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ON THE REPRESENTATION OF SEMIMARTINGALES

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We show that L^1-bounded semimartingales (quasi-martingales, F-processes) correspond to finite signed measures on the σ-field of previsible sets. This representation of semimartingales as signed measures is used to derive in a unified manner the main decomposition theorems for semi- and supermartingales.

0. Introduction. Just as supermartingales may be viewed as the stochastic analogue of decreasing functions on the real line, the quasi-martingales of Fisk [2] and Rao [11] resp. Orey's F-processes [9] resp. the semimartingales in the sense of Meyer [6] are the natural counterpart of functions of bounded variation. This is not only suggested by Fisk's original definition. It is also apparent from Rao's representation of a L^1-bounded semimartingales as the difference of two nonnegative supermartingales. Here we add another aspect: semimartingales may be viewed as signed measures. More precisely: to any L^1-bounded semimartingale $X = (X_t)_{t\geqq 0}$ over a nice system $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ corresponds a finite signed measure P^X on the previsible sets in $\Omega \times (0, \infty]$ such that

$$P^X[A \times (t, \infty]] = E[X_t; A] \qquad (t \geqq 0, A \in \mathscr{F}_t). \tag{0.1}$$

This representation of X by a signed measure is an immediate consequence of the Rao decomposition of X and the author's construction in [3] (resp. Meyer's in [8]) for a positive supermartingale—which in turn is based on the Itô–Watanabe factorization (resp. the Doob–Meyer decomposition). The purpose of this paper is to develop a converse procedure. The signed measure P^X is constructed directly from the definition of a L^1-bounded semimartingale or, as we shall call it, a "process of bounded variation" (Section 1). The measure is then used to derive some of the main facts on semi- and supermartingales (Section 2). The Rao decomposition of a semimartingale appears as the Jordan decomposition of P^X, the various Riesz decompositions follow by splitting P^X on suitable subsets of $\Omega \times (0, \infty]$, and the Doob–Meyer decomposition and the Itô–Watanabe factorization are obtained by using certain projections of P^X to Ω. Let us recall that a part of this program, namely the Doob–Meyer decomposition of a potential of class (D) via a measure on $\Omega \times (0, \infty]$, has already been carried out by C. Doléans–Dade in [1].

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1. Processes of bounded variation and signed measures. Let us fix a probability space $(\Omega, \mathscr{F}, P)$ and an increasing right-continuous family $(\mathscr{F}_t)_{t \geqq 0}$ of σ-fields which generates $\mathscr{F}$. We do not assume that the fields are complete, and the reason will become apparent in (1.4).

(1.1) DEFINITION. Let $X = (X_t)_{t \geqq 0}$ be a real-valued process adapted to $(\mathscr{F}_t)_{t \geqq 0}$ such that the random variables X_t are integrable and $E[X_t]$ is right-continuous in t. We say that X is a *process of bounded variation* if

$$\operatorname{Var}(X) \equiv \sup \textstyle\sum_{i=0}^{n} E[|X_{t_i} - E[X_{t_{i+1}} | \mathscr{F}_{t_i}]|] + E[|X_{t_n}|] < \infty$$

where the supremum is taken over all finite sequences $0 = t_0 < t_1 < \cdots < t_n < \infty$. Note that this definition does not depend on the specific version of X.

(1.2) REMARKS. 1. Our use of the term "bounded variation" is motivated by (1.6) below which extends the classical relation between functions of bounded variation and finite signed measures on the real line. For a martingale our terminology reduces to Krickeberg's in [4]: it is a process of bounded variation if and only if it is bounded in L^1. Let us emphasize that a process of bounded variation in our sense may have paths which, considered as functions of a real variable, are not at all of bounded variation (take Brownian motion stopped at time 1).

2. It is easy to check that any process, which can be written as the difference of two nonnegative supermartingales, is a process of bounded variation. It is less obvious that the converse holds as well: this is the Rao decomposition (2.1) below.

Orey showed in [9] that processes of bounded variation have the same path regularities as supermartingales, and the following theorem is in essence due to him.

