On the Relationships between Scott Domains, Synchronization Trees, and Metric Spaces

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We use Scott's idea of information systems to provide a complete partial order semantics for concurrency involving Milner's synchronization tree model. Several connections are investigated between different models; our principal technique in establishing these connections is the use of compact metric space methods.   © 1985 Academic Press, Inc.

1. INTRODUCTION

Scott's theory of information systems (Scott, 1982) is intended to provide an easy way to define partial order structures (domains) for denotational semantics. This paper illustrates the new method by considering a simple modal logic, due to Hennessy and Milner (1980), as an example of an information system. The models of formulas in this logic are the rigid synchronization trees of Milner (1980). We characterize the domain defined by the Hennessy–Milner information system as the complete partial order of *synchronization forests*: nonempty closed sets of synchronization trees. "Closed" means closed with respect to a natural metric distance on synchronization trees, first defined by de Bakker and Zucker (1982) and characterized by Golson and Rounds (1983).

After notational preliminaries and background results, Section 3 treats the Hennessy–Milner information system. The background results (Brookes and Rounds, 1983; Golson and Rounds, 1983) are used as lemmas in the characterization of the partial order. Section 4 then shows how to use metric space methods to extend certain natural tree operations to forests. These operations become continuous in the partial order sense when so extended, and therefore can be used to provide a denotational semantics for concurrency which allows the full power of least fixed point methods for recursion (Sect. 5).

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From the results in this paper, we conclude that the information system approach to denotational semantics shows real promise. We began by investigating the Hennessy–Milner information system as a way to construct complete partial orders involving synchronization trees. The outcome was a surprisingly natural construction of such an order, which we might not have found without using the tool.

2. NOTATION AND PREVIOUS WORK

DEFINITION 2.1.   Let Σ be a finite alphabet. A Σ-tree is a tree graph on a nonempty finite or countable set of nodes, with arc labels from Σ. No ordering on the arcs leaving a node is presumed, and more than one arc may have the same label. Nodes are unlabeled, although leaf nodes are considered to be the one-node tree **nil**.

Σ-trees are the rigid synchronization trees of Milner (1980). They correspond to "unfoldings" of state graphs for nondeterministic transition systems. Milner develops an algebraic system based on these trees and their generalizations, suitable for a semantics for communicating systems. Our purpose here is to show a way of associating a Scott order structure (domain) with Σ-trees. First we need to recall some definitions.

Notation.   Suppose t is the Σ-tree represented by
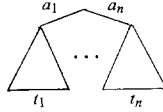
We then write $t = \sum a_i t_i$. The same notation will suffice for a tree with countably many arcs from the root and other nodes. For each $a \in \Sigma$ we define the binary transition relation $\to^a$ on the set T_Σ of all Σ-trees by $t \to^a u$ iff $t = \sum a_i t_i$ for some i, and $u = t_i$.

DEFINITION 2.2 (Weak observational equivalence).   Let a series W_k $(k \geqslant 0)$ of equivalence relations on T_Σ be given as follows:

$t \, W_0 \, u$ always;

$$t \, W_{k+1} \, u \Leftrightarrow (\forall t', a) \left[t \xrightarrow{\ a\ } t' \Rightarrow (\exists u') \left(u \xrightarrow{\ a\ } u' \text{ and } u' \, W_k \, t' \right) \right]$$

and vice versa. The weak observational equivalence W on T_Σ is given by

$$t \, W \, u \Leftrightarrow (\forall k \geqslant 0)(t \, W_k \, u).$$

EXAMPLES. Let $\Sigma = \{a, b, c\}$. Using the obvious notation, we have

$$ab + a(b + b) \ W \ ab;$$

$$a(b + c) \neg W_2 \ ab + ac.$$

Hennessy–Milner logic (HML) appeared in (Hennessy and Milner, 1980) as a language for describing synchronization trees or transition systems.

DEFINITION 2.3. HML is the least class of formulas containing the Boolean constants tt and ff, and closed under the Boolean connectives $\wedge$, $\vee$, $\neg$, and under the application of the (unary) modal operators $\langle a \rangle$ for each $a \in \Sigma$.

EXAMPLES. $\langle a \rangle (\langle b \rangle \ tt \wedge \langle c \rangle \ tt)$, $\neg \langle a \rangle (\langle b \rangle \ tt \vee \langle c \rangle \ tt)$.

DEFINITION 2.4 (Semantics of HML). We define the binary relation $\models$ between T_Σ and HML to be the smallest relation such that

1. $t \in T_\Sigma \Rightarrow t \models \phi = tt$;
2. $t \models \phi$ and $t \models \theta \Rightarrow t \models \phi \wedge \theta$;
3. $t \models \phi$ or $t \models \theta \Rightarrow t \models \phi \vee \theta$;
4. not $(t \models \phi) \Rightarrow t \models \neg \phi$;
5. $(\exists u)(t \to^a u \wedge u \models \theta) \Rightarrow t \models \langle a \rangle \ \theta$.

This definition is read "t satisfies ϕ" or "t is a tree model of ϕ."

EXAMPLES. $a(b + c) \models \theta = \langle a \rangle (\langle b \rangle \ tt \wedge \langle c \rangle \ tt)$; $ab + ac \models \neg \theta$.

DEFINITION 2.5. The modal rank $r(\phi)$ of a formula $\phi \in$ HML is defined inductively:

1. $r(tt) = r(ff) = 0$;
2. $r(\phi \vee \theta) = r(\phi \wedge \theta) = \max(r(\phi), r(\theta))$;
3. $r(\neg \theta) = r(\theta)$;
4. $r(\langle a \rangle \ \theta) = 1 + r(\theta)$.

Let $\text{HML}_k = \{\phi \in \text{HML} \mid r(\phi) \leqslant k\}$.

DEFINITION 2.6. Define the equivalences E_k and E on T_Σ by

$$t \ E_k \ u \Leftrightarrow (\forall \phi \in \text{HML}_k)(t \models \phi \Leftrightarrow u \models \phi)$$

and

$$t \, E \, u \Leftrightarrow (\forall k)(t \, E_k \, u).$$

Finally we have the notion of logical equivalence.

DEFINITION 2.7. θ is logically equivalent to ϕ iff $\forall t(t \models \theta \Leftrightarrow t \models \phi)$.

The proofs of the following facts can be found in (Brookes and Rounds, 1983; Golson and Rounds, 1983). Note that Σ is finite.

