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ON THE RELEVANCE OF THE *r*-MODE INSTABILITY FOR ACCRETING NEUTRON STARS AND WHITE DWARFS

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ABSTRACT

We present a case study for the relevance of the *r*-mode instability for accreting compact stars. Our estimates are based on approximations that facilitate "back of the envelope" calculations. We discuss two different cases. (1) For recycled millisecond pulsars, we argue that the *r*-mode instability may be active at rotation periods longer than the Kepler period (which provides the dynamical limit on rotation) as long as the core temperature is larger than about 2×10^5 K. Our estimates suggest that the instability may have played a role in the evolution of the fastest spinning pulsars and that it may be presently active in the recently discovered 2.49 ms X-ray pulsar, SAX J1808.4-3658, as well as the rapidly spinning neutron stars observed in low-mass X-ray binaries (LMXBs). This provides a new explanation for the remarkably similar rotation periods inferred from kilohertz, quasi-periodic oscillations in the LMXBs. The possibility that the rotation of recycled pulsars may be gravitational-radiation–limited is interesting, because the gravitational waves from a neutron star rotating at the instability limit may well be detectable with the new generation of interferometric detectors. (2) We also consider white dwarfs and find that the *r*-mode instability may possibly be active in short-period white dwarfs. Our order-of-magnitude estimates (for a white dwarf of $M = M_\odot$ and $R = 0.01\ R_\odot$ composed of C^{12}) show that the instability could be operating for rotational periods shorter than $P \approx 27$–33 s. This number is in interesting agreement with the observed periods (greater than 28 s) of the rapidly spinning DQ Herculis stars. However, we find that the instability grows too slowly to affect the rotation of these stars significantly.
Subject headings: accretion, accretion disks — pulsars: general — stars: neutron — stars: oscillations — white dwarfs

1. INTRODUCTION

The recent discovery of a new instability in rotating, relativistic stars has revived interest in problems concerning mechanisms that may limit the rotation rate of neutron stars. As was first shown by one of us (Andersson 1998), the so-called *r*-modes (Papaloizou & Pringle 1978) are formally unstable at all rates of rotation in a perfect-fluid neutron star (see also Friedman & Morsink 1998). The mechanism behind the instability is the familiar one that was discovered by Chandrasekhar (1970) and Friedman & Schutz (1978): the modes are unstable because of the emission of gravitational waves (see Stergioulas 1998 for a review on non-axisymmetric instabilities in rotating, relativistic stars). Even more surprising than the actual existence of the instability are the results of recent estimates of the timescale at which the *r*-mode instability will grow (Lindblom, Owen, & Morsink 1998; Andersson, Kokkotas, & Schutz 1999). It seems that the *r*-mode instability has the potential to slow down newly born neutron stars dramatically, and the *r*-mode instability is considerably stronger than the previously considered one for the *f*-mode of the star (see the results of Lindblom 1995; Stergioulas & Friedman 1998). The new results suggest that, assuming that a neutron star is born spinning as fast as it possibly can (at the Kepler frequency $\Omega_K \approx 0.67\sqrt{\pi G \bar{\rho}}$, i.e., with a period $P_K \approx 0.5$–2 ms), the *r*-mode instability will force the star to spin down to a period of roughly 15 ms in the first year or so. This prediction is in good agreement with the inferred initial period for the Crab pulsar (19 ms). The uncertainties in the present models may also accommodate the newly discovered 16 ms pulsar in N157B (Marshall et al. 1998). The birth-spin period for this rapidly spinning pulsar is less certain, but one can estimate that it ought to have been shorter than 9 ms (this follows by assuming the braking index to be in the range observed for young pulsars together with the youngest likely age of 5000 yr). Anyway, if the *r*-mode instability is active in a newly born neutron star, it will force the star to spin down. The large fraction of the initial rotational energy that is then radiated away as gravitational waves may well be detectable with, for example, LIGO (Owen et al. 1998).

These are undoubtedly exciting suggestions, which indicate that the *r*-mode instability plays an important role in astrophysics. Given the promise of the suggested scenario for newly born neutron stars, we are inspired to investigate whether the instability can be relevant also in other situations. In this paper we assess the potential relevance of the instability for accreting stars, both neutron stars and white dwarfs. Specifically, we provide rough estimates of the timescales involved and ask whether the *r*-mode instability provides an upper limit on the rotation of stars spun up by accretion. We compare our estimates with observed data for the recycled millisecond pulsars (MSPs) and the rapidly spinning neutron stars in low-mass X-ray binaries (LMXBs), as well as the DQ Herculis white dwarfs.

2. "BACK OF THE ENVELOPE" TIMESCALES

At the present time there are no fully relativistic studies of the dynamics of perturbed rotating stars (although the neutral modes that signal the onset of instability of the f-mode have been calculated; see Stergioulas & Friedman 1998 for a fully relativistic calculation and Yoshida & Eriguchi 1997 for a calculation in the relativistic Cowling approximation). Until such calculations become available, we will not know the detailed effects of a gravitational-wave instability. However, we can obtain useful estimates that should be qualitatively correct and may not differ too much from the quantitative truth in a straightforward way (Lindblom et al. 1998; Andersson et al. 1999). In this paper we will use the estimates obtained by Kokkotas & Stergioulas (1999) in the context of uniform-density stars to assess the relevance of the r-mode in various situations. We prefer to work within this approximation, since the way that the various results scale with the parameters of the star (mass and radius) is then clear.

