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## Numbam

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# On the relative lengths of excursions derived from a stable subordinator* 

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#### Abstract

Results are obtained concerning the distribution of ranked relative lengths of excursions of a recurrent Markov process from a point in its state space whose inverse local time process is a stable subordinator. It is shown that for a large class of random times $T$ the distribution of relative excursion lengths prior to $T$ is the same as if $T$ were a fixed time. It follows that the generalized arc-sine laws of Lamperti extend to such random times $T$. For some other random times $T$, absolute continuity relations are obtained which relate the law of the relative lengths at time $T$ to the law at a fixed time.


## 1 Introduction

Following Lamperti [10], Wendel [24], Kingman [7], Knight [8], Perman-Pitman-Yor [ $12,13,15]$, consider the sequence

$$
\begin{equation*}
V_{1}(T) \geq V_{2}(T) \geq \cdots \tag{1}
\end{equation*}
$$

of ranked lengths of component intervals of the set $[0, T] \backslash Z$, where $T$ is a strictly positive random time, and $Z$ is the zero set of a Markov process $X$ started at zero, such as a Brownian motion or Bessel process, for which the inverse ( $\tau_{s}, s \geq 0$ ) of the local time process of $X$ at zero is a stable $(\alpha)$ subordinator, that is an increasing process with stationary independent increments and Lévy measure $\Lambda_{\alpha}$ where

$$
\begin{equation*}
\Lambda_{\alpha}(x, \infty)=C x^{-\alpha} \quad(x>0) \tag{2}
\end{equation*}
$$

for some constant $C>0$, and $0<\alpha<1$. That is, for $\lambda>0$

$$
\begin{equation*}
E\left[\exp \left(-\lambda \tau_{s}\right)\right]=\exp \left(-s K \lambda^{\alpha}\right) \text { where } K=C \Gamma(1-\alpha) \tag{3}
\end{equation*}
$$

[^0]It was shown in [15] that for all $t>0$ and $s>0$

$$
\begin{equation*}
\left(\frac{V_{1}(t)}{t}, \frac{V_{2}(t)}{t}, \ldots\right) \stackrel{d}{=}\left(\frac{V_{1}\left(\tau_{s}\right)}{\tau_{s}}, \frac{V_{2}\left(\tau_{s}\right)}{\tau_{s}}, \ldots\right) \tag{4}
\end{equation*}
$$

where $\stackrel{d}{=}$ denotes equality in distribution. Write simply $V_{n}$ instead of $V_{n}(1)$, so $\left(V_{1}, V_{2}, \ldots\right)$ is a convenient notation for a sequence of random variables with the common joint distribution of the sequences displayed in (4) for all $s>0$ and $t>0$. The distribution of $\left(V_{n}\right)$ of course depends on $\alpha$, but we suppress $\alpha$ in the notation. Note that

$$
\begin{equation*}
V_{1}>V_{2}>\cdots>0 \text { a.s. and } \sum_{n} V_{n}=1 \text { a.s. } \tag{5}
\end{equation*}
$$

For a detailed account of features of the distribution of ( $V_{n}$ ) with a parameter $0<$ $\alpha<1$, references to earlier work, and connections with Kingman's [7] Poisson-Dirichlet distribution, see [16]. Our main purpose in this paper is to point out that beyond the fixed times $t$ and inverse local times $\tau_{s}$ featured in (4), there are many more random times $T$ such that

$$
\begin{equation*}
\left(\frac{V_{1}(T)}{T}, \frac{V_{2}(T)}{T}, \ldots\right) \stackrel{d}{=}\left(V_{1}, V_{2}, \ldots\right) \tag{6}
\end{equation*}
$$

Definition 1 Call $T$ admissible, or to be more precise admissible for $Z$, if (6) holds. Call $T$ inadmissible otherwise.
Note that Definition 1 makes sense for any random closed subset $Z$ of $\mathbb{R}^{+}$, and any $\mathbb{R}^{+}$valued random variable $T$, with $V_{n}(T)$ defined as the $n$th longest component interval of $[0, T] \backslash Z$ and $V_{n}:=V_{n}(1)$. In this paper we obtain some general results which clarify the relation between stability properties of $Z$ and admissibility of various random times $T$ for $Z$. But for the rest of the introduction we continue to assume that $Z$ is the closure of the range of a stable $(\alpha)$ subordinator.

We showed in [16] by direct calculation that

$$
\begin{equation*}
H_{m}:=\inf \left\{t: V_{m}(t) \geq 1\right\} \text { is admissible for each } m=1,2, \ldots \tag{7}
\end{equation*}
$$

Here we provide a criterion for a random time $T$ to be admissible, which yields a large family of random times, including the times $t, \tau_{s}$ and $H_{m}$ mentioned above, which are admissible for $Z$ derived from a stable $(\alpha)$ subordinator. Let

$$
\begin{equation*}
G_{t}=\sup (Z \cap[0, t)) ; \quad D_{t}=\inf (Z \cap[t, \infty)) \tag{8}
\end{equation*}
$$

The admissibility of $H_{m}$ turns out to be intimately connected with the following sampling property of $Z$, established in [15], which finds several applications in this paper:

$$
\begin{equation*}
P\left(1-G_{1}=V_{n} \mid V_{1}, V_{2}, \ldots\right)=V_{n} \quad(n=1,2, \ldots) \tag{9}
\end{equation*}
$$

See $[14,17]$ for further discussion of this property and related results.
The rest of this paper is organized as follows. The main results for the range of a stable subordinator are presented in Section 2 and proved in Section 3. Besides finding times that are admissible, we show for some inadmissible random times $T$, in particular for $T=G_{t}$ and $T=D_{t}$ for a fixed time $t$, that the distribution of the sequence on the left side of (6) has a simple density relative to that of ( $V_{1}, V_{2}, \ldots$ ). In Section 4 we relate our study of admissible times to the generalized arc-sine laws of Lamperti $[9,10]$, studied also in $[2,15,23]$. In particular, we describe the distribution of time spent positive by a skew Bessel process or skew Bessel bridge.

## 2 Results for a Stable Subordinator

Throughout this section, let $0<\alpha<1$, and let $E_{\alpha}$ denote expectation with respect to a probability distribution $P_{\alpha}$ which governs ( $\tau_{s}, s \geq 0$ ) as a stable ( $\alpha$ ) subordinator, and let $Z$ be the closure of the range of $\left(\tau_{s}\right)$. Let ( $S_{t}, t \geq 0$ ) denote the continuous local time process defined by $S_{t}=\inf \left\{s: \tau_{s}>t\right\}$. While many approximations of local time are known [4], a useful one in the present setting is the following:

Proposition 2 For each $t>0$,

$$
\begin{equation*}
n^{1 / \alpha} V_{n}(t) \rightarrow\left(C S_{t}\right)^{1 / \alpha} \text { almost surely }\left(P_{\alpha}\right) \text { as } n \rightarrow \infty . \tag{10}
\end{equation*}
$$

where the limit holds uniformly in $0 \leq t \leq t_{0}$ almost surely $\left(P_{\alpha}\right)$ for every $t_{0}>0$, and also in pth mean for every $p>0$.

Proof. The convergence both a.s. and in $p$ th mean for a fixed $t>0$ is established in Proposition 10 of [16]. As observed by Kingman [7], (10) holds almost surely with the random time $\tau_{s}$ substituted instead of the fixed time $t$, and $S_{\tau_{s}}=s$ instead of $S_{t}$. Since $\left(V_{n}(t), t \geq 0\right)$ is an increasing process in $t$ for each $n$, and ( $S_{t}, t \geq 0$ ) is a continuous increasing process, the claimed almost sure convergence can be deduced by a standard argument. See for instance Lemma 2.5 of [5].