LEMMA 2.1. *For all* $t, u \in T_\Sigma$, *and* $k \geqslant 0$, $t \, W_k \, u \Leftrightarrow t \, E_k \, u$.

As a corollary, $W = E$.

LEMMA 2.2. *Among the formulas of* HML_k, *there are only finitely many logically distinct ones.*

LEMMA 2.3 (Master formula theorem for HML). *For each* Σ-*tree* t *and each* $k \geqslant 0$ *there is a formula* $\phi(k, t)$ *in* HML_k *such that*

   1. *$t \models \phi(k, t)$;*

   2. *for all u, if $u \models \phi(k, t)$ then $u \, W_k \, t$.*

LEMMA 2.4 (Compactness theorem for HML). *Let* $\Gamma \subseteq HML$. *If every finite subset of* Γ *has a tree model, then so does* Γ.

Let $\mathrm{Mod}(\Gamma)$ be the set of tree models of Γ; that is,

$$\mathrm{Mod}(\Gamma) = \{ t \mid (\forall \phi \in \Gamma)(t \models \phi) \}.$$

The compactness theorem states that if for all finite $F \subseteq \Gamma$, $\mathrm{Mod}(F) \neq \varnothing$, then $\mathrm{Mod}(\Gamma) \neq \varnothing$. We note that Lemma 2.4 holds even for infinite Σ. Finally, we recall some facts aboput the *Golson metric* d_w on T_Σ.

DEFINITION 2.8. $d_w(t, u) = \inf\{2^{-n} \mid t \, W_n \, u\}$.

This distance is actually a *pseudo-metric* because two trees may have distance zero without being identical. Clearly it is a metric on the set of W-equivalence classes.

DEFINITION 2.9. The k-section t^k of a tree t is defined to be the set of nodes at distance k from the root (counting number of arcs) and including the relevant arcs. The 0-section is then just the one-node root.

LEMMA 2.5. $t \, W_k \, u \Leftrightarrow t^k \, W \, u^k$.

LEMMA 2.6. $d_w(t^k, t) \to 0$ *as* $k \to \infty$.

LEMMA 2.7. $\langle T_\Sigma/W, d_w \rangle$ is a compact metric space.

Recall that a compact space is one where every covering by open sets has a finite subcovering. For a metric space it is equivalent to saying that every infinite sequence has a convergent subsequence.

3. INFORMATION SYSTEMS AND HENNESSY–MILNER LOGIC

First we recall the general definition of information systems from Scott (1982).

DEFINITION 3.1.   An information system is a structure $\langle D, \phi_0, \text{Con}, \vdash \rangle$, where D is a set of "propositions," $\phi_0 \in D$ is the *least informative* proposition, Con is a collection of finite subsets of D (the finite *consistent* sets), and $\vdash$ is the *entailment relation*, a subset of $\text{Con} \times D$. The following axioms hold:

1. $\Gamma \in \text{Con}$ and $\Delta \subseteq \Gamma \Rightarrow \Delta \in \text{Con}$;
2. $\{\phi\} \in \text{Con}$ for all $\phi \in D$;
3. $\Gamma \vdash \phi$ and $\Gamma \in \text{Con} \Rightarrow \Gamma \cup \{\phi\} \in \text{Con}$;
4. $\Gamma \in \text{Con} \Rightarrow \Gamma \vdash \phi_0$;
5. $\Gamma \in \text{Con}$ and $\Phi \in \Gamma \Rightarrow \Gamma \vdash \Phi$;
6. If $\Delta \vdash \theta$ and $\Gamma \vdash \phi$ for all $\phi \in \Delta$ then $\Gamma \vdash \theta$.

An information system is a way of giving "facts," expressed in D, about abstract structures. The more "facts" we know, the more "well defined" the structure becomes. The set of propositions D allows us to express this idea (partial information about a structure) using just the subsets of D.

DEFINITION 3.2.   The *ideal elements* defined by the information system D are those subsets Γ of D satisfying

1. Γ is *consistent*: Every finite subset of Γ is a member of Con;
2. Γ is *deductively closed*: $\Delta \subseteq \Gamma$, $\Delta \in \text{Con}$, and $\Delta \vdash \theta \Rightarrow \theta \in \Gamma$.

An ideal element Γ is *total* iff it is maximal respect to the inclusion ordering on the collection I_D of ideal elements of D, and is *partial* otherwise.

LEMMA 3.1.   *The ideal elements I_D of an information system D form a* complete *partial order* (cpo) *under ordinary inclusion.*

For our purposes, all we need to know about complete partial orders is that every chain Γ_i has a supremum $\bigsqcup \Gamma_i$. In our case the union of the Γ_i

sets is the obvious supremum. It can be shown that the cpo's defined by information systems are exactly the *consistently complete, algebraic* cpo's. See (Scott, 1982) for details.

HML provides a natural example of an information system describing "partial" Σ-trees. It turns out (Theorem 3.1) that a partial tree is represented as a set of trees.

DEFINITION 3.3. The HML information system is given by:

1. $D = \{\phi \in \mathrm{HML} \mid (\exists t)(t \models \phi)\}$;
2. $\mathrm{Con} = \{\varDelta \subseteq \mathrm{HML} \mid \varDelta$ is finite and $\mathrm{Mod}(\varDelta) \neq \varnothing\}$;
3. $\phi_0 = tt$;
4. $\varDelta \vdash \theta \Leftrightarrow (\forall t)[(\forall \psi \in \varDelta(t \models \psi)) \Rightarrow t \models \theta]$.

Part (4) can be rephrased: if $\varDelta = \{\theta_1, ..., \theta_n\}$ then $\theta_1 \wedge \cdots \wedge \theta_n \to \theta$ is valid.

We want to characterize the abstract cpo $\langle I_{\mathrm{HML}}, \subseteq \rangle$ in terms of Σ-trees. To do this we need two simple definitions.

DEFINITION 3.4. Let E be an equivalence relation on T_Σ. A set $K \subseteq T_\Sigma$ is E-closed iff $u \in K$ and $t \, E \, u$ implies $t \in K$.

DEFINITION 3.5. $K \subseteq T_\Sigma$ is metric-closed iff $t_i \in K$ and $d_w(t_i, t) \to 0$ imply $t \, W \, \bar{t}$ for some $\bar{t} \in K$.

Let $P_c(T_\Sigma)$ be the collection of all nonempty W-closed, metric-closed subsets of T_Σ. (The elements of this collection will be called Σ-forests.)