The uniform density approximation leads to the gravitational radiation growth time,

$$t_{\rm gw} \approx 22\left(\frac{1.4\ M_\odot}{M}\right)\left(\frac{10\ {\rm km}}{R}\right)^4\left(\frac{P}{1\ {\rm ms}}\right)^6\ {\rm s}\,, \tag{1}$$

for the $l = m = 2$ r-mode (which leads to the strongest instability). M, R, and P are the mass, radius, and period of the star, respectively. This estimate of $t_{\rm gw}$ is roughly a factor of 2 smaller than the results for the $N = 1.0$ polytropic equation of state and the specific stellar parameters ($M = 1.5\ M_\odot$ and $R \approx 12.5$ km) considered by both Lindblom et al. (1998) and Andersson et al. (1999). In a way, this discrepancy illustrates the uncertainties associated with the present studies of the r-mode instability. Furthermore, since models of realistic neutron stars typically have a mean polytropic index in the range 0.5–0.8, the difference of a factor of 2 or so (between polytropes of index 0 and 1) in the gravitational radiation timescale can be seen to represent the uncertainty in the real equation of state of neutron star matter. Also, as we will argue below, the difference of a factor of 2 has little effect on the conclusions of the present discussion. Consequently, we will work with the uniform density approximation (eq. [1]) in the remainder of this paper.

The damping timescale due to shear viscosity in a normal fluid star is

$$t_{\rm sv} \approx 1.2 \times 10^4\left(\frac{1.4\ M_\odot}{M}\right)^{5/4}\left(\frac{R}{10\ {\rm km}}\right)^{23/4}\left(\frac{T}{10^7\ {\rm K}}\right)^2\ {\rm s}\,, \tag{2}$$

where T is the temperature of the star, and we have used the shear viscosity coefficient

$$\eta_{\rm nf} = 2 \times 10^{22}\left(\frac{\rho}{10^{15}\ {\rm g\ cm^{-3}}}\right)^{9/4}\left(\frac{T}{10^7\ {\rm K}}\right)^{-2}\ {\rm g\ cm^{-1}\ s^{-1}}\,. \tag{3}$$

This assumes that the main contribution to the shear viscosity arises from neutron-neutron scattering, which should be a valid approximation at temperatures above which the star becomes superfluid (Flowers & Itoh 1979). In a similar way to the gravitational-wave timescale, the normal fluid-shear viscosity timescale (eq. [2]) has a factor of 2 uncertainty.

When the outer core of the star has become superfluid, the main contribution to the viscous dissipation is due to electron-electron scattering. Then the relevant viscosity coefficient and timescale are

$$\eta_{\rm sf} = 6 \times 10^{22}\left(\frac{\rho}{10^{15}\ {\rm g\ cm^{-3}}}\right)^{2}\left(\frac{T}{10^7\ {\rm K}}\right)^{-2}\ {\rm g\ cm^{-1}\ s^{-1}}\,, \tag{4}$$

$$t_{\rm sv} \approx 3.6 \times 10^3\left(\frac{1.4\ M_\odot}{M}\right)\left(\frac{R}{10\ {\rm km}}\right)^{5}\left(\frac{T}{10^7\ {\rm K}}\right)^2\ {\rm s}\,. \tag{5}$$

For a superfluid star we should also be concerned with the so-called mutual friction. Specifically, it has been suggested that mutual friction will completely suppress the instability of the f-mode (Lindblom & Mendell 1995). At present there is no similar calculation for the r-mode instability, but it would not be surprising if mutual friction also has a strong effect on the r-modes. However, Mendell (1991) has suggested that mutual friction is dominated by the shear viscosity (as given by eq. [4]) at temperatures below about 10^7 K. This would indicate that the effect may not be dominant in the oldest stars. Given that results for the effect of mutual friction on the unstable r-modes are outstanding, we will not include this effect in the present discussion. For superfluid stars we will only account for the shear viscosity due to electron-electron scattering. This means that our results may overestimate the strength of the instability in the superfluid case.

On the other hand, one would not expect the entire star to become superfluid: a cold neutron star will have an extended solid crust. This may be highly relevant for the r-mode estimates: the mode used in the calculation of the estimated timescales $t_{\rm gw}$ and $t_{\rm sv}$ is in many ways similar to the f-mode of the star (albeit with a toroidal rather than spheroidal angular dependence). Specifically, both modes are well described by eigenfunctions that grow as r^l toward the surface (in this sense they are "global" modes). For this reason, we expect them to be affected in a similar way by the presence of a crust and, as was shown by McDermott, Van Horn, & Hansen (1988), the crust has hardly any effect on the frequency of the f-mode (basically since the mode propagates at the speed of sound, which is generally much higher than the speed of a shear wave in the crust). We expect that this is also likely to hold for the r-mode under consideration, and therefore the above timescales may in fact also be reasonable for a star with a superfluid core and a solid crust.

As long as we are mainly interested in relatively low temperatures and other, more exotic dissipation mechanisms do not play a role, the effect of the r-mode instability can be inferred from $t_{\rm gw}$ and $t_{\rm sv}$. At high temperatures (above 10^9 K) one expects the bulk viscosity of the fluid to play a significant role; see the estimates of Lindblom et al. (1998) and Andersson et al. (1999). However, in accreting systems the temperature should always be low enough that it is sufficient to include only $t_{\rm gw}$ and $t_{\rm sv}$. When this is the case, one can easily deduce the critical rotation period above which the r-mode instability affects the rotation of the star (when the gravitational-wave growth time is shorter than the viscosity damping time). We thus find that the mode grows if the period is shorter than