(1.3) THEOREM (Orey). *Any process of bounded variation has a right-continuous version which is adapted to* $(\mathscr{F}_t)$.

PROOF. We have only to combine Theorems 2.2 and 2.3 in [9] with the usual construction for supermartingales as presented in [5] VI 3, 4. Note however that a slight modification is needed since our σ-fields are not complete. The details of this are the same as in [3] (1.1).

Let us now introduce the product space $\bar{\Omega} := \Omega \times (0, \infty]$ together with the σ-field $\mathscr{P}$ of previsible sets in $\bar{\Omega}$. $\mathscr{P}$ is generated by the processes which are adapted to $(\mathscr{F}_t)$ and have left-continuous paths or, equivalently, by the sets $A \times (t, \infty]$ with $t \geqq 0$ and $A \in \mathscr{F}_t$ (cf. [7] Appendix 3). We are going to associate to any process X of bounded variation a signed measure on $\mathscr{P}$, and this requires some regularity condition on the underlying σ-fields.

(1.4) ASSUMPTION. $(\mathscr{F}_t)$ is the right-continuous modification of a standard system $(\mathscr{F}_t^0)$, i.e. $\mathscr{F}_t = \bigcap_{s>t} F_s^0$ for any $t \geqq 0$. The notion of a standard system is taken from Parthasarathy (cf. [10] V). It means that

(i) each space $(\Omega, \mathscr{F}_t^0)$ is a standard Borel space, and that

(ii) $\bigcap A_i \neq \varnothing$ whenever $(A_i)_{i=1,2,\cdots}$ is a decreasing sequence of sets such that A_i is an atom of $\mathscr{F}_{t_i}^0$ for some increasing sequence $(t_i)_{i=1,2,\cdots}$.

Actually we could do without (ii), but then we would have to replace $\bar{\Omega}$ by a less transparent inverse limit space (as suggested by [10] V Theorem 3.2). Let us rather try to justify our assumption by the following "canonical."

(1.5) EXAMPLE. Let Ω be the set of all right-continuous paths on a nice state space with absorbing point Δ, which have left limits at least before being absorbed by Δ. Taking the usual σ-fields $\mathscr{F}_t^0$ which describe the path behavior up to time t, we obtain indeed a standard system (cf. [3] or [8]).

Let us now fix a right-continuous process $X = (X_t)$ adapted to $(\mathscr{F}_t)$.

(1.6) THEOREM. *X is a process of bounded variation if and only if there is a signed measure P^X on $\mathscr{P}$ such that*

$$P^X[A \times (t, \infty]] = E[X_t; A] \qquad (t \geqq 0,\ A \in \mathscr{F}_t)\,. \tag{1.7}$$

P^X is uniquely determined by X, and its total variation $|P^X|$ satisfies $|P^X|(\bar{\Omega}) = \mathrm{Var}\,(X)$.

(1.8) REMARK. The theorem establishes a 1–1 correspondence between processes of bounded variation and those finite signed measures Q on $\mathscr{P}$ whose projections $Q_t(A) \equiv Q[A \times (t, \infty]]$ $(A \in \mathscr{F}_t)$ are absolutely continuous with respect to P (to Q corresponds the process dQ_t/dP).

PROOF. (i) Assume that X is of bounded variation. For each $n = 1, 2, \cdots$ let us partition the closed interval $[0, \infty]$ into $n2^n + 1$ dyadic intervals $J_{n,k}$ such that $J_{n,k} = (k2^{-n}, (k+1)2^{-n}]$ for $0 < k < n2^n$. Write $\tilde{\Omega} = \Omega \times \tilde{R} = \bar{\Omega} + \Omega \times \tilde{D}$ where $\tilde{R} = (0, \infty] + \tilde{D}$ denotes the disjoint union of $(0, \infty]$ and an extra copy $\tilde{D}$ of the dyadic rationals. Now let $\mathscr{P}_n$ be the σ-field on $\tilde{\Omega}$ generated by the sets $A \times \tilde{J}_{n,k}$ $(A \in \mathscr{F}_{k2^{-n}}^0,\ 0 \leqq k \leqq n2^n)$ where $\tilde{J}_{n,k}$ is the disjoint union of $J_{n,k}$ and the point $\tilde{d} \in \tilde{D}$ which represents the dyadic rational $d = k2^{-n}$. Setting