THEOREM 3.1. $\langle I_{\mathrm{HML}}, \subseteq \rangle$ *is isomorphic as a cpo to* $\langle P_c(T_\Sigma), \supseteq \rangle$.

Proof. Consider the map $\Gamma \to \mathrm{Mod}(\Gamma)$ from I_{HML} to the collection of subsets of T_Σ. We verify that this map is the required isomorphism:

(i) $\Gamma \subseteq \Gamma' \Leftrightarrow \mathrm{Mod}(\Gamma) \supseteq \mathrm{Mod}(\Gamma')$. The $\Rightarrow$ direction is trivial. Consider the reverse implication. Suppose that $\mathrm{Mod}(\Gamma) \supseteq \mathrm{Mod}(\Gamma')$ and let $\theta \in \Gamma$. By Definition 3.2.2, we need only find a finite $\varDelta \subseteq \Gamma'$ such that $\varDelta \vdash \theta$. Assume not: for every finite $\varDelta \subseteq \Gamma'$, we have $\neg(\varDelta \vdash \theta)$. Then for each such $\varDelta$ we have that $\varDelta \cup \{\neg \theta\}$ has a tree model. Thus every finite subset of $\Gamma' \cup \{\neg \theta\}$ has a tree model, and by compactness (Lemma 2.4) $\Gamma' \cup \{\neg \theta\}$ has a tree model. If t is such a model, then by hypothesis $t \in \mathrm{Mod}(\Gamma)$. But $\theta \in \Gamma$, and we conclude that $t \models \theta$ and $t \models \neg \theta$, a contradiction. This proves (i).

(ii) $\mathrm{Mod}(\Gamma)$ is metrically closed and W-closed. Certainly $\mathrm{Mod}(\Gamma)$ is W-closed by the corollary to Lemma 2.1. Let $d_w(t_n, t) \to 0$ where $t_n \in \mathrm{Mod}(\Gamma)$. Suppose that t is not W-equivalent to any $u \in \mathrm{Mod}(\Gamma)$. Then there

is some $\theta \in \Gamma$ such that $t \models \neg \theta$. Let p be the modal rank of $\neg \theta$. Choose n such that $d_w(t_n, t) < 2^{-p}$. Then $t_n W_p t$, so by Lemma 2.1, $t_n E_p t$. But $t_n \models \theta$ and $t \models \neg \theta$, a contradiction.

(iii) If K is metrically closed and W-closed, then for some Γ, $K = \text{Mod}(\Gamma)$.

Let

$$\Gamma = \{\theta \mid (\forall t \in K)(t \models \theta)\}.$$

It is easy to check that Γ is consistent and deductively closed. Certainly $K \subseteq \text{Mod}(\Gamma)$ by definition. We assert $\text{Mod}(\Gamma) \subseteq K$. To show this we let $t \in \text{Mod}(\Gamma)$ and construct a sequence $\langle s_n \rangle$ such that $s_n \in K$ and $d_w(s_n, t) \to 0$.

Fix $n \geq 0$. For each $s \in K$ let $\phi(n, s)$ be the master formula in HML_n satisfied by s (Lemma 2.3.) By Lemma 2.2 there are only a finite number of logically inequivalent $\phi(n, s)$ as s ranges over K.

Let $\phi_n = \bigvee_{s \in k} \phi(n, s)$. This is a finite disjunction, and for all $s \in K$, $s \models \phi_n$, so $\phi_n \in \Gamma$. Since $t \in \text{Mod}(\Gamma)$, we have $t \models \phi_n$. By definition of ϕ_n, there is an $s_n \in K$ such that $t \models \phi(n, s_n)$. The s_n form the required sequence, since by 2.3 $t W_n s_n$ and by 2.6 and 2.7, $d(t, s_n) \to 0$. This proves (iii).

Now (i) shows that the map $\Gamma \to \text{Mod}(\Gamma)$ is one-to-one and order-preserving, and (ii) and (iii) show that it is onto $P_c(T_\Sigma)$. This completes the proof of the theorem.

COROLLARY. *The maximal elements of $\langle P_c(T_\Sigma), \supseteq \rangle$ are the equivalence classes of single Σ-trees; the bottom element is the set T_Σ itself.*

It is an instructive exercise to show this corollary directly from the definition of $\langle I_{\text{HML}}, \subseteq \rangle$.

4. FORESTS AND OPERATIONS ON FORESTS

DEFINITION 4.1. A Σ-forest is a metrically closed, W-closed subset of T_Σ.

We recall the notion of *Hausdorff distance* between closed subsets of a metric space.

DEFINITION 4.2. Let $\langle X, d \rangle$ be a metric space, and Y, Z closed subsets of X.

$$d_H(Y, Z) = \max \left\{ \sup_{y \in Y} (d(y, Z)), \sup_{z \in Z} (d(z, Y)) \right\}$$

where

$$d(y, Z) = \inf_{z \in Z} (d(y, z)).$$

Intuitively, $d_H(Y, Z)$ is the maximum distance any point in Y must travel to enter Z, and vice versa.

We would like to characterize d_H when d is the Golson metric d_w. To do this we extend the W_k and W relations to forests in the expected way.

DEFINITION 4.3. Let H and H' be forests:

1. $H \, W_k \, H' \Leftrightarrow (\forall t \in H)(\exists t' \in H')(t \, W_k \, t')$ and conversely;

2. $H \, W \, H' \Leftrightarrow (\forall t \in H)(\exists t' \in H')(t \, W \, t')$ and conversely.

LEMMA 4.1. $H = H' \Leftrightarrow H \, W \, H' \Leftrightarrow (\forall k)(H \, W_k \, H')$.

Proof. The $(\Rightarrow)$ directions are trivial. Let $H \, W_k \, H'$ for each k. If $t \in H$, then for each k there is a $t_k \in H'$ such that $t \, W_k \, t_k$. Thus $d_w(t_k, t) \to 0$, which implies $t \in H'$ because H' is a forest. Similarly $H' \subseteq H$, completing the proof.

LEMMA 4.2. $H \, W_k \, H'^k$, where $H^k = \{t^k \mid t \in H\}$ and t^k is the k-section of t (Definition 2.9).

Proof. Routine.

DEFINITION 4.4 (Golson metric on forests).

$$d_w(H, H') = \inf\{2^{-n} \mid H \, W_n \, H'\}.$$

LEMMA 4.3. d_w is a metric on forests.