$$P_c \approx 2.8\left(\frac{R}{10\ \mathrm{km}}\right)^{39/24}\left(\frac{1.4\ M_\odot}{M}\right)^{1/24}\left(\frac{T}{10^7\ \mathrm{K}}\right)^{1/3}\ \mathrm{ms} \quad (6)$$

for a normal fluid star, and

$$P_c \approx 2.3\left(\frac{R}{10\ \mathrm{km}}\right)^{3/2}\left(\frac{T}{10^7\ \mathrm{K}}\right)^{1/3}\ \mathrm{ms} \quad (7)$$

when we use the viscosity due to electron-electron scattering in a superfluid. Interestingly, these critical periods are not strongly dependent on the mass of the star. Furthermore, the uncertainties in equations (1), (2), and (5) have little effect on the critical period. For example, the uncertain factors of 2 in t_{gw} and t_{sv} individually lead to an uncertainty of 12% in P_c. When combined, the uncertainties suggest that we may be (over)estimating the critical period at the 25% level. Considering uncertainties associated with the various realistic equations of state for supranuclear matter and the many approximations on which our present understanding of the *r*-mode instability is based, we feel that it is acceptable to work at this level of accuracy.

3. IMPLICATIONS FOR MSPs

We will now discuss the possibility that the *r*-mode instability may be relevant for the period evolution of the fastest observed pulsars. All observed MSPs have periods larger than the 1.56 ms of PSR 1937+21, and it is relevant to ask whether there is a mechanism that prevents a neutron star from being spun up farther (e.g., to the Kepler limit) by accretion. Specifically, we are interested in the possibility that the *r*-mode instability plays such a role. Before proceeding with our discussion, we recall that Andersson et al. (1999) have already pointed out that the instability has implications for the formation of MSPs (albeit in an indirect way). Specifically, the strength of the *r*-mode instability seems to rule out the scenario in which MSPs (with $P < 5$–10 ms) are formed as an immediate result of accretion-induced collapse of white dwarfs. Continued accretion would be needed to reach the shortest observed periods. In other words, all MSPs with periods shorter than (say) 5–10 ms should be recycled.

Our main question here is whether it is realistic to expect the instability to be relevant also for older (and in consequence much colder) neutron stars. Even though the critical period is much shorter for a cold star, our estimates (eqs. [6] and [7]) are still above the Kepler period (≈ 0.8 ms for our canonical star), which suggests that the instability could be relevant. As an attempt to answer the question, we will confront our rough approximations with observed data for MSPs and the neutron stars in LMXBs.

3.1. *The MSPs*

In this section we discuss the *r*-mode instability in the context of the recycled MSPs. These stars are no longer accreting, and supposing that they have been cooling for some time, they should not be affected by the instability at present. Our main question is whether the observed data is in conflict with a picture in which the *r*-mode instability halted accretion-driven spin-up at some point in the past.

Our estimates show that the rotation will be limited by the Kepler frequency (using $P_{\mathrm{K}} \approx 0.8$ ms for a canonical star) if the interior of the star is colder than $T \approx 2 \times 10^5$ K. Also, it is straightforward to show that in order to "rule out" the instability (to lead to a critical period equal to the Kepler period at, e.g., temperature 4×10^8 K), the dissipation coefficient of the shear viscosity (or any other dissipation mechanism) must be almost 6 orders of magnitude stronger than equation (4).

Our inferred critical periods (eqs. [6] and [7], for a canonical neutron star) are illustrated and compared with observed periods and upper limits on the surface temperatures (from *ROSAT* observations; see data given by Reisenegger 1997) for the fastest MSPs in Figure 1. In the figure we also indicate the associated upper limits on the core temperatures as estimated using equation (8) of Gudmundsson, Pethick, & Epstein (1982).

The illustrated *r*-mode instability estimates would be in conflict with the MSP observations if the interior temperature of a certain star were such that it was placed considerably below the critical period for the relevant temperature. Basically, an accreting star whose spin is limited by the *r*-mode instability would not be able to spin up far beyond the critical period, since the instability would radiate away any excess accreted angular momentum. As the accretion phase ends, the star will both cool down and spin down (the timescales for these two processes, photon cooling and magnetic dipole braking, are such that an MSP would evolve almost horizontally toward the left in Fig. 1).

Given the uncertainties in the available data, we do not think the possibility that the *r*-mode instability may have played a role in the period evolution of the fastest MSPs can be ruled out. First of all, it must be remembered that the *ROSAT* data only provide upper limits on the surface temperature, and the true temperature may well be considerably lower than this. If the true core temperatures of the fastest spinning pulsars were roughly 1 order of magnitude

