and D. Greenberg, "A global illumination solution for general reflectance distributions," *Comput. Graph.* **25**(4), 187–196 (1991).
25. J. Koenderink and A. van Doorn, "Phenomenological description of bidirectional surface reflection," *J. Opt. Soc. Am. A* **15**, 2903–2912 (1998).