THEOREM 4.1. $d_w = d_H$.

Proof. (i). $d_H \leqslant d_w$. Let j be the largest integer k such that $H \, W_k \, H'$ (possibly $j = \infty$). Then $d_w(H, H') = 2^{-j}$:

$$\Rightarrow (\forall t \in H)(\exists t' \in H')(d_w(t, t') \leqslant 2^{-j})$$

$$\Rightarrow (\forall t \in H)(d_w(t, H') \leqslant 2^{-j})$$

$$\Rightarrow \sup_{t \in H} d_w(t, H') \leqslant 2^{-j}.$$

Similarly

$$\sup_{t' \in H} d_w(t', H) \leqslant 2^{-j}$$

and the inequality holds.

(ii). $d_w \leqslant d_H$. Again let $d_w(H, H') = 2^{-j}$ with j as above. If $j = \infty$ there is nothing to show. Therefore $j < \infty$ and not $H W_{j+1} H'$. This implies

$$(\exists t \in H)(\forall t' \in H')(d_w(t, t') > 2^{-(j+1)})$$

or the same assertion with the roles of H and H' reversed. In the first case, which can be assumed without loss of generality, we have

$$(\exists t \in H)(\forall t' \in H')(d_w(t, t') \geqslant 2^{-j})$$

because d_w takes only discrete values. Therefore

$$\inf_{t' \in H'} (d_w(t, t')) = d_w(t, H') \geqslant 2^{-j}$$

and so

$$\sup_{t \in H} (d_w(t, H')) \geqslant 2^{-j}$$

and the inequality follows as in (i). This completes the proof of the theorem.

The space T_Σ of trees admits a number of operations suitable for defining semantics for concurrency. deBakker and Zucker, in particular, consider the operations of "sum"—joining trees at the root; "shuffle"—interleaving trees nondeterministically; and "composition"—grafting one tree to terminating nodes of another. They prove these operations to be continuous in the metric topology of T_Σ. We would like to extend these operations to forests in a manner analogous to extending string-valued functions to languages. There is a standard general theorem (Kuratowski, 1966, p. 414) which allows this in any compact metric space. We present this theorem, and extend it to show that it characterizes continuity of the direct image mapping on forests with respect to the Hausdorff metric.

In what follows, X is a compact metric space, and H and K are closed (hence compact) nonempty subsets of X.

LEMMA 4.4. *If H, $K \subseteq X$, then there is a $k_0 \in K$ such that $d(K, H) = d(k_0, H)$. Also, for any $k \in K$, there is an $h_0 \in H$ such that $d(k, H) = d(k, h_0)$.*

Proof. Standard, using the property that infinite sequences in compact metric spaces have convergent subsequences.

LEMMA 4.5. *Let $\langle H_i \rangle$ be a decreasing chain of nonempty closed subsets of X and let $h_i \in K_i$ for each i. Then there is an $h_0 \in \bigcap_i H_i$ and a subsequence $\langle h_{ij} \rangle \to h_0$.*

Proof. Again standard.

Finally, we need a lemma on Hausdorff distances.

LEMMA 4.6. *Let H_i be a decreasing chain of nonempty closed subsets of X, and $H = \bigcap_i H_i$. Then $d(H_i, H) \to 0$.*

Proof. We show $\sup_{h \in H_i}(d(h, H)) \to 0$, from which the result follows. By Lemma 4.4, choose $h_i \in H$ such that $\sup_{h \in H_i}(d(h, H)) = d(h_i, H)$. By Lemma 4.5 the h_i have a convergent subsequence $h_{i_j} \to h_0 \in H$. Now let $\varepsilon > 0$. Choose i_k such that $d(h_{i_k}, h_0) < \varepsilon$. Then for any $j \geqslant i_k$,

$$\sup_{h \in H_j}(d(h, H)) \leqslant \sup_{h \in H_{i_k}}(d(h, H)) = d(h_{i_k}, H).$$

But

$$d(h_{i_k}, H) \leqslant d(h_{i_k}, h_0) < \varepsilon$$

which completes the proof.

DEFINITION 4.5. Let $f: X \to Y$. The *direct image* function is the map $f[\cdot]: 2^X \to 2^Y$ given by $f[K] = \{f(k) \mid k \in K\}$.

THEOREM 4.2. *Let f be a function from a compact metric space X to a metric space Y. The following are equivalent:*

1. *f is continuous;*

2. *$f[H]$ is closed for all closed H, and $f[\bigcap_i H_i] = \bigcap_i f[H_i]$ for all decreasing chains H_i of closed nonempty subsets of X;*

3. *$f[\cdot]$ is continuous in the Hausdorff metric.*

Proof. We show $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. Note: $(1) \Rightarrow (2)$ is the standard result from $[K]$.

$(1) \Rightarrow (3)$. Let $\varepsilon > 0$. We find a δ such that

$$\forall H, H' : d(H, H') < \delta \Rightarrow d(f[H], f[H']) \leqslant \varepsilon$$

(i.e., $f[\cdot]$ is uniformly continuous.) Since X is compact and (1) holds, we know by a standard result that f is already uniformly continuous. Therefore, choose δ such that

$$\forall h, h'(d(h, h') < \delta \Rightarrow d(f(h), f(h')) < \varepsilon.$$

Assume that $d(H, H') < \delta$:

$$\Rightarrow \sup_{h \in H} (d(h, H')) < \delta$$

$$\Rightarrow (\forall h \in H)(d(h, H') < \delta)$$

$$\Rightarrow (\forall h \in H)[\inf_{h' \in H'} d(h, h') < \delta]$$

$$\Rightarrow (\forall h \in H)(\exists h' \in H')(d(h, h') < \delta) \qquad \text{(by 4.4)}$$

$$\Rightarrow (\forall h \in H)(\exists h' \in H')(d(f(h), f(h')) < \varepsilon) \qquad \text{(by uniform continuity)}$$

$$\Rightarrow (\forall h \in H)[\inf_{h' \in H'} d(f(h), f(h')) < \varepsilon]$$

$$\Rightarrow \sup_{h \in H} (d(f(h), f[H'])) \leqslant \varepsilon.$$

Similarly

$$\sup_{h' \in H'} (d(f[H], f(h'))) \leqslant \varepsilon$$

and (3) follows.