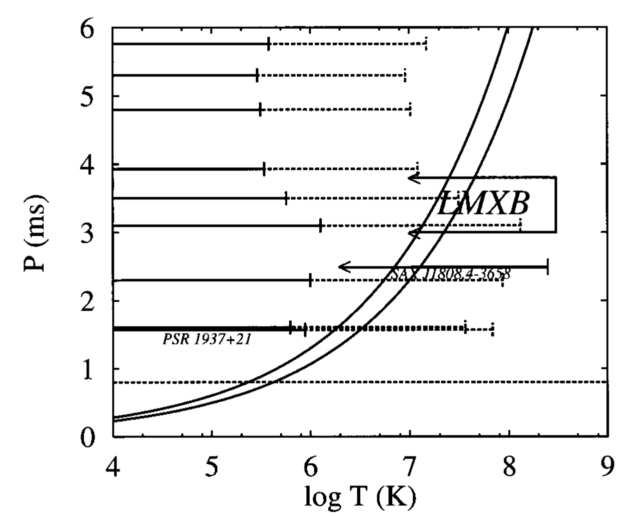

FIG. 1.—Inferred critical period for the *r*-mode instability at temperatures relevant to older neutron stars (*solid lines*). The upper line is for a normal fluid star, while the lower one is for a superfluid star (only taking electron-electron scattering into account; see text for discussion). The data is for a neutron star with $M = 1.4\ M_\odot$ and $R = 10$ km. The Kepler limit, which corresponds to $P \approx 0.8$ ms for our canonical star, is shown as a horizontal dashed line. We compare our theoretical result with (1) the observed periods and temperatures of the most rapidly spinning MSPs (see Reisenegger 1997 for the data): the surface temperatures are indicated as solid vertical lines: the dashed continuation of each line indicates the estimated core temperature; (2) the observed/inferred periods and temperatures for accreting neutron stars in LMXBs; and (3) the recently discovered 2.49 ms X-ray pulsar SAX J1808.4-3658.

lower than the ones indicated in Figure 1, our inferred lower limit on the spin period would in fact be in quite good agreement with observations. Second, one must also recall the many simplifying assumptions that went into our estimates of the critical period. (Recall that we have already suggested that the uncertainties in the critical period may be at the level of 25%.) The most crucial of these assumptions regards the detailed role of superfluidity. It is not at all unlikely that superfluid mutual friction, or some other exotic dissipation mechanism, serves to suppress the instability and decrease the critical period. In conclusion, Figure 1 is suggestive. It seems plausible that the *r*-mode instability is relevant for neutron stars spun up to millisecond periods. Furthermore, the figure indicates how improved temperature observations should be able to test present and future (more detailed) models for the *r*-mode instability.

3.2. *Neutron Stars in LMXBs*

In the standard scenario it is assumed that MSPs have evolved from LMXBs, in that they spin up to the present rotation rate by accreting matter from their low-mass (less than 0.4 $M_\odot$) companions. If the accretion rate is close to the Eddington limit ($10^{-8}\ M_\odot\ \mathrm{yr}^{-1}$), it would typically take 10^7 yr to spin a neutron star up to a period of a few milliseconds. Recent observations have provided interesting support for this model. First of all, the observed kilohertz quasi-periodic oscillations (QPOs) detected by the *Rossi X-Ray Timing Explorer* (see van der Klis 1996 for a review) show that the neutron stars in LMXBs are spinning rapidly. Secondly, the recent discovery of the 2.49 ms X-ray pulsar, SAX J1808.4-3658 (Wijnands & van der Klis 1998), provides the first evidence of an evolutionary link between LMXBs and MSPs.

The observations of kilohertz QPOs from systems containing rotating neutron stars provide a wealth of interesting information. Of specific relevance for our present discussion is the suggestion that many neutron stars in LMXBs are spinning with almost identical periods. This suggestion follows from the fact that the separation between the two kilohertz QPOs that are commonly observed stays constant, even though the individual peak frequencies vary. The proposed explanation—the beat-frequency model (see van der Klis 1996 for references)—is that this robust frequency is the underlying rotation frequency (or a multiple thereof). Assuming that this is the case, one finds that the observed QPOs indicate a typical rotation frequency in the range 260–330 Hz (P in range 3–3.8 ms). (Interestingly, Backer 1998 recently suggested that MSPs are typically born with a period of 3 ms after spin-up.) There is, however, one major caveat to this interpretation of the QPOs: for Scorpius X-1 the peak separation varies considerably with the luminosity of the system. It has, in fact, been argued that the available data is consistent with such a variation in all LMXBs (Psaltis et al. 1998). If this is the case, the peak separation cannot correspond to the rotation frequency (since the rotation of Sco X-1 would then have to vary between 3.2 and 4.4 ms on short timescales), but even so it seems reasonable that the observed beat frequency is close (but not equal) to the rotation frequency. Hence we believe it makes sense to assume that the neutron stars in the LMXBs are actually spinning at almost identical periods and try to understand why this is the case.

Two explanations for this phenomenon have been proposed. In the first model (White & Zhang 1997), the star reaches an equilibrium period when the spin-up torque due to accretion is balanced by the spin-down torque due to the magnetic field. For a magnetic field dominated by the dipole the standard equilibrium period follows from

$$P_{\mathrm{eq}} = 0.86\left(\frac{B}{4\times 10^8\ \mathrm{G}}\right)^{6/7}\left(\frac{M}{1.4\ M_\odot}\right)^{-5/7} \times \left(\frac{10^{-8}\ M_\odot\ \mathrm{yr}^{-1}}{\dot{M}}\right)^{3/7}\left(\frac{R}{10\ \mathrm{km}}\right)^{16/7}\ \mathrm{ms}\ . \quad (8)$$

Within this model one can use the observed periods and luminosities (which imply the accretion rates) and deduce a maximum magnetic field for each star. This results (see Table 1 of White & Zhang 1997) in magnetic fields very similar to those found for MSPs. This is suggestive, but the magnetic fields deduced for the LMXBs will only compare favorably with the observed values for the MSPs if the field does not decay as the neutron star evolves. However, this is in accord with the favored model at present: the magnetic field is only expected to decay considerably during the accretion phase (Romani 1990; Urpin, Geppert, & Konenkov 1998). Hence the values that White & Zhang (1997) infer for B are reasonable as long as the low-field systems are close to the end of the accretion phase (which could well be the case, since they are the ones with the lowest X-ray luminosity).