$$\begin{aligned} Q_n[A \times J_{n,k}] &\equiv E[X_{k2^{-n}} - X_{(k+1)2^{-n}}; A] \qquad &&\text{for } 0 \leqq k < n2^n \\ &\equiv E[X_n; A] &&\text{for } k = n2^n \end{aligned}$$

we obtain a signed measure Q_n on $\mathscr{P}_n$ whose total variation $|Q_n|$ satisfies

$$|Q_n|[\tilde{\Omega}] \leqq \mathrm{Var}\,(X) \tag{1.9}$$

since

$$|Q_n[A \times J_{n,k}]| \leqq |E[X_{k2^{-n}} - E[X_{(k+1)2^{-n}} \,|\, \mathscr{F}_{k2^{-n}}]; A]|$$

for $0 \leqq k < n2^n$ and $A \in \mathscr{F}_{k2^{-n}}^0$, and similarly for $k = n2^n$. It is easy to check that $(Q_n)_{n=1,2,\cdots}$ is a consistent sequence of measures. Our assumption (1.4) implies that $(\mathscr{P}_n)_{n=1,2,\cdots}$ is a standard system[1] as required in the extension Theorem 4.1 in [10] V. We may thus conclude: there is a finite signed measure Q on $(\tilde{\Omega}, \bigvee_{n=1}^{\infty} \mathscr{P}_n)$ such that the restriction of Q to $\mathscr{P}_n$ coincides with Q_n. Moreover we may conclude that Q is concentrated on $\bar{\Omega}$: For $\tilde{d} \in \tilde{D}$ and $A \in \mathscr{F}_d^0$ we have

$$Q[A \times \{\tilde{d}\}] = \lim_n Q[A \times (\{\tilde{d}\} + (d, d + 2^{-n}])]$$
$$= \lim_n E[X_{d+2^{-n}} - X_d; A] ,$$

and this is 0 by right-continuity of X and by uniform integrability of $(X_{d+2^{-n}})_{n=1,2,\cdots}$ (cf. Theorem 2.3 in [9]). Now note that the restriction of $\bigvee_{n=1}^{\infty} \mathscr{P}_n$ to $\bar{\Omega}$ is just $\mathscr{P}$, and define P^X as the restriction of Q to $(\bar{\Omega}, \mathscr{P})$. It is clear from our construction that P^X satisfies (1.7) at least when t is a dyadic rational and $A \in \mathscr{F}_t^0$; the rest of (1.7) is obtained by approximation, and this is done in Lemma (1.10) below. We should mention that the extension Theorem 4.1 in [10] is only formulated for probability measures. But our situation is easily reduced to this case: because of (1.9) we may write $Q_n = Q_n' - Q_n''$ where (Q_n') and (Q_n'') are consistent sequences of nonnegative measures (cf. e.g. [4]), and now we may as well assume $Q_n'(\tilde{\Omega}) = Q_n''(\tilde{\Omega}) = 1$.

(ii) Suppose that P^X is a finite signed measure on $\mathscr{P}$ such that (1.7) holds. Note first that (1.7) implies the right-continuity of $E[X_t] = P^X[\Omega \times (t, \infty]]$. Now take $0 \leqq s < t \leqq \infty$ and $A = \{X_s > E[X_t \mid \mathscr{F}_s]\}$ (we set $X_\infty \equiv 0$). Then we have

$$|P^X| [\Omega \times (s, t]] \geqq |P^X[A \times (s, t]]| + |P^X[A^c \times (s, t]]|$$
$$= E[|X_s - E[X_t \mid \mathscr{F}_s]|] ,$$

and this shows that X is of bounded variation with $\mathrm{Var}\,(X) \leqq |P^X|(\bar{\Omega})$. The converse inequality follows via (1.9). To show the uniqueness of P^X we have only to note that (1.7) determines P^X on the sets $A \times (s, t]$ $(s < t, A \in \mathscr{F}_s)$ which generate $\mathscr{P}$.