(3) $\Rightarrow$ (2). Let H_i be a decreasing sequence of nonempty closed subsets of X and let $H = \bigcap_i H_i$. Then by Lemma 4.6, $d(H_i, H) \to 0$ as $i \to \infty$. By (3), $d(f[H_i], f[H]) \to 0$. But $f[H_i]$ is again a decreasing sequence of closed sets, so $d(f[H_i], \bigcap_i f[h_i]) \to 0$ by Lemma 4.6. This proves (2) because limits are unique.

(2) $\Rightarrow$ (1). Let $d(t_n, t) \to 0$. We must prove using (2) that $d(f(t_n), f(t)) \to 0$. Define for $k \geqslant 0$,

$$S_k = \{u \mid d(t, u) \leqslant 1/(k+1)\}.$$

The sets S_k form a decreasing sequence of nonempty closed sets, and $\bigcap_k S_k = \{t\}$. By (2) we have

$$f\left[\bigcap_k S_k\right] = [t] = \bigcap_k f[S_k].$$

Again by Lemma 4.6, since $f[S_k]$ decrease, $d(f[S_k], \bigcap_k f[S_k]) \to 0$. Now let $\varepsilon > 0$. Choose K such that

$$d\left(f[S_K], \bigcap_k f[S_k]\right) < \varepsilon.$$

Then since $t_n \rightarrow t$, there is an N_K such that for all $n \geqslant N_K$, we have $t_n \in S_K$ and $f(t_n) \in f[S_K]$. Thus for $n \geqslant N_K$,

$$d(f(t_n), f(t)) = d(f(t_n), f[\{t\}]) = d\left(f\left(t_n, \bigcap_k f[S_K]\right)\right)$$

$$\leqslant d\left(f[S_K], \bigcap_k f[S_k]\right)$$

and by the above, this last quantity is less than ε. This completes the proof of Theorem 4.2.

In order to apply the results of Theorem 4.2 we present several tree operations. These can be used to define the semantics of appropriate combinations of processes, as in (de Bakker and Zucker, 1982). We call on the lemmas of *ibid.*, in fact, to establish that certain operations are metrically continuous. Theorem 4.2 then applies to show that the extensions to forests are *sup-continuous*, i.e., satisfy the condition of Theorem 4.2.2. We first define the operations for finite trees, and then extend them to infinite trees using the Cauchy limit technique from *ibid.*

DEFINITION 4.6 (Alternative choice). Let s and t be trees. Then $s + t$ is the result of joining s and t at the root.

DEFINITION 4.7 (Δ-synchronized shuffle). Suppose $\Delta \subseteq \Sigma$. We want a tree operation which matches two trees (glues them together) at events in Δ and interleaves events outside Δ. The appropriate definition is inductive. Define

$$\mathbf{nil} \parallel_\Delta s = s \parallel_\Delta \mathbf{nil} = s;$$

$$\sum a_i s_i \parallel_\Delta \sum b_j t_j = u + v + w$$

where

$$u = \sum_{a_i = b_j \in \Delta} a_i(s_i \parallel_\Delta t_j);$$

$$v = \sum_{a_i \notin \Delta} a_i\left(s_i \parallel_\Delta \sum b_j t_j\right);$$

$$w = \sum_{b_j \notin \Delta} b_j\left(\sum a_i s_i \parallel_\Delta t_j\right).$$

For the purposes of the next definition, we change the definition of trees. (We did not do this at the beginning of the paper because it would have

complicated the simple intuition of a tree graph.) We would like to model two kinds of stopped processes: one which can continue because it has sucessfully terminated, and one which cannot because it has failed. We will use **nil** to model the second kind, and we introduce a new kind of leaf node or nullary tree called **skip** to model the first kind. We declare **nil** and **skip** to be W_0-inequivalent, and we introduce new propositional constants FAILED and DONE into HML, and extend the definition of satisfaction so that

$$t \models \text{FAILED} \Leftrightarrow t = \textbf{nil}$$

$$t \models \text{DONE} \quad \Leftrightarrow t = \textbf{skip}.$$

One can check that the lemmas of Section 2 still hold. We also need to modify the previous two definitions slightly. For alternative choice we put **nil** + **skip** = **skip**, and we add another base clause to the definition of synchronized shuffle with **skip** replacing **nil** in the existing clause.

DEFINITION 4.8 (Sequential composition). Again the definition is inductive for finite trees:

$$\textbf{nil} \circ t = \textbf{nil}$$

$$\textbf{skip} \circ t = t;$$

$$\left(\sum a_i s_i \right) \circ t = \sum a_i (s_i \circ t).$$

DEFINITION 4.9 (Renaming). We wish to rename events so that, for example, they are removed from the synchronization alphabet. This is done by letting h be a map from Σ to Σ. Define

$$h(\textbf{nil}) = \textbf{nil};$$

$$h(\textbf{skip}) = \textbf{skip};$$

$$h\left(\sum a_i s_i \right) = \sum h(a_i) h(s_i).$$

Now we extend these definitions to infinite trees. The definition of alternative choice needs no change. For the others, we take Cauchy limits as in (de Bakker and Zucker, 1982):

$$s \parallel_A t = \lim_{k \to \infty} s^k \parallel_A t^k;$$

$$s \circ t = \lim_{k \to \infty} s^k \circ t;$$

$$h(t) = \lim_{k \to \infty} h(t^k).$$

We need to check that the above limits exist in the metric sense. This fact is a corollary of the next lemma, which can be used to show that the above sequences are Cauchy sequences.

LEMMA 4.7.   *Let* s, s', t, *and* t' *be finite trees, and* $M = \max(d(s, s'), d(t, t'))$. *Then*

1. $d(s + t, s' + t') \leqslant M$;
2. $d(s \parallel_\Delta t, s' \parallel_\Delta t') \leqslant M$;
3. $d(s \circ t, s' \circ t') \leqslant M$; *and*
4. $d(h(s), h(s')) \leqslant d(s, s')$.

Proof.   de Bakker and Zucker show (1) and (3). They also show (2) when $\Delta = \varnothing$. The proof of (2) when $\Delta \neq \varnothing$ is a tedious but straightforward extension of their proof, and (4) is an easy exercise. We thus omit the details of the lemma.

COROLLARY.   *The Cauchy limits above exist, modulo W-equivalence.*

LEMMA 4.8.   *The inequalities of Lemma* 4.7 *hold for infinite trees.*

Proof.   We show only 4.7.2 as the proof works the same way in the other cases. We also drop the Δ subscript on the parallel operator. Consider first a special case: $d_w(s, s') = 0$. Then we must show

$$d_w(s \parallel t, s' \parallel t') \leqslant d_w(t, t').$$

Incidentally, this shows that W is a congruence with respect to $\parallel$.