However, the model of White & Zhang only leads to similar equilibrium periods if there is some intrinsic association between $\dot{M}$ and B^2 (see eq. [8]), and, as was pointed out by Bildsten (1998), no such relation is seen in the data for the (slower rotating) highly magnetized X-ray pulsars. In view of this, Bildsten proposed an alternative explanation for the similar rotation periods in the LMXBs. He argued that the rotation may be limited by gravitational radiation in these systems. Specifically, he suggested that temperature gradients in the star lead to a time-dependent quadrupole moment, i.e., emission of gravitational waves that balance the spin-up torque due to continued accretion. This is an interesting suggestion, especially since the resultant gravitational waves may well be detectable.

It is interesting to compare the data for the LMXBs with our inferred critical period for the *r*-mode instability. In order to do this, we need to estimate the interior temperature of the neutron stars in LMXBs. For a star that is accreting close to the Eddington limit, hydrogen (and possibly helium) burning of the accreted material can heat the core of the star up to a temperature considerably higher than the typical one for nonaccreting neutron stars (Fujimoto et al. 1984; Brown & Bildsten 1998). The suggested core temperature for a rapidly accreting neutron star is in the range $1\text{–}3\times 10^8$ K. We use this estimate as an upper limit and compare the resultant data with our critical periods in Figure 1. The figure suggests that the *r*-mode instability provides a possible explanation for the inferred periods in the LMXBs.

Regarding the comparison between our critical periods and the LMXB data in Figure 1, it is worthwhile making a few remarks. First of all, it is not impossible that the interior temperature of the LMXB neutron stars is actually lower than the $1\text{–}3\times 10^8$ K we used as our upper limit. As was shown by, for example, Fujimoto et al. (1984), a pion condensate in the core of the star can act as a heat sink and keep the bulk of the star at 10^7 K or so, even though the outer layers of the star are heated up further by hydrogen burning. It is relevant to point out that our critical period would be in perfect agreement with such a model. Second, it is worth pointing out that the shear viscosity must be 200 times stronger than our superfluid value (eq. [4]) in order for the r-mode instability not to be active in these systems. If this were the case, we would have $P_c \approx 3$ ms at $T \approx 3 \times 10^8$, and the r-modes would be stable in all indicated LMXBs. Finally, it should be noted that the r-mode explanation is in qualitative agreement with a possible trend seen in the LMXB: the neutron star can only spin up further as the accretion slows down. In our picture (and also in Bildsten's explanation), the star must accrete more slowly and cool down in order to be able to spin up. This means that the evolution toward periods shorter than 3 ms should take place on a cooling timescale. In contrast, in the model of White & Zhang (1997) the star would spin up on the timescale that the magnetic field decays. In reality, these two timescales are, of course, likely to be linked in some way.

Having suggested an alternative explanation for the clustering of the observed rotation periods in LMXBs (that the r-mode instability prevents the neutron stars from spinning up once they reach the critical period), one would perhaps expect us to attempt to rule out the other two models. However, given the uncertainties in all the suggested scenarios, we do not believe that it is meaningful to rule out either possibility at the present time. To be able to do so, we need more detailed modeling. In fact, we can see no reason why several of these mechanisms could not be operating at the same time or be relevant in different individual systems. As one can immediately deduce from equations (7) and (8), the final outcome depends on a delicate balance. For example, in order for the r-mode instability to dominate over the magnetic dipole, we need $P_{eq} < P_c$. Using equations (7) and (8) we can translate this into a relation between the temperature, the accretion rate, and the magnetic field strength. From this relation we find (for our canonical star): (1) for accretion at the Eddington rate and a temperature of 10^8 K, the r-mode instability dominates for $B < 3 \times 10^9$ G, i.e., for most of the observed systems; and (2) for a lower accretion rate, 1% of the Eddington limit, at 10^7 K we find that the r-modes would be dominant for $B < 1.3 \times 10^8$ G, i.e., only a few of the observed systems. This simple example shows that the two mechanisms are active at roughly the same level and that neither of them should be ruled out at the present time.

3.3. *Gravitational-Wave Estimates*

As we have seen, the rotation of the fastest observed neutron stars in accreting systems may be limited by the r-mode instability. Hence it is meaningful to estimate the amplitude of the gravitational waves that could be emitted from these systems. As was first pointed out by Wagoner (1984), a gravitational-wave instability in an accreting star will lead to periodic gravitational waves (assuming that the system reaches a steady state where the excess angular momentum of the accreted matter is radiated as gravitational waves). Following Wagoner (1984), we can deduce the gravitational-wave amplitude by using the angular momentum gained by accretion:

$$\dot{J} = \sqrt{GMR}\dot{M} = -\frac{m}{\omega}\dot{E} \tag{9}$$

(where ω is the angular frequency of the mode), in the standard flux formula

$$h^2 = \frac{4G}{c^3}\left(\frac{1}{\omega r}\right)^2 |\dot{E}| , \tag{10}$$

where r is the distance to the source, and h is the (dimensionless) gravitational-wave amplitude. Using these relations together with our r-mode estimates, we find

$$h \approx 2.3 \times 10^{-26}\left(\frac{P_c}{1\ \mathrm{ms}}\right)^{1/2}\left(\frac{M}{1.4\ M_\odot}\right)^{1/4}\left(\frac{R}{10\ \mathrm{km}}\right)^{1/4} \times \left(\frac{\dot{M}}{10^{-8}\ M_\odot\ \mathrm{yr}^{-1}}\right)^{1/2}\left(\frac{1\ \mathrm{kpc}}{r}\right). \tag{11}$$