For the rest of this section we assume that X is a right-continuous process of bounded variation. Whenever T is a stopping time (i.e. $\{T \leqq t\} \in \mathscr{F}_t$ for any $t \geqq 0$), we write $A \times (T, \infty]$ instead of $\{(\omega, t) \mid \omega \in A, t > T(\omega)\}$ and $(T, \infty]$ instead of $\Omega \times (T, \infty]$.

(1.10) LEMMA. *If T is a stopping time and $A \in \mathscr{F}_T$* (cf. [5] IV, D35) *then we have*

$$P^X[A \times (T, \infty]] = E[X_T; A \cap \{T < \infty\}] . \tag{1.11}$$

PROOF. Both sides make sense: $A \times (T, \infty]$ is previsible since its indicator function is a left-continuous process adapted to $(\mathscr{F}_t)$, and X_T is measurable by the right-continuity of (X_t). Our constraction of P^X shows that (1.11) holds at least for all those $(\mathscr{F}_t^0)$-stopping times which assume only dyadic values (split into the sets where such a stopping time is constant). Now take a general T and approximate it from above by a decreasing sequence of dyadic $\mathscr{F}_t^0$-stopping times T_n (cf. [5] IV D43). Then we may conclude

$$\begin{aligned} P^X[A \times (T, \infty]] &= \lim_{a\uparrow\infty} P^X[(A \cap \{T < a\}) \times (T, \infty]] \\ &= \lim_{a\uparrow\infty} \lim_{n\uparrow\infty} P^X[(A \cap \{T < a\}) \times (T_n \wedge a, \infty]] \\ &= \lim_{a\uparrow\infty} \lim_{n\uparrow\infty} E[X_{T_n\wedge a}; A \cap \{T < a\}] \\ &= \lim_{a\uparrow\infty} E[X_T; A \cap \{T < a\}] \\ &= E[X_T; A \cap \{T < \infty\}] \,. \end{aligned}$$

In order to justify the fourth step we have to show that each sequence $(X_{T_n\wedge a})_{n=1,2,\ldots}$ is uniformly integrable. This is settled by Orey's argument for Theorem 2.3 in [9]: if uniform integrability were violated we could derive

$$\sup_K E[\textstyle\sum_{k=1}^K |E[X_{T_{n_k}\wedge a} - X_{T_{n_{k+1}}\wedge a} \,|\, \mathscr{F}_{T_{n_{k+1}}\wedge a}]|] = \infty$$

for a suitable subsequence (n_k) (same proof as for (14) in [9]). But on the other hand, since (1.11) is valid for the $(\mathscr{F}_t^0)$-stopping times $T_{n_k} \wedge a$, the left side is majorized by

$$\sup_K \textstyle\sum_{k=1}^K |P^X| \, [(T_{n_{k-1}} \wedge a, T_{n_k} \wedge a]] \leqq |P^X| \, [\bar{\Omega}] < \infty$$

(cf. (ii) of the previous proof), and this is the desired contradiction.

Let us shortly illustrate how properties of the process X can be read off from the measure P^X; cf. [3] for a continuation of this list, and also (2.5) below.

(1.12) Proposition. *X is a martingale* iff *P^X is concentrated on $\Omega \times \{\infty\}$, a supermartingale* iff *the restriction of P^X to $\Omega \times (0, \infty)$ is nonnegative, a positive supermartingale* iff *P^X is a positive measure, and a potential* iff *P^X is nonnegative and concentrated on* $(0, \Omega)$.

Proof. Immediate from (1.7).

2. The decomposition theorems. Let $X = (X_t)$ be a right-continuous process of bounded variation in the sense of (1.1), and let P^X be the associated signed measure on $\mathscr{P}$.

(2.1) Theorem (Rao decomposition). *X can be written as the difference of two nonnegative supermartingales.*

Proof. Let $P^X = (P^X)^+ - (P^X)^-$ be the Jordan decomposition of the signed measure P^X into its positive and negative part. $(P^X)^+$ and $(P^X)^-$ satisfy the requirements in (1.8) and thus correspond to two processes of bounded variation X^+ and X^-, which are in fact nonnegative supermartingales due to (1.12). The decomposition $X = X^+ - X^-$ follows via (1.7).