We know that for each k

$$d(s^k \parallel t^k, s'^k \parallel t'^k) \leqslant d(t^k, t'^k)$$

because $s^k W s^k$ for each k. As $k \to \infty$, we have $d(t^k, t'^k) \to d(t, t')$, and the left side of the above inequality approaches $d(s \parallel t, s' \parallel t')$. Therefore the desired inequality holds.

We can now assume that $s \neg W s'$, and similarly for t and t'. Let $\varepsilon > 0$. Choose k so large that

$$d(s^k, s'^k) = d(s, s');$$

$$d(t^k, t'^k) = d(t, t');$$

$$d(s \parallel t, s^k \parallel t^k) \leqslant \varepsilon/2$$

and

$$d(s'^k \parallel t'^k, s' \parallel t') \leqslant \varepsilon/2.$$

Then by the triangle inequality

$$d(s\|t, s'\|t') \leqslant d(s\|t, s^k\|t^k) + d(s^k\|t^k, s'^k\|t'^k) + d(s'^k\|t'^k\|t'^k, s'\|t')$$

$$\leqslant \varepsilon/2 + \varepsilon/2 + \max\{d(s^k, s'^k), d(t^k, t'^k)\}$$

$$= \varepsilon + \max(d(s, s'), d(t, t')).$$

But ε was arbitrary so the lemma follows.

COROLLARY. *The various operators are (jointly) continuous in the d_w metric.*

COROLLARY. *The extended operators are sup-continuous on forests, by Theorem 4.2.*

Unfortunately, the sequential composition operator is not quite the right one to lift to forests. That is,

$$G \circ H = \{t \circ u \mid t \in G, u \in H\}$$

requires that the *same* tree u be substituted for occurrences of **skip** in t. If the root nodes of trees in a forest represent possible initial states of a process, then to capture a full notion of nondeterminism, we should allow the process H to start in any of its possible initial states whenever the process G has successfully terminated. Thus we want an operator which will allow substitution of any tree from H for **skip** nodes in trees of G.

We give a special definition for the operator $G \circ H$. We begin with

DEFINITION 4.10.   Let G be a forest and t be a finite tree. The forest $t \circ G$ is given inductively:

$$\mathbf{nil} \circ G = \mathbf{nil};$$

$$\mathbf{skip} \circ G = G;$$

$$\left(\sum a_i t_i\right) \circ G = \left\{\sum a_i u_i \mid u_i \in t_i \circ G\right\}.$$

It is easy to check that $t \circ G$ is closed for each t. Now let

$$t \circ G = \lim_{k \to \infty} t^k \circ G$$

for infinite t. Again we need to check that this limit exists.

LEMMA 4.9.   *For finite t and u*

$$t \; W_k \, u \Rightarrow t \circ G \; W_k \, u \circ G$$

where W_k is given by Definition 4.3.

Proof. We use induction on k. The case $k = 0$ is easy. By 4.3 we need to check that (in the inductive case) if $t W_{k+1} u$, then for all $w \in t \circ G$ there is a $z \in u \circ G$ such that $w W_{k+1} z$. Let $w = \sum a_i w_i \in t \circ G$, and $w \to^{a_i} w_i$. By definition $w_i \in t \circ G$, where $t = \sum a_i t_i$. Therefore $t \to^{a_i} t_i$, and since $t W_{k+1} u$, we have $u \to^{a_i} u_i$ for some $u_i W_k t_i$. If $w = \sum a_i w_i$, let $z = \sum a_i z_i$ for each such a_i, where $z_i \in u_i \circ G$ is chosen so that $z_i W_k w_i$ (this is possible by the induction hypothesis and the fact that $u_i W_k t_i$). Then $z W_{k+1} w$ as desired. The reverse implication is similar, completing the proof.

COROLLARY. $d(t \circ G, u \circ G) \leqslant d(t, u)$, *where the Hausdorff metric is used on the left, and* t *and* u *are finite trees.*

Proof. Apply Theorem 4.1 and the definitions at the beginning of Section 4.

COROLLARY. *The limit* $t^k \circ G$ *exists as* $k \to \infty$.

DEFINITION 4.11. Let G and H be forests:

$$H \circ G = \bigcup_{t \in H} t \circ G.$$

Our objective is to prove that $\lambda G H . H \circ G$ is sup-continuous in G and in H. This will be established by a series of independently interesting lemmas. We begin with

LEMMA 4.10. *The map* $\lambda t . t \circ G$ *is metrically continuous in* t.

Proof. By the triangle inequality, for any $k \geqslant 0$,

$$d(t \circ G, u \circ G) \leqslant d(t \circ G, t^k \circ G) + d(t^k \circ G, u^k \circ G) + d(u^k \circ G, u \circ G).$$

Let $\varepsilon > 0$. Choose k large enough so that

$$d(t \circ G, t^k \circ G) < \varepsilon/3$$
$$d(t^k \circ G, u^k \circ G) \leqslant d(t^k, u^k) \leqslant d(t, u) + \varepsilon/3;$$
$$d(u^k \circ G, u \circ G) < \varepsilon/3.$$

Then $d(t \circ G, u \circ G) \leqslant d(t, u) + \varepsilon$, which implies that $d(t \circ G, u \circ G) \leqslant d(t, u)$ since ε was arbitrary. The result follows directly.

Next, we prove

LEMMA 4.11. *If* t *is finite, then*

$$t \circ \bigcap_i G_i = \bigcap_i (t \circ G_i);$$

that is, for finite trees t, *the map* $\lambda G . t \circ G$ *is sup-continuous in* G.

Proof. We use induction on t. The result is clear if $t = \textbf{skip}$ or $t = \textbf{nil}$. Also, the left side is clearly included in the right side. Thus let $t = \sum_j a_j t_j$, and let w be any tree in the right-hand set. Then $w = \sum_j a_j w_j$, where for each i, $w_j \in t_j \circ G_i$. By induction hypothesis, $w_j \in t_j \circ \bigcap_i G_i$. Thus $w \in t \circ \bigcap_i G_i$, completing the proof.

Our aim is to extend Lemma 4.11 to infinite t. This can be done directly using the definitions, but it is still interesting to take a roundabout approach.