### 2.1 Admissible Times

Proposition 3 Given $c_{n} \geq 0$ with $\sup _{n} c_{n}<\infty$ and $c \geq 0$, let

$$
\begin{equation*}
A_{t}:=\sum_{n} c_{n} V_{n}(t)+c S_{t}^{1 / \alpha} \tag{11}
\end{equation*}
$$

and for $u>0$ let

$$
\begin{equation*}
\alpha_{u}:=\inf \left\{t: A_{t}>u\right\} \tag{12}
\end{equation*}
$$

Then $\alpha_{u}$ is an admissible time.
Proposition 2 has the following immediate corollary:
Corollary 4 If $T$ is admissible then

$$
\begin{equation*}
\left(\frac{S_{T}}{T^{\alpha}}, \frac{V_{1}(T)}{T}, \frac{V_{2}(T)}{T}, \ldots\right) \stackrel{d}{=}\left(S_{1}, V_{1}, V_{2}, \ldots\right) \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{1}:=C^{-1} \lim _{n} n V_{n}^{\alpha} \text { almost surely }\left(P_{\alpha}\right) \text { and in pth mean for all } p>0 \tag{14}
\end{equation*}
$$

### 2.2 Inadmissible Times

Corollary 4 implies that if $T$ is an admissible time such that $P_{\alpha}\left(G_{T}<T\right)>0$, then $G_{T}$ is not admissible. Indeed

$$
\frac{S_{G_{r}}}{G_{T}^{\alpha}}=\frac{S_{T}}{G_{T}^{\alpha}} \geq \frac{S_{T}}{T^{\alpha}}
$$

and the inequality is strict on the event $\left(G_{T}<T\right)$. So $S_{G_{T}} / G_{T}^{\alpha}$ cannot have the same distribution as $S_{T} / T^{\alpha}$ if $P_{\alpha}\left(G_{T}<T\right)>0$. Similar remarks apply to $D_{T}$. For a constant time $t$, the sequence $\left(\frac{V_{1}\left(G_{t}\right)}{G_{t}}, \frac{V_{2}\left(G_{t}\right)}{G_{t}}, \ldots\right)$ is independent of $G_{t}$ with the same distribution as the sequence of ranked lengths of excursion intervals of the corresponding bridge of length 1 . This follows from the fact (easily verified using the invariance of Bessel processes under time inversion [22]) that if ( $R_{t}, t \geq 0$ ) is a Bessel process of dimension $2-2 \alpha$ starting at 0 , then $\left(G_{t}^{-1 / 2} R_{u G_{t}}, 0 \leq u \leq 1\right)$ is a standard Bessel bridge of the same dimension independent of $G_{t}$. From Theorem 5.3 of [15], there is the following density formula relative to the distribution of $\left(V_{1}, V_{2}, \ldots\right)$ : for all non-negative product measurable $f$

$$
\begin{equation*}
E_{\alpha}\left[f\left(\frac{V_{1}\left(G_{t}\right)}{G_{t}}, \frac{V_{2}\left(G_{t}\right)}{G_{t}}, \ldots\right)\right]=\frac{E_{\alpha}\left[S_{1} f\left(V_{1}, V_{2}, \ldots\right)\right]}{E_{\alpha}\left(S_{1}\right)} \tag{15}
\end{equation*}
$$

Let $N_{t}$ be the rank of the meander length $t-G_{t}$ in the sequence of excursion lengths $V_{1}(t)>V_{2}(t)>\cdots$, so $t-G_{t}=V_{N_{t}}(t)$. Formula (9) amounts to the formula

$$
\begin{equation*}
E_{\alpha}\left[1\left(N_{t}=n\right) f\left(\frac{V_{1}(t)}{t}, \frac{V_{2}(t)}{t}, \ldots\right)\right]=E_{\alpha}\left[V_{n} f\left(V_{1}, V_{2}, \ldots\right)\right] \tag{16}
\end{equation*}
$$

for all $n=1,2, \ldots$ and all non-negative product measurable functions $f$. Consider now $N_{D_{t}}$, the rank of the excursion length $D_{t}-G_{t}$ straddling $t$ in the sequence of complete excursion lengths $V_{1}\left(D_{t}\right)>V_{2}\left(D_{t}\right)>\cdots$. So $N_{t}-1$ is the number of excursions completed by time $t$ whose lengths exceed $t-G_{t}$, and $N_{D_{t}}-1$ is the smaller number of such excursions whose lengths exceed $D_{t}-G_{t}$.

Proposition 5 For each $t>0$ and $n=1,2, \ldots$,

$$
\begin{equation*}
E_{\alpha}\left[1\left(N_{D_{\mathrm{t}}}=n\right) f\left(\frac{V_{1}\left(D_{t}\right)}{D_{t}}, \frac{V_{2}\left(D_{t}\right)}{D_{t}}, \ldots\right)\right]=E_{\alpha}\left[-\alpha \log \left(1-V_{n}\right) f\left(V_{1}, V_{2}, \ldots\right)\right] \tag{17}
\end{equation*}
$$

Immediately from Proposition 5, we draw the following consequences. First, summing over $n$ gives

$$
\begin{equation*}
E_{\alpha}\left[f\left(\frac{V_{1}\left(D_{t}\right)}{D_{t}}, \frac{V_{2}\left(D_{t}\right)}{D_{t}}, \ldots\right)\right]=E_{\alpha}\left[\left(-\sum_{n} \alpha \log \left(1-V_{n}\right)\right) f\left(V_{1}, V_{2}, \ldots\right)\right] \tag{18}
\end{equation*}
$$

which is the analog of (15) for $D_{t}$ instead of $G_{t}$. Next, an analog of (9) for $D_{t}$ instead of $t$ can be read from (17) as follows: for each $n=1,2, \ldots$

$$
\begin{equation*}
P_{\alpha}\left(D_{t}-G_{t}=V_{n}\left(D_{t}\right) \left\lvert\, \frac{V_{m}\left(D_{t}\right)}{D_{t}}=u_{m}\right., m=1,2, \ldots\right)=\frac{\log \left(1-u_{n}\right)}{\sum_{m} \log \left(1-u_{m}\right)} \tag{19}
\end{equation*}
$$

Note the remarkable fact that, just as in (9), the conditional distribution does not depend on $\alpha$.

Finally, by taking $f=1$ in (17), we obtain the formula

$$
\begin{equation*}
P_{\alpha}\left(N_{D_{\mathbf{t}}}=n\right)=E_{\alpha}\left[-\alpha \log \left(1-V_{n}\right)\right] \tag{20}
\end{equation*}
$$

As noted in [16], combined with (14) and (16) this allows the asymptotic evaluations as $n \rightarrow \infty$ :

$$
\begin{equation*}
P_{\alpha}\left(N_{D_{t}}=n\right) \sim \alpha P_{\alpha}\left(N_{t}=n\right) \sim \frac{\alpha \Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma(1-\alpha)^{1 / \alpha}} \frac{1}{n^{1 / \alpha}} \tag{21}
\end{equation*}
$$

where $a(n) \sim b(n)$ means $a(n) / b(n) \rightarrow 1$ as $n \rightarrow \infty$. See [19, 16] for integral expressions for the distributions of $N_{t}$ and $N_{D_{t}}$, and some numerical values.

In (15) and (17) we have described the law of $\left(V_{1}(T) / T, V_{2}(T) / T, \ldots\right)$ for $T=G_{t}$ and for $T=D_{t}$ by a change of measure relative to the law of this random vector for a fixed time $T$. By similar arguments we obtain change of measure formulae for $T=G_{H_{n}}$ and $T=D_{H_{n}}$. We now give these descriptions for $n=1$.

Proposition 6 For each non-negative product measurable function $f$,

$$
\begin{gather*}
E_{\alpha}\left[f\left(\frac{V_{1}\left(G_{H_{1}}\right)}{G_{H_{1}}}, \frac{V_{2}\left(G_{H_{1}}\right)}{G_{H_{1}}}, \ldots\right)\right]=E_{\alpha}\left[\left(\frac{S_{1}}{V_{1}^{\alpha}}\right) f\left(V_{1}, V_{2}, \ldots\right)\right]  \tag{22}\\
E_{\alpha}\left[f\left(\frac{V_{1}\left(D_{H_{1}}\right)}{D_{H_{1}}}, \frac{V_{2}\left(D_{H_{1}}\right)}{D_{H_{1}}}, \ldots\right)\right]=E_{\alpha}\left[\left(\alpha \log \frac{V_{1}}{V_{2}}\right) f\left(V_{1}, V_{2}, \ldots\right)\right] \tag{23}
\end{gather*}
$$

As checks, we recall from [16, Props. 10 and 8$]$ that under $P_{\alpha}$ the distribution of $S_{1} / V_{1}^{\alpha}$ is standard exponential, whereas the distribution of $V_{2} / V_{1}$ is beta $(\alpha, 1)$. Therefore, both $S_{1} / V_{1}^{\alpha}$ and $\alpha \log \left(V_{1} / V_{2}\right)$ are random variables whose $P_{\alpha}$ expectation equals 1, as implied by (22) and (23) for $f=1$.