To illustrate what this means, let us provide a specific example. Since it would provide one of the strongest sources, we consider the particular case of Sco X-1 (see Bildsten 1998). Assume that the true rotation period of Sco X-1 is 4 ms and that the interior temperature is such that the r-mode instability is active and provides a limit on the spin of the star (this corresponds to assuming that $T \approx 10^8$ K; see Fig. 1). Given this, the relevant data for Sco X-1 ($r = 0.7$ kpc and an average accretion rate of $\dot{M} \approx 3 \times 10^{-9}\ M_\odot\ \mathrm{yr}^{-1}$; Lang 1992), we find that equation (11)implies $h \approx 3.5 \times 10^{-26}$ for a star with canonical mass and radius. The effective amplitude achievable after matched filtering improves as the square root of the number of detected cycles, so we get $h_{\mathrm{eff}} = \sqrt{\omega t_{\mathrm{obs}}/2\pi}h \sim 10^{-21}$ after about 2 weeks of integration. After 1 yr of integration, a similar source at the center of our Galaxy (at 10 kpc) would reach the same effective amplitude. In other words, the gravitational waves from Galactic neutron stars whose rotation is limited by the r-mode instability should be observable by, for example, LIGO. The detection of this kind of essentially continuous signals is obviously more complicated than we have indicated here, but since the data-analysis issues are identical to those for gravitational waves from a slightly deformed rotating neutron star, we refer the interested reader to detailed studies of that problem (Jaranowski, Krolak, & Schutz 1998; Brady et al. 1998).

Interestingly, given a detection of gravitational waves from the LMXBs one would easily be able to distinguish between the r-mode instability and radiation from the time-dependent quadrupole model suggested by Bildsten (1998). The r-modes radiate at $\frac{4}{3}$ of the rotation frequency, while a rotating quadrupole should generate radiation of (mainly) twice the rotation frequency. Supposing that the rotation frequency is known from a QPO, one should have no trouble distinguishing between the two mechanisms.

Finally, we should also mention that there may exist systems that are radiating gravitational waves according to the prescribed scenario but are otherwise invisible. As was suggested by Schutz (1997), the accretion rate may be close to the Eddington limit if the neutron star is part of a so-called Thorne-Zytkow object, i.e., when it spirals inside a giant star at the late stages of binary evolution. Such systems should be similar to the example we discussed above and would likely be detectable through the radiated gravitational waves.

4. IMPLICATIONS FOR WHITE DWARFS

Although it is clear that the r-modes would also be formally unstable in less compact, perfect-fluid stars, it is not at all obvious that the timescales will work out in such a way that the instability has any relevance. Hence it is interesting to attempt to extend our estimates to white dwarfs. To achieve a rough estimate for the critical period is rather straightforward. We first rescale our expression for the growth timescale due to gravitational waves (eq. [1]) to canonical values for a white dwarf. This leads to

$$t_{\rm gw} \approx 3.1 \times 10^9 \left(\frac{M_\odot}{M}\right)\left(\frac{0.01\ R_\odot}{R}\right)^4 \left(\frac{P}{30\ {\rm s}}\right)^6 . \quad (12)$$

To estimate the role of the viscosity, we need to replace equation (3) with an expression that is relevant at white dwarf densities. As a rough approximation, we will use (for a star composed of C^{12})

$$\eta = 9.7 \times 10^5 \left(\frac{\rho}{10^6\ {\rm g\ cm^{-3}}}\right)^{7/4} \left(\frac{10^5\ {\rm K}}{T}\right)^{1/4} {\rm g\ cm^{-1}\ s^{-1}} . \quad (13)$$

This expression is deduced from data used by Durisen (1973) and estimates the electron contribution to the shear viscosity at densities $10^4 \leq \rho({\rm g\ cm^{-3}}) \leq 10^6$. We have used the results for nonrelativistic electrons, since they lead to a larger value of η, in order not to underestimate the role of the viscosity. We have compared our approximation with more recent (and detailed) results of Itoh, Kohyama, & Takeuchi (1987). The comparison suggests that our expression (eq. [13]) is a slight overestimate of the strength of the electron viscosity (by a factor of less than 5 in the temperature/density range of interest). Furthermore, we have checked that the electron contribution dominates that of the ions in the temperature/density range of interest. This is true as long as the density is above 10^4 g cm^{-3} or so. At lower densities, the results of Itoh et al. (1987) indicate that the ion contribution will dominate over the electron contribution. Then we expect equation (13) to be an underestimate by (again) less than a factor of 5. In conclusion, we think that estimates obtained using equation (13) are reasonable, but the potential erring factors of 5 should be kept in mind. Such errors would affect the critical rotation period at the 25% level. Since the main objective of the present estimates is to assess whether a more detailed study of the role of the r-mode instability in white dwarfs is warranted, we believe it is reasonable to work at this level of accuracy.

Using equation (13), we find

$$t_{\rm sv} \approx 1.7 \times 10^9 \left(\frac{M_\odot}{M}\right)^{3/4} \left(\frac{R}{0.01\ R_\odot}\right)^{17/4} \left(\frac{T}{10^5\ {\rm K}}\right)^{1/4} {\rm yr} , \quad (14)$$

and a critical period of

$$P_c \approx 27 \left(\frac{M}{M_\odot}\right)^{1/24} \left(\frac{R}{0.01\ R_\odot}\right)^{11/8} \left(\frac{T}{10^5\ {\rm K}}\right)^{1/24} {\rm s} . \quad (15)$$

As in the case of neutron stars, the critical period is not strongly dependent on the mass. Furthermore, in this case it is also weakly dependent on the temperature. The question is: does the estimated critical period indicate that the r-mode instability is relevant also for white dwarfs?