(2.2) Definition. Let us call the supermartingales X^+ and X^- above the *positive* resp. the *negative variation* of X, and the supermartingale $|X| \equiv X^+ + X^-$, which corresponds to $|P^X|$, the *total variation* of X.

Let us now assume that X is a nonnegative supermartingale.

(2.3) THEOREM (Riesz decomposition I). *X can be written as the sum of a martingale and a potential.*

PROOF. Split P^X into its restrictions on $\Omega \times \{\infty\}$ and $\Omega \times (0, \infty)$.

Let us recall that a right-continuous process (Z_t) adapted to is said to be in *class* (D) if the family of random variables Z_T (T a finite stopping time) is uniformly integrable. (Z_t) is called a *local martingale* if there in an increasing sequence of stopping times T_n with $\sup T_n = \infty$ P-a.s. such that the stopped processes $(Z_{T_n \wedge t})_{t \geqq 0}$ are all martingales of class (D).

(2.4) THEOREM (Riesz decomposition II). *X can be written as the sum of a local martingale and a potential of class* (D).

PROOF. In view of (2.3) we may assume that X is a potential. Take the stopping times $R_n \equiv \inf\{t > 0 \mid X_t > n\}$ and define K as the previsible set $\bigcap_{n=1}^{\infty} (R_n, \infty]$. Split P^X into its restrictions P^Y on K and P^Z on K^c and denote by $Y = (Y_t)$ and $Z = (Z_t)$ right-continuous versions of the corresponding supermartingales. Y is a local martingale since any of the stopped processes $(Y_{R_n \wedge t})$ is a martingale:

$$\begin{aligned} E[Y_{R_n \wedge s}; A] &= P^X[K \cap (A \times (R_n \wedge s, \infty])] \\ &= P^X[K \cap (A \times (R_n \wedge t, \infty])] = E[Y_{R_n \wedge t}; A] \end{aligned}$$

for $s < t$ and $A \in \mathscr{F}_s$. As to Z, define $S_n \equiv \inf\{t > 0 \mid Z_t > n\} \geqq R_n$ and note

$$\lim E[Z_{S_n}] \leqq \lim E[Z_{R_n}] = \lim P^X[K^c \cap (R_n, \infty]] = 0$$

which shows that Z is a potential of class (D) ([5] VI T 20).

For the deeper decompositions we need the following criterion. Let us say that a set $K \in \mathscr{P}$ is *evanescent* if its projection on Ω has P-measure 0.

(2.5) THEOREM. *X is of class* (D) *if and only if P^X vanishes on evanescent sets.*

PROOF. (i) Assume that P^X vanishes on evanescent sets. Let us also assume that X is a potential (otherwise use (2.3) and note that the conclusion is immediate for martingales). It is then enough to show $\lim E[X_{R_n}] = 0$ where R_n is defined as in the proof of (2.4). Fix $\varepsilon > 0$, take a > 0 such that $E[X_a] < \varepsilon$ and define $K \equiv \bigcap_{n=1}^{\infty} (R_n, a]$. The projection of K on Ω is the set $\bigcap_{n=1}^{\infty} \{R_n < a\}$ which has P-measure 0 since the paths of a right-continuous supermartingale do not "explode." Hence

$$0 = P^X[K] = \lim P^X[(R_n \wedge a, a]] = \lim E[X_{R_n \wedge a} - X_a]$$

which implies

$$\lim E[X_{R_n}] \leqq \lim E[X_{R_n \wedge a}] = E[X_a] < \varepsilon .$$

(ii) Now let X be of class (D). For a martingale the conclusion is again immediate, and so we may assume that X is a potential. Take an evanescent set $E \in \mathscr{P}$ and suppose $P^X[E] > 0$. Then there is a previsible subset F of E with $P^X[F] > 0$ such that the "debut of F" $T(\omega) = \inf\{t > 0 \mid (\omega, t) \in F\}$ $(\omega \in \Omega)$ is previsible in the following sense: there is an increasing sequence of stopping times T_n which satisfy $T_n < T$ everywhere and $T_n \uparrow T$ P-a.s. (the proof is the same as in [7] 214, if we replace the capacity used there by the measure P^X). F is evanescent since E is, and thus we may conclude $T_n \uparrow \infty$ P-a.s. Now we obtain the desired contradiction: since

$$0 < P^X[F] = \lim_n P^X[(T_n, \infty]] = \lim_n E[X_{T_n}] ,$$

X cannot be a potential of class (D) (cf. [5] VI, T 20).