## 3 Proofs

### 3.1 Admissible times

The foundation for the proof of Proposition 3 is a scaling argument which may prove useful in other contexts. The following theorem presents the conclusion of this argument in a fairly general setting.

Recall that a real or vector-valued process $\left(X_{t}, t>0\right)$ is called $\beta$-self-similar for some $\beta \in \mathbb{R}$ if for every $c>0$

$$
\begin{equation*}
\left(X_{c t}, t>0\right) \stackrel{d}{=}\left(c^{\beta} X_{t}, t>0\right) \tag{24}
\end{equation*}
$$

See [20] for a survey of the literature of these processes. Note that $\left(X_{t}\right)$ is $\beta$-selfsimilar iff the process ( $Y_{t}$ ) defined by $Y_{t}=t^{-\beta} X_{t}$ is 0 -self-similar, that is to say, for every $c>0$

$$
\begin{equation*}
\left(Y_{c t}, t>0\right) \stackrel{d}{=}\left(Y_{t}, t>0\right) \tag{25}
\end{equation*}
$$

This definition of 0 -self-similarity makes sense even for $Y$ with values in an abstract measurable space where there is no notion of scalar multiplication. Suppose now that $X$ is viewed as a measurable map from the basic probability space to a suitable path space $(S, \mathcal{S})$, e.g. $S=C[0, \infty)$ and $\mathcal{S}$ the $\sigma$-field generated by coordinate maps, assuming $X$ has continuous paths. Suppose $\left(X_{t}\right)$ is $\beta$-self-similar. Let ( $\mathbf{X}_{t}, t>0$ ) denote the path valued process defined by letting $\mathbf{X}_{\boldsymbol{t}}$ be the rescaling of $\mathbf{X}$ that maps time $t$ to time 1 , that is

$$
\begin{equation*}
\mathbf{X}_{t}(s)=t^{-\beta} X_{s t} \quad(s \geq 0) \tag{26}
\end{equation*}
$$

Then it is easily verified that $\left(\mathbf{X}_{t}, t>0\right)$ is 0 -self-similar.
It is this kind of 0 -self-similar process which we have in mind for applications of the following theorem.

Theorem 7 Let $\left(\mathbf{X}_{t}, t>0\right)$ be a jointly measurable 0-self-similar process with values in an arbitrary measurable space $(S, \mathcal{S})$. Let $\theta_{s}=\Theta\left(\mathbf{X}_{s}\right)$ for a non-negative $\mathcal{S}$-measurable function $\Theta$ defined on $S$, let

$$
\begin{gather*}
A_{t}=\int_{0}^{t} \theta_{s} d s, \quad(t \geq 0)  \tag{27}\\
\alpha_{u}=\inf \left\{t: A_{t}>u\right\} \tag{28}
\end{gather*} \quad(u \geq 0)
$$

Suppose that $0<A_{1}<\infty$ a.s. Then $0<\alpha_{u}<\infty$ a.s. for every $u>0$, and for all non-negative product measurable $\psi$ defined on $S \times[0, \infty)$

$$
\begin{equation*}
E\left[\psi\left(\mathbf{X}_{\alpha_{1}}, 1 / \alpha_{1}\right)\right]=E\left[\psi\left(\mathbf{X}_{1}, A_{1}\right) \frac{\theta_{1}}{A_{1}}\right] \tag{29}
\end{equation*}
$$

Remarks. According to (29), the law of $\left(\mathbf{X}_{\alpha_{1}}, 1 / \alpha_{1}\right)$ on the product space $S \times[0, \infty)$ is absolutely continuous with respect to that of ( $\mathbf{X}_{1}, A_{1}$ ), with Radon-Nikodym density $g$ defined by

$$
g\left(\mathbf{X}_{1}, A_{1}\right)=\frac{E\left[\theta_{1} \mid \mathbf{X}_{1}, A_{1}\right]}{A_{1}}
$$

It follows that for an arbitrary product measurable map $\Psi$ whose range can be any measurable space,

$$
\begin{equation*}
\Psi\left(\mathbf{X}_{\alpha_{1}}, 1 / \alpha_{1}\right) \stackrel{d}{=} \Psi\left(\mathbf{X}_{1}, A_{1}\right) \text { iff } E\left[\left.\frac{\theta_{1}}{A_{1}} \right\rvert\, \Psi\left(\mathbf{X}_{1}, A_{1}\right)\right]=1 \tag{30}
\end{equation*}
$$

For $\Psi(\mathbf{x}, a)$ such that $a$ can be recovered as a measurable function of $\Psi(\mathbf{x}, a)$, condition (30) reduces to

$$
\begin{equation*}
E\left[\theta_{1} \mid \Psi\left(\mathbf{X}_{\mathbf{1}}, A_{\mathbf{1}}\right)\right]=A_{1} \tag{31}
\end{equation*}
$$

In particular, since it follows immediately from the 0 -self-similarity of the process $\left(\theta_{s}\right)$ that

$$
\begin{equation*}
1 / \alpha_{1} \stackrel{d}{=} A_{1} \tag{32}
\end{equation*}
$$

we learn from (30) that

$$
\begin{equation*}
E\left[\theta_{1} \mid A_{1}\right]=A_{1} \tag{33}
\end{equation*}
$$

Taking $\mathbf{X}_{t}=\theta_{t}$ shows that the identity (33) holds for an arbitrary non-negative 0 -self-similar process $\left(\theta_{t}\right)$ and $A_{1}=\int_{0}^{1} \theta_{s} d s$. See $[18,17]$ for further developments and applications of this identity. Formula (29) is an abstract version of a result of Yor [26] in the case that $\left(\mathrm{X}_{t}\right)$ is the path-valued process derived by the scaling transformation (26) starting from a Brownian motion ( $X_{t}$ ). A consequence of (29) is the following variation of the result of [26] for Brownian motion.

Corollary 8 Let $\left(X_{t}, t \geq 0\right)$ a $\beta$-self-similar process and let $\left(\theta_{t}, t \geq 0\right)$ be such that for each $c>0$

$$
\begin{equation*}
\left(X_{c t}, \theta_{c t} ; t \geq 0\right) \stackrel{d}{=}\left(c^{\beta} X_{t}, \theta_{t} ; t \geq 0\right) \tag{34}
\end{equation*}
$$

Then, with $A_{1}$ and $\alpha_{1}$ defined as in (27) and (28), for all non-negative measurable functions $F$ defined on the path space

$$
\begin{equation*}
E\left[F\left(\frac{X_{t \alpha_{1}}}{\alpha_{1}^{\beta}} ; t \geq 0\right)\right]=E\left[\frac{\theta_{1}}{A_{1}} F\left(X_{t} ; t \geq 0\right)\right] \tag{35}
\end{equation*}
$$

Proof of Theorem 7. The following proof of (29) is a simple adaptation of the argument in [26]. Since the bivariate process $\left(\left(\mathbf{X}_{t}, \frac{A_{t}}{t}\right), t \geq 0\right)$ is also 0 -self-similar, it suffices to prove (29) for $\psi$ of the form $\psi(\mathbf{x}, a)=\phi(\mathbf{x})$ for an arbitrary non-negative $\mathcal{S}$ measurable function $\phi$. For $h$ a non-negative Borel function with $\int_{0}^{\infty} s^{-1} h(s) d s<\infty$, consider the quantity

$$
\begin{equation*}
Q=\int_{0}^{\infty} d s h(s) E\left[\frac{\theta_{s}}{A_{s}} \phi\left(\mathbf{X}_{s}\right)\right] \tag{36}
\end{equation*}
$$