First of all, we should compare the deduced critical period with the mass-shedding limit provided by the Kepler frequency. The value of $P_c \approx 27$ s for a canonical white dwarf (at $T = 10^5$ K) should then be compared with the Kepler period for a white dwarf: $P_{\rm K} \approx 17$ s (that follows from $\Omega_{\rm K} = 0.67\sqrt{\pi G \bar{\rho}}$, but if the star is differentially rotating at the relevant temperature, the limiting period can be much smaller than this value). The fact that $P_c > P_{\rm K}$ indicates that the r-mode instability should become active before a white dwarf that is spinning up reaches the mass-shedding limit.

Second, a comparison of our result with the available observations of rapidly spinning white dwarfs is suggestive. There are no observations of white dwarfs rotating faster than the limit set by equation (15). But the fastest spinning white dwarfs, the so-called DQ Her stars (which are magnetized, accreting cataclysmic variables; Patterson 1994), have periods quite close to P_c. The shortest definite rotation period that has been observed is the 33 s of AE Aquarii. A likely candidate for slightly faster rotation is WZ Sagittae, with a period of 28 s. The estimated surface temperature of these stars is 15,000 K, and an upper limit (above which the color of the stars would change considerably) is 50,000 K (J. Patterson 1998, private communication). Just as in the case of neutron stars, the interior temperature (which should be used in the r-mode estimates) is likely to be considerably higher (maybe 2 orders of magnitude) than the surface value. However, since the critical period P_c is very weakly dependent on T, an increase of the temperature by 2 orders of magnitude only changes the critical period by 20% (at $T = 10^7$ K we get $P_c \approx 33$ s). In Figure 2 we compare our estimated critical period with the observations.

Our estimates suggest that the fastest spinning white dwarfs that have been observed are close to the limit where the r-modes would be unstable. This is an interesting result, but it is clear from equation (12) that the r-modes would grow very slowly in these stars. To see whether we should

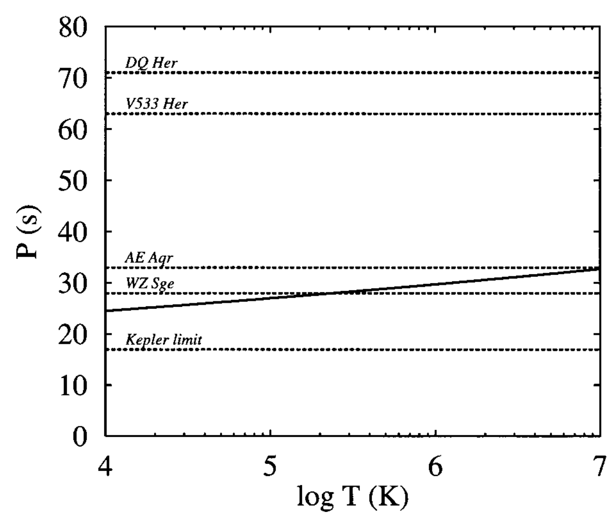

FIG. 2.—Inferred critical period for the r-mode instability at temperatures relevant for white dwarfs (*thick solid line*). The data is for a (carbon-based) white dwarf with $M = M_\odot$ and $R = 0.01\ R_\odot$. The theoretical result is compared with the Kepler limit (*thick dashed line*), which corresponds to $P \approx 17$ s, and the observed periods of the most rapidly spinning DQ Her stars (*dashed lines*; see Patterson 1994). In order from below, these stars are WZ Sge, AE Aqr, V533 Her, and DQ Her. The likely surface temperature of the fastest rotators is 15,000 K, and the core temperature (which is the relevant one for the r-mode estimates) is likely to be something like 2 orders of magnitude higher.

expect the instability to be able to prevent the star from spinning up, we compare the growth timescale with the estimated spin-up time for a white dwarf accreting at the rates observed in DQ Her. The spin-up timescale can be approximated by

$$t_{su} \approx \frac{J}{\dot{J}} \approx \frac{2MR^2\Omega}{5\sqrt{GMR}} \frac{1}{\dot{M}}, \tag{16}$$

which leads to

$$t_{su} \approx 1.3 \times 10^9 \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{0.01\ R_\odot}\right)^{3/2} \left(\frac{30\ \mathrm{s}}{P}\right) \times \left(\frac{10^{-10}\ M_\odot\ \mathrm{yr}^{-1}}{\dot{M}}\right) \mathrm{yr}\ . \tag{17}$$

Interestingly, the two timescales, t_{gw} and t_{su}, would balance at $P \approx 30$ s for the observed accretion rate of AE Aqr ($\dot{M} = 2 \times 10^{-10}\ M_\odot\ \mathrm{yr}^{-1}$). Still, we believe that the growth rate of the mode is too slow for the instability to be relevant. If the star spins up closer to the Kepler period, the mode grows faster, but at $P = 17$ s we still get $t_{gw} \approx 10^8$ yr. This means that the r-mode has to be able to grow coherently for 100 million yr in order for the instability to affect the rotation of a white dwarf, which seems unlikely. Hence we can probably rule out the r-mode instability from playing any role in the scenario where a weak-field white dwarf is spun up toward the Kepler frequency and then undergoes accretion induced collapse. As was shown by Narayan & Popham (1989), accretion-induced collapse is likely to occur only if the accretion rate is higher than $4 \times 10^{-8}\ M_\odot\ \mathrm{yr}^{-1}$. The white dwarf is expected to reach critical mass after accreting 0.1–0.15 $M_\odot$, which means that it would collapse after 10^7 yr or so. This would clearly not give the r-modes time to grow by the many orders of magnitude required for the instability to affect the rotation of the star.