LEMMA 4.12. *For finite t, arbitrary closed G, and all $n > 0$,*

$$(t \circ G)^n = t^n \circ G)^n.$$

Proof. This will be established by induction on n. First we need an inductive definition of the n-section of a finite tree t:

Case (1) $n = 0$. $(\textbf{skip})^0 = \textbf{skip}$; $(\textbf{nil})^0 = \textbf{nil}$; $(\sum a_i t_i)^0 = \textbf{nil}$.

Case (2) $n > 0$. $(\textbf{skip})^n = \textbf{skip}$; $(\textbf{nil})^n = \textbf{nil}$; $(\sum a_i t_i)^n = \sum a_i (t_i)^{n-1}$.

When $n = 0$, the lemma is clear. Assume it for all values less than n, and consider the case n:

$$(\textbf{skip} \circ G)^n = G^n = \textbf{skip}^n \circ G)^n;$$

$$(\textbf{nil} \circ G)^n = \{\textbf{nil}\} = (\textbf{nil}^n \circ G)^n;$$

$$\left(\left(\sum a_i t_i \right) \circ G \right)^n = \left\{ \left(\sum a_i u_i \right)^n \,\middle|\, u_i \in t_i \circ G \right\}$$

$$= \left\{ \sum a_i u_i^{n-1} \,\middle|\, u_i \in t_i \circ G \right\}$$

$$= \left\{ \sum a_i u_i^{n-1} \,\middle|\, u_i \in t_i^{n-1} \circ G \right\} \qquad \text{by induction hypothesis,}$$

$$= \left\{ \left(\sum a_i u_i \right)^n \,\middle|\, u_i \in t_i^{n-1} \circ G \right\}$$

$$= \left\{ \sum a_i u_i \,\middle|\, u_i \in t_i^{n-1} \circ G \right\}^n$$

$$= \left\{ \left(\sum a_i t_i^{n-1} \right) \circ G \right\}^n$$

$$= \left(\left(\sum a_i t_i \right)^n \circ G \right)^n \qquad \text{as desired.}$$

We would like to establish Lemma 4.12 for infinite t. To do this we recall a familiar fact about continuous functions.

LEMMA 4.13. *Let f and g be continuous functions from a metric space X to a metric space Y. If f and g agree on a dense subset of X, then they coincide everywhere.*

As a corollary, we get

LEMMA 4.14. *The conclusion of Lemma 4.12 holds for infinite t.*

Proof. The map $t \rightarrow t \circ G$ is metrically continuous (Lemma 4.10). The map $F \rightarrow F^n$, where F is a forest and $F^n = \{t^n \mid t \in F\}$, is also metrically continuous. (Proof. $\lambda t.\ t^n$ is obviously m.c., so by Theorem 4.2 $\lambda F.F^n$ is m.c.) Composing these two maps, we get an m.c. map $t \rightarrow (t \circ G)^n$. Similarly the map $t \rightarrow (t^n \circ G)^n$ is m.c., and by Lemma 4.12 these maps agree on the dense subset of finite trees. This completes the proof.

LEMMA 4.15. *Lemma 4.11 holds for arbitrary trees t.*

Proof. Let G_i be a decreasing chain. We claim for each i

$$d\left(t \circ G_i, t \circ \bigcap_i G_i \right) \leqslant d\left(t \circ G_i, \bigcap_i t \circ G_i \right).$$

To see that the result follows from the claim, notice that the sequence $t \circ G_i$ is also closed and decreasing. By Lemma 4.6, $d(t \circ G_i, \bigcap_i t \circ G_i) \rightarrow 0$ as $i \rightarrow \infty$. But we also have, by the claim, $d(t \circ G_i,\ t \circ \bigcap_i G_i) \rightarrow 0$. Therefore $t \circ \bigcap_i G_i = \bigcap_i t \circ G_i$ by uniqueness of limits.

Proof of Claim. For any $k \geqslant 0$,

$$d\left(t \circ G_i, t \circ \bigcap_i G_i \right) \leqslant d\left((t \circ G_i)^k, \left(t \circ \bigcap_i G_i \right)^k \right)$$

by properties of W-equivalence on forests. Let $F_i = t \circ G_i$. Then

$$d\left(F_i^k, \left(t \circ \bigcap_i G_i \right)^k \right) = d\left(F_i^k, \left(t^k \circ \bigcap_i G_i \right)^k \right) \qquad \text{by 4.14}$$

$$= d\left(F_i^k, \left(\bigcap_i t^k \circ G_i \right)^k \right) \qquad \text{by 4.11 for finite trees}$$

$$= d\left(F_i^k, \bigcap_i (t^k \circ G_i)^k \right) \qquad \text{by sup-continuity of } \lambda F.F^n$$

$$= d\left(F_i^k, \bigcap_i (t \circ G_i)^k \right) \qquad \text{by 4.14.}$$

Now as $k \to \infty$, $F^k \to F$, and d is continuous. Thus for any ε, we can choose a k such that

$$d\left(F_i^k, \left(\bigcap_i t \circ G_i\right)^k\right) \leqslant d\left(F_i, \bigcap_i t \circ G_i\right) + \varepsilon.$$

Therefore

$$d\left(F_i, t \circ \bigcap_i G_i\right) \leqslant d\left(F_i, \bigcap_i t \circ G_i\right) + \varepsilon.$$

The inequality of the claim follows since ε was arbitrary. This completes the proof of the lemma.

We are almost ready to show that the sequential composition operator on forests is sup-continuous. This will be a consequence of another general fact about metric spaces.

LEMMA 4.16. *Let f be a metrically continuous function from the compact space X to the space $P_c(Y)$ endowed with the Hausdorff metric from Y. Define for closed $H \subseteq X$:*

$$f^*[H] = \bigcup_{x \in H} f(x).$$

Then f^*: $P_c(X) \to P_c(Y)$ is sup-continuous.

Proof. We leave to the reader the proof that $f^*[H]$ is closed. We then must show:

$$f^*\left[\bigcap_i H_i\right] = \bigcap_i f^*[H_i].$$

The inclusion of the left side in the right is obvious. Let $y \in \bigcap_i f^*[H_i]$. Then for each i, there is an $x_i \in H_i$ such that $y \in f(x_i)$. The x_i have a convergent subsequence x_{i_k} approaching some $x_0 \in \bigcap_i H_i$ by 4.5. We want $y \in f(x_0)$. But $d(f(x_{i_k}), f(x_0)) \to 0$ as $k \to \infty$, and $d(y, f(x_0)) \leqslant d(f(x_{i_k}), f(x_0))$ for each k. So $d(y, f(x_0)) = 0$. Thus $y \in f(x_0)$ since $f(x_0)$ is closed. This completes the proof.