On the one hand, the assumption that $\left(\mathbf{X}_{s}\right)$ is 0 -self-similar and the definitions of $\theta_{s}$ and $A_{s}$ imply that $\left(\left(\theta_{s}, A_{s} / s, \mathbf{X}_{s}\right), s>0\right)$ is 0 -self-similar. So $\left(\theta_{s}, A_{s}, \mathbf{X}_{s}\right) \stackrel{d}{=}\left(\theta_{1}, s A_{1}, \mathbf{X}_{1}\right)$ and we can compute

$$
\begin{equation*}
Q=\left(\int_{0}^{\infty} \frac{d s}{s} h(s)\right) E\left[\frac{\theta_{\mathbf{1}}}{A_{1}} \phi\left(\mathbf{X}_{1}\right)\right] \tag{37}
\end{equation*}
$$

On the other hand, using Fubini, a time change, and using scaling again to see that $\left(\alpha_{t}, \mathbf{X}_{\alpha_{t}}\right) \stackrel{d}{=}\left(t \alpha_{1}, \mathbf{X}_{\alpha_{1}}\right)$, we can compute

$$
\begin{align*}
Q & =E\left[\int_{0}^{\infty} \frac{d t}{t} h\left(\alpha_{t}\right) \phi\left(\mathbf{X}_{\alpha_{t}}\right)\right] \\
& =E\left[\int_{0}^{\infty} \frac{d t}{t} h\left(t \alpha_{1}\right) \phi\left(\mathbf{X}_{\alpha_{1}}\right)\right] \\
& =\left(\int_{0}^{\infty} \frac{d s}{s} h(s)\right) E\left[\phi\left(\mathbf{X}_{\alpha_{1}}\right)\right] \tag{38}
\end{align*}
$$

Comparison of (38) with (37) yields (29) for $\psi(\mathbf{x}, a)=\phi(\mathbf{x})$, as was to be proved.
Proposition 9 Suppose that $Z$ is the closure of the random set of zeros of a $\beta$-selfsimilar process $\left(X_{t}, t \geq 0\right)$, and assume that the Lebesgue measure of $Z$ is 0 almost surely. Let $V_{1}(t) \geq V_{2}(t) \geq \cdots$ be the ranked lengths of the component intervals of $[0, t] \backslash Z$, and put $V_{n}=V_{n}(1)$. Let $\mathbf{X}_{t}$ be the 0 -self-similar path valued process defined as in (26) by $\mathbf{X}_{t}(s)=t^{-\beta} X_{s t}, s \geq 0$, let $\theta_{s}=\Theta\left(\mathbf{X}_{s}\right)$ for a non-negative $\mathcal{S}$-measurable function $\Theta$, and for $t \geq 0$ and $u \geq 0$, let $A_{t}=\int_{0}^{t} \theta_{s} d s$, assume that $0<A_{1}<\infty$ almost surely, and let $\alpha_{u}=\inf \left\{t: A_{t}>u\right\}$. Then

$$
\begin{equation*}
E\left[F\left(\frac{V_{n}\left(\alpha_{1}\right)}{\alpha_{1}}, n \geq 1\right)\right]=E\left[F\left(V_{n}, n \geq 1\right) \frac{\theta_{1}}{A_{1}}\right] \tag{39}
\end{equation*}
$$

for all non-negative product measurable functions $F$. Consequently, $\alpha_{1}$ is admissible, meaning

$$
\begin{equation*}
\left(\frac{V_{1}\left(\alpha_{1}\right)}{\alpha_{1}}, \frac{V_{2}\left(\alpha_{1}\right)}{\alpha_{1}}, \ldots\right) \stackrel{d}{=}\left(V_{1}, V_{2}, \ldots\right) \tag{40}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
E\left[\left.\frac{\theta_{1}}{A_{1}} \right\rvert\, V_{1}, V_{2}, \ldots\right]=1 \tag{41}
\end{equation*}
$$

Proof. Since for each $n$, and every $t>0, V_{n}(t) / t=f_{n}\left(\mathbf{X}_{t}\right)$ for a measurable function $f_{n}$ which does not depend on $t$, formula (39) follows immediately from the previous theorem.

Note that in case $A_{1}$ is a measurable function of $\left(V_{n}, n \geq 1\right)$, the condition (41) becomes

$$
\begin{equation*}
E\left[\theta_{1} \mid V_{1}, V_{2}, \ldots\right]=A_{1} . \tag{42}
\end{equation*}
$$

Corollary 10 Let $A_{t}$ be the time spent positive by a standard Brownian motion $B$ up to time $t$, so $\alpha_{1}$ is the first instant that $B$ has spent time 1 positive. Then $\alpha_{1}$ is admissible for the zero set of $B$.

Proof. We show that (41) holds. Clearly, it suffices to show that

$$
\begin{equation*}
E\left[\theta_{1} \mid A_{1}, V_{1}, V_{2}, \ldots\right]=A_{1} \tag{43}
\end{equation*}
$$

where $\theta_{1}=1\left(B_{1}>0\right)$. Let $\varepsilon_{n}$ be the indicator of the event that $B$ is positive on the interval whose length is $V_{n}$. Since the $V_{n}$ are a.s. all distinct, there are a.s. no quibbles about the definition of the $\varepsilon_{n}$. By Itô's excursion theory, the $\varepsilon_{n}$ are independent Bernoulli $\left(\frac{1}{2}\right)$ variables, independent of $\left(G_{1}, V_{1}, V_{2}, V_{3}, \ldots\right)$, and by definition

$$
\theta_{1}=\sum_{n} \varepsilon_{n} 1\left(1-G_{1}=V_{n}\right) \text { and } A_{1}=\sum_{n} \varepsilon_{n} V_{n}
$$

so we have, by the sampling property (9),

$$
\begin{gathered}
E\left(\theta_{1} \mid \varepsilon_{1}, \varepsilon_{2}, \ldots, V_{1}, V_{2}, \ldots\right)=\sum_{n} \varepsilon_{n} P\left(1-G_{1}=V_{n} \mid V_{1}, V_{2}, \ldots\right) \\
=\sum_{n} \varepsilon_{n} V_{n}=A_{1}
\end{gathered}
$$

and (43) follows.
Remark 11 It is clear from the above proof that the conclusion of Corollary 10 holds just as well for $B$ a skew Brownian motion or a skew Bessel process, as discussed in Section 4.

Remark 12 As a companion to (43) we note that the sampling property (9) and [25, Exercise 3.4] imply that if $V_{1}, V_{2}, \ldots$ are the ranked interval lengths generated by the zero set of a Bessel process ( $R_{t}, 0 \leq t \leq 1$ ) of dimension $2-2 \alpha$ started at $R_{0}=0$ then for $x>0$

$$
P\left(R_{1} \in d x \mid V_{1}, V_{2}, \ldots\right)=x d x \sum_{n=1}^{\infty} \exp \left(-\frac{x^{2}}{2 V_{n}}\right)
$$

Corollary 13 In the setting of Proposition 9, the random time $H_{n}:=\inf \left\{t: V_{n}(t) \geq 1\right\}$ is admissible for $Z$ iff

$$
\begin{equation*}
P\left(1-G_{1}=V_{n} \mid V_{1}, V_{2}, \ldots\right)=V_{n} \tag{44}
\end{equation*}
$$

Proof. Observe that for each $n$ the process

$$
\begin{equation*}
\theta_{s}:=1\left(s-G_{s}=V_{n}(s)\right) \tag{45}
\end{equation*}
$$

is of the form $\theta_{s}=\Theta\left(\mathbf{X}_{s}\right)$ required in Theorem 7 and Proposition 9. Moreover, as observed in [17], the corresponding $A_{t}$ is just

$$
\begin{equation*}
V_{n}(t)=\int_{0}^{t} d s 1\left(s-G_{s}=V_{n}(s)\right) \tag{46}
\end{equation*}
$$

so the corresponding $\alpha_{1}$ equals $H_{n}$ as defined in (7).
In particular, $H_{n}$ is admissible for every $n$ iff (44) holds for every $n$. We then say that $Z$ has the sampling property. For $Z$ the range of a stable $(\alpha)$ subordinator, the sampling property of $Z$ was established in [15] while the admissibility of $H_{n}$ for all $n$ was shown in [16]. Neither of these results seems obvious without some calculation. In [17] we give examples of various 0 -self-similar sets $Z$, some with and some without the sampling property. It would be interesting to characterize all 0 -self-similar sets $Z$ with the sampling property, but we have no idea how to do this.
Proof of Proposition 3. Note first that if $\left(T_{n}\right)$ is a sequence of admissible times, and $T_{n}$ converges in probability as $n \rightarrow \infty$ to $T$ with $T>0$ a.s., then $T$ is admissible. By this observation and Proposition 2, it suffices to prove Proposition 3 for

$$
A_{t}=\sum_{k=1}^{p} c_{k} V_{k}(t)
$$

In this case we have from (46)

$$
\theta_{t}=\sum_{k=1}^{p} c_{k} 1\left(t-G_{t}=V_{k}(t)\right)
$$

so the sampling property and linearity of conditional expectations imply (42).
The class of admissible times is preserved under certain homogeneous transformations described in the following proposition.