Finally, a word of caution is in order. Our estimate for the viscosity assumed that the stellar fluid was in the liquid phase. At low temperatures this may not be the case, and the viscosity of the resultant lattice may be much higher than equation (3). We estimate that the instability becomes active unless the true viscosity of white dwarfs is more than 400 times larger than equation (13).

5. FINAL REMARKS

We have presented rough estimates that suggest that the r-mode instability may be active in accreting neutron stars and white dwarfs. Our "back of the envelope" calculations lead to two main conclusions.

First, the r-mode instability can limit the rotation of recycled MSPs, provided that they are hotter than 2×10^5 K. We have shown that this suggestion cannot be ruled out by present observations for MSP periods and temperatures. We have also compared our predicted critical periods with observed data for LMXBs. In particular, we considered the systems for which a rotation period in the narrow range 3–3.8 ms has been inferred from the beat-frequency model for QPOs. This comparison strongly suggests that the r-mode instability may be active in these systems and that it can potentially be the agent that limits the rate of rotation to the observed periods. Furthermore, we have estimated the amplitude of the gravitational waves from a neutron star rotating at the r-mode instability limit and accreting at the Eddington limit (typical examples could be the fastest accreting LMXBs or the so-called Thorne-Zytkow objects). The result suggests that this kind of source (from within our Galaxy) could be detected by the new generation of gravitational-wave detectors.

There are, of course, many uncertainties associated with these suggestions. The most important one concerns the basically unknown role of superfluidity. A better understanding of neutron star superfluidity and its effect on the r-mode instability is urgently needed to firm up the present estimates.

Second, we find that the r-mode instability may be active in short-period white dwarfs. Our order-of-magnitude estimates (for a C^{12} white dwarf of $M = M_\odot$ and $R = 0.01\ R_\odot$) yield a critical period of $P \approx 27$–33 s. This number is in interesting agreement with the observed periods (greater than 28 s) of the rapidly spinning DQ Her stars. It is, however, only an order-of-magnitude estimate that indicates that the instability should also be taken seriously for white dwarfs and that a more detailed study should be attempted.

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REFERENCES

Andersson, N. 1998, ApJ, 502, 708
Andersson, N., Kokkotas, K. D., & Schutz, B. F. 1999, ApJ, 510, 846
Backer, D. C. 1998, ApJ, 493, 873
Bildsten, L. 1998, ApJ, 501, L89
Brady, P., Creighton, T., Cutler, C., & Schutz, B. F. 1998, Phys. Rev. D, 57, 1
Brown, E. F., & Bildsten, L. 1998, ApJ, 496, 915
Chandrasekhar, S. 1970, Phys. Rev. Lett., 24, 611
Durisen, R. H. 1973, ApJ, 183, 205
Flowers, E., & Itoh, N. 1979, ApJ, 230, 847
Friedman, J. L., & Morsink, S. 1998, ApJ, 502, 714
Friedman, J. L., & Schutz, B. F. 1978, ApJ, 222, 281
Fujimoto, M. Y., Hanawa, T., Iben, I., Jr., & Richardson, M. B. 1984, ApJ, 278, 813
Gudmundsson, E. H., Pethick, C. J., & Epstein, R. I. 1982, ApJ, 259, L19
Itoh, N., Kohyama, Y., & Takeuchi, H. 1987, ApJ, 317, 733
Jaranowski, P., Krolak, A., & Schutz, B. F. 1998, Phys. Rev. D, 58, 063001
Kokkotas, K. D., & Stergioulas, N. 1999, A&A, 341, 110
Lang, K. R. 1992, Astrophysical Data (New York: Springer)
Lindblom, L. 1995, ApJ, 438, 265
Lindblom, L., & Mendell, G. 1995, ApJ, 444, 804
Lindblom, L., Owen, B. J., & Morsink, S. M. 1998, Phys. Rev. Lett., 80, 4843
Marshall, F. E., Gotthelf, E. V., Zhang, W., Middleditch, J., & Wang, Q. D. 1998, ApJ, 499, L179
McDermott, P. N., Van Horn, H. M., & Hansen, C. J. 1988, ApJ, 325, 725
Mendell, G. 1991, ApJ, 380, 530
Narayan, R., & Popham, R. 1989, ApJ, 346, L25
Owen, B., Lindblom, L., Cutler, C., Schutz, B. F., Vecchio, A., & Andersson, N. 1998, Phys. Rev. D, 58, 084020
Papaloizou, J., & Pringle, J. E. 1978, MNRAS, 182, 423
Patterson, J. 1994, PASP, 106, 209
Psaltis D., et al. 1998, ApJ, 501, L95
Reisenegger, A. 1997, ApJ, 485, 313

Romani, R. W. 1990, Nature, 347, 741
Schutz, B. F. 1997, in Mathematics and Gravitation. II. Gravitational Wave Detection, ed. A. Krolak (Warsaw: Banach Center), 11
Stergioulas, N. 1998, in Living Reviews in Relativity, 1, 1998-8, http://www.livingreviews.org
Stergioulas, N., & Friedman, J. L. 1998, ApJ, 492, 301
Urpin, V., Geppert, U., & Konenkov, D. 1998, MNRAS, 295, 907
van der Klis, M. 1996, in The Many Faces of Neutron Stars, ed. R. Buccheri, J. Paradijs, & M. A. van Alper (Dordrecht: Kluwer), 337
Wagoner, R. V. 1984, ApJ, 278, 345
White, N. E., & Zhang, W. 1997, ApJ, 490, L87
Wijnands, R., & van der Klis, M. 1998, Nature, 394, 344
Yoshida, S., & Eriguchi, Y. 1997, ApJ, 490, 779