As a corollary,

THEOREM 4.3. *The map λGH. $H \circ G$ is sup-continuous in both arguments.*

Proof. Recall that $H \circ G = \bigcup_{t \in H} t \circ G$. Sup-continuity in H follows from

Lemmas 4.10 and 4.16. Sup-continuity in G is a simple calculation using Lemma 4.15 and the distributive law for intersection over union.

To close the section, we notice that the operation of union of two forests is a sup-continuous operation, by the distributive law.

5. Applications and Conclusions

The operator of Section 4 can be used to give a denotational semantics to a CSP-like language, with two versions of nondeterminism. (See Brookes, Hoare, and Roscoe, 1984, for the reasons to consider these two versions.) We consider two syntactic operators: the "fat-bar" operator ▯ of Dijkstra, and the "nondeterministic or" operator ⊓ from op. cit. The operator ▯ will be interpreted as $+$ on forests. When applied to singleton forests of the form

$$\{at\} \text{ and } \{bu\}$$

it produces the forest

$$\{at + bu\}.$$

On the other hand, the operator ⊓ will be interpreted as union of forests. When applied to the above two singletons, it produces

$$\{at, bu\}$$

which can be thought of as a "tree"

$$\{\varepsilon at + \varepsilon bu\}$$

with fictitious ε arcs joining the two roots. In a sense the ⊓ operator introduces a hidden choice as in CCS. It should not be confused with Milner's use of the silent transition τ, however. For example, in our semantics

$$a(\varepsilon t + \varepsilon u) = \{at, au\} = \varepsilon at + \varepsilon au$$

but for Milner

$$a(\tau t + \tau u) \neq \tau at + \tau au$$

where τ is the silent transition.

Consider the CSP-like language L, where S is a statement or command:

$$\langle S \rangle := \langle \text{stmtvariable} \rangle \; |\textbf{skip}|\textbf{fail}| \; a \rightarrow S$$

$$|S_1 ; S_2| \; S_1 \; \square \; S_2 \; |S_1 \sqcap S_2| \; S_1 \|_A S_2 \; |h(S)| \; \mu x S$$

$$\langle \text{stmtvariable} \rangle := x \; |y| \; z...$$

where in the construct $a \rightarrow S$ we let a range over Σ; in the construct $h(S)$, h is a given renaming on Σ, and in the construct $\mu x S$, x is a statement variable in S.

We interpret L in the space of maps $[\text{Env} \rightarrow P_c(T_\Sigma)]$, where

$$\text{Env} = [\text{Stmtvariable} \rightarrow P_c(T_\Sigma)]$$

is the set of environments, or assignments of forests to free statement variables. Let the metavariable ρ range over environments. We define the semantic map M from L to $[\text{Env} \rightarrow P_c(T_\Sigma)]$ inductively:

1. $M[\![x]\!] \; \rho = \rho(x)$ for x a statement variable;
2. $M[\![\textbf{skip}]\!] \; \rho = \{\textbf{skip}\}$;
3. $M[\![\textbf{fail}]\!] \; \rho = \{\textbf{nil}\}$;
4. $M[\![a \rightarrow S]\!] \; \rho = (a \cdot \textbf{skip}) \circ (M[\![S]\!] \; \rho)$;
5. $M[\![S_1 ; S_2]\!] \; \rho = M[\![S_1]\!] \; \rho \circ M[\![S_2]\!] \; \rho$;
6. $M[\![S_1 \; \square \; S_2]\!] \; \rho = M[\![S_1]\!] \; \rho + M[\![S_2]\!] \; \rho$;
7. $M[\![S_1 \sqcap S_2]\!] \; \rho = M[\![S_1]\!] \; \rho \cup M[\![S_2]\!] \; \rho$;
8. $M[\![S_1 \|_A S_2]\!] \; \rho = M[\![S_1]\!] \; \rho \|_A M[\![S_2]\!] \; \rho$;
9. $M[\![h(S)]\!] \; \rho = h[M[\![S]\!] \; \rho]$;
10. $M[\![\mu x S]\!] \; \rho = \text{least fixedpoint of } \phi$

where ϕ is the map $\lambda K. \; M[\![S]\!] \; \rho(K/x)$, and $\rho(K/x)$ is the same as ρ except that x is assigned the forest K.

EXAMPLE. Consider

$$\mu x(a \rightarrow x \; \square \; b \rightarrow x).$$

The interpretation of this expression is given by the familiar formula

$$\bigcap_{n \geqslant 0} \phi^n(\bot)$$

where $\bot = T_\Sigma$, and ϕ^n is the nth iterate of

$$\phi(K) = aK + bK.$$

The first two iterates of ϕ are

$$\phi^1(\bot) = a\bot + b\bot = \{at + bu \mid t, u \in T_\Sigma\};$$
$$\phi^2(\bot) = a(a\bot + b\bot) + b(a\bot + b\bot).$$

The least fixedpoint is therefore the singleton set consisting of the full infinite binary tree over $\{a, b\}$.

EXAMPLE. Two more syntactic constructs can be introduced into L by definition. Let S be a statement expression not containing x as a free statement variable. Then we define

$$S^\omega = \mu x(S; x)$$

and

$$S^* = \mu x((S; x) \sqcap \text{skip}).$$

These operators give the infinite and indefinite repetition of S, respectively.

In conclusion, we have shown how the information system approach leads to a natural cpo for the semantics of concurrency. This cpo is closely related to the metric spaces introduced in (de Bakker and Zucker, 1982), and also to the structures explored in (de Bakker, Bergstra, Klop, and Meyer, 1983), where the cpo of closed *languages* is used as a linear time semantic structure. The work of (Bergstra and Klop, 1982) on projective limits gives another approach, although in this work no use is made of cpo methods.

It should be remarked that our theory is connected closely to the work of Courcelle (1983). The contrast here is that we work with *unordered* trees, and with countably branching trees. However, our metric space (T_Σ, d_w) can probably be obtained as a quotient of the free F-magma studied in op. cit. We have seen that *compactness* plays a key role for the definition of operators on our space $P_c(T_\Sigma)$, which seems to be a natural candidate for further study. One interesting problem is to relax the hypothesis of finite alphabet and still obtain cpo continuity properties.

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