Proposition 14 In the setting of Proposition 9, with $Z$ the closure of the random set of zeros of a $\beta$-self-similar process $\left(X_{t}, t \geq 0\right)$, the Lebesgue measure of $Z$ equal to 0 almost surely, and $\mathbf{X}_{t}$ the 0 -self-similar path valued process defined by $\mathbf{X}_{t}(s)=$ $t^{-\beta} X_{s t}, s \geq 0$, suppose for each $1 \leq j \leq k$ that $\theta_{s}^{(j)}=\Theta^{(j)}\left(\mathbf{X}_{s}\right)$ for a non-negative $\mathcal{S}$-measurable function $\Theta^{(j)}$, and for $t \geq 0$ and $u \geq 0$ let $A_{t}^{(j)}=\int_{0}^{t} \theta_{s}^{(j)} d s$ be such that $0<A_{1}^{(j)}<\infty$ almost surely, and define $\alpha_{u}^{(j)}=\inf \left\{t: A_{i}^{(j)}>u\right\}$. Suppose further for each $1 \leq j \leq k$ that $A_{1}^{(j)}$ is $\mathcal{V}$-measurable, where $\mathcal{V}$ is the $\sigma$-field generated by $V_{1}, V_{2}, \ldots$, and that $\alpha_{1}^{(j)}$ is admissible for $Z$. Let $f: \mathbb{R}_{+}^{k} \rightarrow \mathbb{R}_{+}$be an increasing function in each variable such that

$$
\begin{equation*}
f\left(c x_{1}, c x_{2}, \ldots, c x_{k}\right)=c f\left(x_{1}, x_{2}, \ldots, x_{k}\right) \tag{47}
\end{equation*}
$$

and $f$ is differentiable on $(0, \infty)^{k}$, and let $A_{t}:=f\left(A_{t}^{(1)}, \ldots, A_{t}^{(k)}\right)$.
Then $\alpha_{1}:=\inf \left\{t: A_{t}>1\right\}$ is admissible.
Proof. By calculus $A_{t}=\int_{0}^{t} \theta_{s} d s$ where

$$
\theta_{s}=\sum_{i=1}^{k} f_{i}^{\prime}\left(A_{s}^{(1)}, \ldots, A_{s}^{(k)}\right) \theta_{s}^{(i)}
$$

Thus we can compute

$$
\begin{aligned}
E\left[\theta_{1} \mid \mathcal{V}\right] & =\sum_{i=1}^{k} f_{i}^{\prime}\left(A_{1}^{(1)}, \ldots, A_{1}^{(k)}\right) E\left[\theta_{1}^{(i)} \mid \mathcal{V}\right] \\
& =\sum_{i=1}^{k} f_{i}^{\prime}\left(A_{1}^{(1)}, \ldots, A_{1}^{(k)}\right) A_{1}^{(i)}
\end{aligned}
$$

by (42). But, from the hypotheses on $f$ we deduce that $\sum_{i=1}^{k} f_{i}^{\prime}\left(x_{1}, \ldots, x_{k}\right) x_{i}=$ $f\left(x_{1}, \ldots, x_{k}\right)$ so we obtain $E\left[\theta_{1} \mid \mathcal{V}\right]=A_{1}$, as in (42). Therefore, $\alpha_{1}$ is admissible.

Note that the class of functions $f$ considered above is much larger than the class of functions of the form $f(x)=\sum_{i=1}^{k} c_{i} x_{i}$. For instance, one can take

$$
f_{p}\left(x_{1}, \ldots, x_{k}\right)=\left(\sum_{i=1}^{k}\left(c_{i} x_{i}\right)^{p}\right)^{1 / p}
$$

for $p>0$ and positive constants $c_{i}$. By passage to the limit, it can be deduced that the conclusion of Proposition 14 also holds for

$$
f\left(x_{1}, \ldots, x_{k}\right)=\max _{1 \leq i \leq k} x_{i}
$$

### 3.2 The lengths at time $D_{t}$

Proof of Proposition 5. Let $\mathbf{V}(T)=\left(V_{1}(T), V_{2}(T), \ldots\right)$ denote the sequence of ranked lengths of component intervals of $[0, T] \backslash Z$ for $Z$ the closed range of a stable subordinator ( $\tau_{s}$ ). By scaling, the distribution of $\mathbf{V}\left(D_{t}\right) / D_{t}$ for fixed $t>0$ does not depend on $t$. So let us write simply $D$ for $D_{1}$ and $G$ for $G_{1}$, and compute the law of $\mathbf{V}(D) / D$. Recall that the sequence $\mathbf{V}(1)$ contains the term $1-G$ as $1-G=V_{N}(1)$ for a random index $N$. The sequence $\mathbf{V}(D)$ is derived from $\mathbf{V}(1)$ by first substituting $D-G$ for this term, then reranking. Let ( $S_{t}$ ) be the local time inverse of ( $\tau_{s}$ ). Let $S=S_{1}$. So $S^{-1 / \alpha} \stackrel{d}{=} \tau_{1}$. Consider the three point processes $N_{1}, N_{G}$, and $N_{D}$ on $(0, \infty)$ defined as follows for $T=1, T=G$ or $T=D$ :

$$
N_{T}(\cdot)=\sum_{n} 1\left(S^{-1 / \alpha} V_{n}(T) \in \cdot\right)
$$

Let $X:=S^{-1 / \alpha}(1-G)$ and $Y:=S^{-1 / \alpha}(D-G)$. Then

$$
N_{G}=N_{1}-\delta_{X}=N_{D}-\delta_{Y}
$$

where $\delta_{W}(\cdot)=1(W \in \cdot)$. According to Theorems 2.1 and 1.2 of [15], $P_{\alpha}$ governs $N_{1}$ as a Poisson random measure with intensity measure $\Lambda_{\alpha}$ on $(0, \infty)$ where $\Lambda_{\alpha}$ is the stable $(\alpha)$ Lévy measure, and given $N_{1}$ the point $X$ is a size-biased pick from the points of $N_{1}$. That is to say

$$
\begin{equation*}
P_{\alpha}\left(N_{G} \in d n, X \in d x\right)=\frac{x}{\Sigma n+x} P_{\alpha}\left(N_{1} \in d n\right) \Lambda_{\alpha}(d x) \tag{48}
\end{equation*}
$$

where for a counting measure $n$ on $(0, \infty), \Sigma n=\int_{0}^{\infty} x n(d x)$ is the sum of locations of the points of $n$. Let

$$
R:=\frac{Y}{X}=\frac{D-G}{1-G}
$$

From asymptotic renewal theory [3], or by the last exit decomposition at time $G$, there is the formula

$$
\begin{equation*}
P_{\alpha}(G \in d x, D \in d y)=\frac{\alpha}{\Gamma(\alpha) \Gamma(1-\alpha)} \frac{x^{\alpha-1}}{(y-x)^{\alpha+1}} d x d y \quad(0<x<1<y<\infty) \tag{49}
\end{equation*}
$$

which implies that $G$ and $R$ are independent, with

$$
\begin{equation*}
P_{\alpha}(R \in d r)=\frac{\alpha}{r^{\alpha+1}} d r \quad(r>1) \tag{50}
\end{equation*}
$$

The last exit decomposition at time $G$ and scaling imply further that $G, N_{G}$ and $R$ are mutually independent. Since $S$ is a measurable function of $G$ and $N_{G}$, so is $X$, and we can compute for $y>x$

$$
\begin{aligned}
P_{\alpha}\left(Y \in d y \mid N_{G}, X=x\right) & =P_{\alpha}\left(X R \in d y \mid N_{G}, X=x\right)=P_{\alpha}(x R \in d y) \\
& =P_{\alpha}\left(R \in \frac{d y}{x}\right)=\alpha\left(\frac{x}{y}\right)^{\alpha+1} \frac{d y}{x}
\end{aligned}
$$

and hence

$$
\begin{aligned}
P_{\alpha}\left(N_{G} \in d n, Y \in d y\right) & =\int_{0}^{y} P_{\alpha}\left(N_{G} \in d n, X \in d x, Y \in d y\right) \\
& =\left(\int_{0}^{y} \frac{x}{\Sigma n+x} \Lambda_{\alpha}(d x) P_{\alpha}\left(Y \in d y \mid N_{G}, X=x\right)\right) P_{\alpha}\left(N_{1} \in d n\right) \\
& =\left(\int_{0}^{y} \frac{x}{\Sigma n+x} \frac{C \alpha d x}{x^{\alpha+1}} \alpha\left(\frac{x}{y}\right)^{\alpha+1} \frac{1}{x}\right) P_{\alpha}\left(N_{1} \in d n\right) d y \\
& =\alpha\left(\int_{0}^{y} \frac{d x}{\Sigma n+x}\right) P_{\alpha}\left(N_{1} \in d n\right) \Lambda_{\alpha}(d y) \\
& =\alpha \log \left(\frac{\Sigma n+y}{\Sigma n}\right) P_{\alpha}\left(N_{1} \in d n\right) \Lambda_{\alpha}(d y)
\end{aligned}
$$

That is to say

$$
\begin{equation*}
P_{\alpha}\left(N_{G} \in d n, Y \in d y\right)=\rho\left(y \mid n+\delta_{y}\right) P_{\alpha}\left(N_{1} \in d n\right) \Lambda_{\alpha}(d y) \tag{51}
\end{equation*}
$$

where for a counting measure $m$

$$
\rho(y \mid m)=\alpha \log \left(\frac{\Sigma m}{\Sigma m-y}\right)
$$

Since $N_{D}=N_{G}+\delta_{Y}$ and $N_{1}$ is a Poisson measure with intensity $\Lambda_{\alpha}$, the Palm formula of [15, Lemma 2.2] shows that (51) can be recast as

$$
\begin{equation*}
P_{\alpha}\left(N_{D} \in d m, Y \in d y\right)=\rho(y \mid m) P_{\alpha}\left(N_{1} \in d m\right) \Lambda_{\alpha}(d y) \tag{52}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
P_{\alpha}\left(N_{D} \in d m\right)=\rho(m) P_{\alpha}\left(N_{\mathbf{1}} \in d m\right) \tag{53}
\end{equation*}
$$

where

$$
\rho(m)=\int \rho(y \mid m) m(d y)=\alpha \sum_{y: m\{y\}=1} \log \left(\frac{\Sigma m}{\Sigma m-y}\right)
$$

Now

$$
\frac{\mathbf{V}(T)}{T}=\frac{S^{-1 / \alpha} \mathbf{V}(T)}{S^{-1 / \alpha} T}
$$

Since for $T=1$ and $T=D$, both $S^{-1 / \alpha} V(T)$ and $S^{-1 / \alpha} T=\sum_{n} S^{-1 / \alpha} V_{n}(T)$ are measurable functions of $N_{T}$, so is $\mathbf{V}(T) / T$. Since also

$$
\begin{equation*}
\rho\left(N_{T}\right)=\alpha \sum_{i} \log \left(\frac{T}{T-V_{i}(T)}\right)=-\alpha \sum_{i} \log \left(1-\frac{V_{i}(T)}{T}\right) \tag{54}
\end{equation*}
$$

is a function of $\mathrm{V}(T) / T$, a change of variables in (53) yields (18). A similar manipulation of (52) yields (17).

As noted in [16], formula (9) implies that for every non-negative measurable function $f$ defined on $[0,1]$,

$$
\begin{equation*}
E_{\alpha}\left[\sum_{n} f\left(V_{n}\right)\right]=E_{\alpha}\left[\frac{f\left(1-G_{1}\right)}{\left(1-G_{1}\right)}\right]=\frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} \int_{0}^{1} d u f(u) u^{-\alpha-1}(1-u)^{\alpha-1} \tag{55}
\end{equation*}
$$

where the last expression is obtained from the $\operatorname{beta}(\alpha, 1-\alpha)$ density of $G_{1}$. The consequence of (18), that

$$
E_{\alpha}\left(-\alpha \sum_{n} \log \left(1-V_{n}\right)\right)=1
$$

therefore amounts to the formula

$$
\begin{equation*}
\frac{\alpha}{\Gamma(\alpha) \Gamma(1-\alpha)} \int_{0}^{1} d u(-\log (1-u)) u^{-\alpha-1}(1-u)^{\alpha-1}=1 \tag{56}
\end{equation*}
$$

This identity can be checked directly as follows. Expanding

$$
-\log (1-u)=u+\frac{u^{2}}{2}+\frac{u^{3}}{3}+\cdots
$$

allows the left side of (56) to be evaluated as

$$
\frac{\alpha}{\Gamma(\alpha) \Gamma(1-\alpha)}\left(B(1-\alpha, \alpha)+\frac{1}{2} B(2-\alpha, \alpha)+\frac{1}{3} B(3-\alpha, \alpha)+\cdots\right)
$$

where $B(a, b)=\Gamma(a) \Gamma(b) / \Gamma(a+b)$ is the beta function, so (56) reduces to

$$
\alpha\left(1+\frac{1-\alpha}{2!}+\frac{(1-\alpha)(2-\alpha)}{3!}+\cdots\right)=1
$$

which can be seen by letting $x \uparrow 1$ in the formula

$$
\begin{equation*}
1-(1-x)^{\alpha}=\alpha x+\alpha(1-\alpha) \frac{x^{2}}{2!}+\alpha(1-\alpha)(2-\alpha) \frac{x^{3}}{3!}+\cdots \tag{57}
\end{equation*}
$$

obtained from the binomial expansion of $(1-x)^{\alpha}$. See [14] for an interpretation in terms of a stable $(\alpha)$ subordinator of the discrete distribution with the generating function (57).

A number of variations of the identity (56) can be obtained as follows. Since $G_{1}$ has beta $(\alpha, 1-\alpha)$ distribution, if $T$ is an independent exponential variable, then $T G_{1}$ has gamma $(\alpha)$ distribution. Therefore, for $\lambda>-1$,

$$
\begin{equation*}
E_{\alpha}\left[\frac{1}{1+\lambda G_{1}}\right]=\int_{0}^{\infty} d t e^{-t} E_{\alpha}\left(e^{-t \lambda G_{1}}\right)=E_{\alpha}\left[\exp \left(-\lambda T G_{1}\right)\right]=(1+\lambda)^{-\alpha} \tag{58}
\end{equation*}
$$

Take $\lambda=(1-x) / x$ in (58) to obtain

$$
\begin{equation*}
E_{\alpha}\left[\left(x+(1-x) G_{1}\right)^{-1}\right]=x^{\alpha-1} \quad(0<\alpha<1, x>0) . \tag{59}
\end{equation*}
$$

Integration of (59) with respect to $d x$ over $0<x<a$ yields the formula

$$
\begin{equation*}
E_{\alpha}\left[\frac{1}{1-G_{1}} \log \left(1+\frac{a\left(1-G_{1}\right)}{G_{1}}\right)\right]=\frac{a^{\alpha}}{\alpha} \tag{60}
\end{equation*}
$$

which reduces to (56) for $a=1$. For later reference, we note also the following elementary formula. For an arbitrary non-negative Borel $f$ :

$$
\begin{equation*}
E_{\alpha}\left[\frac{1}{1-G_{1}} f\left(\frac{1-G_{1}}{G_{1}}\right)\right]=\frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} \int_{0}^{\infty} \frac{d v}{v^{\alpha+1}} f(v) \tag{61}
\end{equation*}
$$

### 3.3 The lengths at times $G_{H_{1}}$ and $D_{H_{1}}$

In this section, we prove Proposition 6. We can assume that $Z$ is the zero set of $\rho:=(\rho(u), u \geq 0)$ where under $P_{\alpha}$ the process $\rho$ is a Bessel process of dimension $2-2 \alpha$ started at $\rho(0)=0$. Let $\pi$ denote the Bessel bridge of dimension $2-2 \alpha$ defined by $\pi_{u}:=\rho\left(u G_{1}\right) / \sqrt{G_{1}}, 0 \leq u \leq 1$ and let $\tilde{\rho}$ be the process defined by $\tilde{\rho}_{u}:=$ $\rho\left(u G_{H_{1}}\right) / \sqrt{G_{H_{1}}}, 0 \leq u \leq 1$.
Proof of (22). This formula is a consequence of (15) and the following absolute continuity relationship between the laws of $\pi$ and $\tilde{\rho}$ on $C[0,1]$ : for every measurable function $F: C[0,1] \rightarrow \mathbb{R}^{+}$

$$
\begin{equation*}
E_{\alpha}[F(\tilde{\rho})]=\gamma_{\alpha} E_{\alpha}\left[\left(V_{1}(\pi)\right)^{-\alpha} F(\pi)\right] \tag{62}
\end{equation*}
$$

where $V_{1}(\pi)$ denotes the longest excursion interval of the bridge $\pi$ and

$$
\begin{equation*}
\gamma_{\alpha}:=1 / E_{\alpha}\left[\left(V_{1}(\pi)\right)^{-\alpha}\right]=E_{\alpha}\left[\left(1-G_{1}\right)^{\alpha}\right]=\frac{1}{\alpha \Gamma(\alpha) \Gamma(1-\alpha)}=\frac{\sin (\pi \alpha)}{\pi \alpha} \tag{63}
\end{equation*}
$$

Formula (62) is a consequence of the following identity, which we obtain from Corollary 8 with the help of (46):

$$
\begin{equation*}
E_{\alpha}\left[F\left(\frac{\rho\left(u H_{1}\right)}{\sqrt{H_{1}}} ; 0 \leq u \leq 1\right)\right]=E_{\alpha}\left[\frac{1\left(1-G_{1}=V_{1}\right)}{1-G_{1}} F(\rho(u) ; 0 \leq u \leq 1)\right] \tag{64}
\end{equation*}
$$

To obtain (62) from (64), observe that $G_{H_{1}} / H_{1}$ is the last zero before time 1 of ( $\rho\left(u H_{1}\right) / \sqrt{H_{1}} ; 0 \leq u \leq 1$ ), and consequently

$$
\begin{equation*}
E_{\alpha}[F(\tilde{\rho})]=E_{\alpha}\left[\frac{1\left(1-G_{1}=V_{1}\right)}{1-G_{1}} F(\pi)\right] \tag{65}
\end{equation*}
$$

Formula (62) now appears as a consequence of

$$
\begin{equation*}
E_{\alpha}\left[\left.\frac{1\left(1-G_{1}=V_{1}\right)}{1-G_{1}} \right\rvert\, \pi\right]=\frac{\gamma_{\alpha}}{\left(V_{1}(\pi)\right)^{\alpha}} \tag{66}
\end{equation*}
$$

To check (66), evaluate the left side of (66) as

$$
E_{\alpha}\left[\left.\frac{1\left\{\left(1-G_{1}\right) / G_{1}>V_{1}(\pi)\right\}}{1-G_{1}} \right\rvert\, \pi\right]=h_{\alpha}\left(V_{1}(\pi)\right)
$$

where

$$
h_{\alpha}(v):=E_{\alpha}\left[\frac{1}{1-G_{1}} 1\left(\frac{1-G_{1}}{G_{1}}>v\right)\right]=\left(\alpha \Gamma(\alpha) \Gamma(1-\alpha) v^{\alpha}\right)^{-1}
$$

the last equality being a consequence of (61).
Proof of (23). For $t>0$ and $n=1,2, \ldots$ let $R_{n}(t):=V_{n+1}(t) / V_{n}(t)$. Since $H_{1}$ is admissible,

$$
\begin{equation*}
\left(R_{1}\left(H_{1}\right), R_{2}\left(H_{1}\right), \ldots\right) \stackrel{d}{=}\left(R_{1}(1), R_{2}(1), \ldots\right) . \tag{67}
\end{equation*}
$$

According to Proposition 8 of [16], the $R_{n}(1)$ are independent, and $R_{n}(1)$ has a $\operatorname{beta}(n \alpha, 1)$ distribution. Now

$$
\begin{equation*}
R_{1}\left(D_{H_{1}}\right)=\frac{V_{2}\left(H_{1}\right)}{D_{H_{1}}-G_{H_{1}}}=R_{1}\left(H_{1}\right)\left(D_{H_{1}}-G_{H_{1}}\right)^{-1} \tag{68}
\end{equation*}
$$

and $R_{m}\left(D_{H_{1}}\right)=R_{m}\left(H_{1}\right)$ for $m \geq 2$. Since $D_{H_{1}}-G_{H_{1}}$ is independent of the sequence $\left(V_{1}\left(H_{1}\right), V_{2}\left(H_{1}\right), \ldots\right)$, for a generic non-negative product measurable $f$, we obtain

$$
\begin{equation*}
E_{\alpha}\left[f\left(V_{1}\left(D_{H_{1}}\right), V_{2}\left(D_{H_{1}}\right), \ldots\right)\right]=E_{\alpha}\left[\xi_{\alpha}\left(R_{1}\left(H_{1}\right)\right) f\left(V_{1}\left(H_{1}\right), V_{2}\left(H_{1}\right), \ldots\right)\right] \tag{69}
\end{equation*}
$$

and hence from (67)

$$
\begin{equation*}
E_{\alpha}\left[f\left(\frac{V_{1}\left(D_{H_{1}}\right)}{D_{H_{1}}}, \frac{V_{2}\left(D_{H_{1}}\right)}{D_{H_{1}}}, \ldots\right)\right]=E_{\alpha}\left[\xi_{\alpha}\left(V_{2} / V_{1}\right) f\left(V_{1}, V_{2}, \ldots\right)\right] \tag{70}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi_{\alpha}(x):=\frac{P_{\alpha}\left(R_{1}\left(D_{H_{1}}\right) \in d x\right)}{P_{\alpha}\left(R_{1}(1) \in d x\right)}=-\alpha \log x \tag{71}
\end{equation*}
$$

The last equality follows by elementary computation from the fact that under $P_{\alpha}$ the distribution of $R_{1}(1)$ is beta $(\alpha, 1)$ while $P_{\alpha}\left(D_{H_{1}}-G_{H_{1}}>t\right)=t^{-\alpha}$ for $t>1$.

To conclude this section we note that there are analogs of the above formulae for $H_{n}$ instead of $H_{1}$. For example, formula (22) is modified by replacing $S_{1} V_{1}^{-\alpha}$ by $S_{1}\left(V_{n}^{-\alpha}-V_{n-1}^{-\alpha}\right)$, which is also exponentially distributed [17, Prop. 10], and formula (62) is modified by replacing $V_{1}^{-\alpha}$ by $V_{n}^{-\alpha}-V_{n-1}^{-\alpha}$.

## 4 Generalized arc-sine laws.

In this section, we assume that $0<\alpha<1,0<p<1$, and let $P_{\alpha, p}$ govern a real-valued process ( $B_{t}, t \geq 0$ ) with continuous paths, such that
(i) the zero set $Z$ of $B$ is the range of a stable ( $\alpha$ ) subordinator, and
(ii) given $|B|$, the signs of excursions of $B$ away from zero are chosen independently of each other to be positive with probability $p$ and negative with probability $q:=1-p$.

For example, $B$ could be any of the following:

- an ordinary Brownian motion ( $\alpha=p=\frac{1}{2}$ ) [11]
- a skew Brownian motion ( $\alpha=\frac{1}{2}, 0<p<1$ ) [21, 6, 2, 1]
- a symmetrized Bessel process of dimension $2-2 \alpha$ [10]
- a skew Bessel process of dimension $2-2 \alpha[2,23]$

For $t>0$ let

$$
\begin{equation*}
A_{t}:=\int_{0}^{t} 1\left(B_{s}>0\right) d s \tag{72}
\end{equation*}
$$

denote the time spent positive by $B$ up to time $t$. See the papers cited above for background and motivation for the study of this process. For any random time $T$ which is a measurable function of $|B|$,

$$
\begin{equation*}
A_{T}=\int_{0}^{T} 1\left(B_{s}>0\right) d s=\sum_{n} \varepsilon_{n}(T) V_{n}(T) \tag{73}
\end{equation*}
$$

where under $P_{\alpha, p}$ the $\varepsilon_{n}(T)$ are independent indicators of events with probability $p$, independent of the sequence of ranked lengths ( $V_{n}(T), n=1,2, \ldots$ ) of component intervals of $[0, T] \backslash Z$. Consequently, the $P_{\alpha, p}$ distribution of $A_{T} / T$ is the same for such $T$ that are admissible for the zero set of $B$, and this common distribution is the $P_{\alpha, p}$ distribution of $A:=A_{1}$. This is Lamperti's [9] generalized arc-sine distribution on $[0,1]$, determined by its Stieltjes transform

$$
\begin{equation*}
E_{\alpha, p}\left[\frac{1}{\lambda+A}\right]=\frac{p(1+\lambda)^{\alpha-1}+q \lambda^{\alpha-1}}{p(1+\lambda)^{\alpha}+q \lambda^{\alpha}} \quad(\lambda>0) \tag{74}
\end{equation*}
$$

Let $P_{\alpha, p}^{\text {br }}$ denote the standard bridge law obtained by conditioning $P_{\alpha, p}$ on $(1 \in Z)$. If $P_{\alpha, p}$ governs $B$ as a skew Bessel process, $P_{\alpha, p}^{\text {br }}$ governs $B$ as a skew Bessel bridge of length 1 . According to formula (4.b') of [2],

$$
\begin{equation*}
E_{\alpha, p}^{\mathrm{br}}\left[\frac{1}{(1+\lambda A)^{\alpha}}\right]=\frac{1}{p(1+\lambda)^{\alpha}+q} \quad(\lambda>0) \tag{75}
\end{equation*}
$$

Lamperti [9] inverted the Stieltjes transform (74) to obtain the corresponding density on $[0,1]$, which is reproduced in [15] and [23]. We do not know how to invert (75) to obtain such an explicit formula in the bridge case for general $\alpha$ with $0<\alpha<1$, but it is a famous result of Lévy [11] that for the standard Brownian bridge, with $\alpha=p=1 / 2$, the distribution of $A$ is simply uniform on $[0,1]$.

We note that the $P_{\alpha, p}$ distribution of $A$ is uniquely determined by formula (75), since by differentiating $k$ times we obtain for $k=1,2, \ldots$

$$
\begin{equation*}
E_{\alpha, p}^{\mathrm{br}}\left[\frac{\alpha(\alpha+1) \cdots(\alpha+k-1) A^{k}}{(1+\lambda A)^{\alpha+k}}\right]=(-1)^{k} \frac{d^{k}}{d \lambda^{k}}\left(\frac{1}{p(1+\lambda)^{\alpha}+q}\right) \quad(\lambda>0) \tag{76}
\end{equation*}
$$

so we recover the moments

$$
\begin{equation*}
E_{\alpha, p}^{\mathrm{br}}\left(A^{k}\right)=\left.\frac{(-1)^{k}}{\alpha(\alpha+1) \cdots(\alpha+k-1)} \frac{d^{k}}{d \lambda^{k}}\left(\frac{1}{p(1+\lambda)^{\alpha}+q}\right)\right|_{\lambda=0} \tag{77}
\end{equation*}
$$

In particular, from (74) and (77), for all $0<\alpha<1$ and $0<p<1$, we obtain the means

$$
\begin{equation*}
E_{\alpha, p}^{\mathrm{br}}(A)=E_{\alpha, p}(A)=p \tag{78}
\end{equation*}
$$

which is also obvious from (72) and $P_{\alpha, p}\left(B_{t}>0\right)=P_{\alpha, p}^{\mathrm{br}}\left(B_{t}>0\right)=p$ for all $0<t<1$, and the variances

$$
\begin{equation*}
\operatorname{Var}_{\alpha, p}^{\mathrm{br}}(A)=\frac{(1-\alpha) p q}{1+\alpha}<(1-\alpha) p q=\operatorname{Var}_{\alpha, p}(A) \tag{79}
\end{equation*}
$$

The inequality between the variances can be understood intuitively as follows. Conditioning to return to zero at time 1 tends to make the intervals smaller and more evenly distributed in length. So there is less variability in the fraction of time spent positive. For fixed $p$, as $\alpha$ increases from $0+$ to $1-$, both variances decrease, from the variance $p q$ of a $\operatorname{Bernoulli}(p)$ variable $\epsilon_{p}$ at $\alpha=0+$, down to variance 0 at $\alpha=1-$. Consequently, under either $P_{\alpha, p}$ or $P_{\alpha, p}^{\mathrm{br}}$

$$
A \xrightarrow{d} \begin{cases}p & \text { as } \alpha \uparrow 1  \tag{80}\\ \epsilon_{p} & \text { as } \alpha \downarrow 0\end{cases}
$$

where $\xrightarrow{d}$ denotes convergence in distribution. This behaviour can also be understood from the representation (73) and the observation that under either $P_{\alpha, p}$ or $P_{\alpha, p}^{\mathrm{br}}$

$$
V_{1}(1) \stackrel{d}{\rightarrow} \begin{cases}0 & \text { as } \alpha \uparrow 1  \tag{81}\\ 1 & \text { as } \alpha \downarrow 0\end{cases}
$$

See [16] for details and further references concerning the exact distribution of $V_{1}(1)$ under $P_{\alpha, p}$ and $P_{\alpha, p}^{\mathrm{br}}$.

Let $G:=G_{1}$ be the time of the last zero of $B$ before time 1. To conclude this section, we record the following proposition which describes the $P_{\alpha, p}$ distribution of $A_{G}$ by a surprisingly simple density relative to the $P_{\alpha, p}$ distribution of $A:=A_{1}$ discussed above. Combined with Lamperti's formula for the density of $A_{1}$, this yields an explicit formula for the density of $A_{G}$ relative to Lebesgue measure.
Proposition 15 For all $0<\alpha<1$ and $0<p<1$,

$$
\begin{equation*}
P_{\alpha, p}\left(A_{G} \in d x\right)=\frac{1-x}{1-p} P_{\alpha, p}\left(A_{1} \in d x\right) \quad(0<x<1) \tag{82}
\end{equation*}
$$

Proof. Write $E$ for $E_{\alpha, p}$. Then for all Borel measurable $f:[0,1] \rightarrow[0, \infty)$

$$
\begin{aligned}
(1-p) E\left[f\left(A_{G}\right)\right] & =E\left[f\left(A_{G}\right) 1_{\left(B_{1} \leq 0\right)}\right] \\
& =E\left[f\left(A_{1}\right) 1_{\left(B_{1} \leq 0\right)}\right] \\
& =E\left[f\left(A_{1}\right)\left(1-A_{1}\right)\right]
\end{aligned}
$$

where the first equality is due to the independence of $A_{G}$ and the event ( $B_{1}<0$ ), the second is obvious, and the third is deduced from the formula

$$
\begin{equation*}
P_{\alpha, p}\left(B_{1} \leq 0 \mid A_{1}\right)=1-A_{1} \tag{83}
\end{equation*}
$$

which, as noted in [15], is an easy consequence of the sampling property (9).

As a consequence of (82), the moments of $A_{G}$ can be expressed simply in terms of those of $A:=A_{1}$ which can be read from (74). Assume now for simplicity that $B$ is a skew Bessel process under $P_{\alpha, p}$. As noted in [2], we can write

$$
\begin{equation*}
A_{G}=G A^{\mathrm{br}} \tag{84}
\end{equation*}
$$

where $G$ has beta $(\alpha, 1-\alpha)$ distribution, and $A^{\mathrm{br}}$ is the fraction of time spent positive by the skew Bessel bridge of length 1 obtained by rescaling of $B$ on the random interval $[0, G]$. So the $P_{\alpha, p}$ distribution of $A^{\mathrm{br}}$ is identical to the $P_{\alpha, p}^{\mathrm{br}}$ distribution of $A:=A_{1}$ discussed before. In principle, (84) determines this distribution of $A^{\text {br }}$ in terms of the distributions of $G$ and $A_{G}$ just described. This gives an alternative formula to (77) for computing moments of $A^{\text {br }}$, hence some tricky algebraic identities, but unfortunately does not seem to yield any more explicit description of the law of $A^{\mathrm{br}}$.

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