(2.6) Remarks. 1. A potential X is a local martingale iff P^X is supported by some evanescent set. To see this write X as the sum of local martingale Y and a potential Z of class (D). If P^X is supported by some evanescent set then P^Z vanishes by (2.5) so that $X = Y$. Conversely, if $X = Y$ then P^X is supported by the evanescent set $K = \bigcap_{n=1}^{\infty} (R_n, \infty]$ (recall the proof of (2.4)).

2. Suppose that X is a potential of class (D); this is the case considered by C. Doléans-Dade in [1]. Let us denote by $(\mathscr{F}_t^*)$ the usual "completion" of $(\mathscr{F}_t)$ (first adjoin all P-nullsets in $\mathscr{F}$ to the fields $\mathscr{F}_t$, then take the right-continuous modification), and by $\mathscr{P}^*$ the σ-field generated by the left-continuous processes adapted to $(\mathscr{F}_t^*)$. (2.5) implies that $\mathscr{P}^*$ is contained in the P^X-completion of $\mathscr{P}$. This shows that P^X is equivalent to Doléans-Dade's measure in [1].

3. Let us denote by $\bar{\mathscr{F}}$ the product of $\mathscr{F}$ with the usual σ-field on $(0, \infty]$. If X is of class (D) then we may extend P^X to $\bar{\mathscr{F}}$ as indicated in [1]: For any bounded $\bar{\mathscr{F}}$-measurable process $Y = (Y_t)$ we define

$$E^X[Y] \equiv E^X[Z] \tag{2.7}$$

where $Z = (Z_t)$ is the projection of Y on $\mathscr{P}$, and this definition is legitimate by (2.5) since different projections differ at most on an evanescent set (cf. [7] 214).

Let us now recall how C. Doléans-Dade obtained the Doob-Meyer decomposition of a potential of class (D) via the measure P^X (cf. [1] and the previous remark). By an *increasing process* $A = (A_t)_{t \geq 0}$ we mean as usual a right-continuous process adapted to $(\mathscr{F}_t)$ with $A_0 = 0$ and increasing paths. A is called *integrable* if $\sup E[A_t] < \infty$, and A is previsible if and only if it is "natural" in the sense of [5] VII D18 (cf. [7] 312).

(2.8) Theorem (Doob-Meyer-Itô-Watanabe decomposition). *X can be written as the difference of a local martingale and an integrable and previsible increasing process.*

PROOF (Doléans–Dade). By (2.4) we may assume that X is a potential of class (D). In view of (2.5) and (2.6) we have only to reproduce Section 3 of the proof in [1]: the random variable A_t of the increasing process A appears as a P-density of the projection $\pi_t(A) \equiv P^X[A \times (0, t]]$ $(A \in \mathscr{F})$, and from (2.5) we may conclude, as indicated in [1] resp. [3] (1.1), that (A_t) is adapted to $(\mathscr{F}_t)$ and also previsible.

Let us conclude our discussion of supermartingales by looking at the factorization of X into a local martingale and a decreasing process.

(2.9) THEOREM (Itô–Watanabe factorization). *X can be written as a product*

$$X_t = M_t(1 - A_t) \qquad (0 \leqq t < \infty) \tag{2.10}$$

P-a.s., *where $M = (M_t)$ is a local martingale and $A = (A_t)$ a previsible increasing process whose paths are bounded by* 1. *This factorization is unique at least up to* $T_X \equiv \inf\{t > 0 \mid X_t = 0\}$.

Suppose we have such a factorization. Assume (stopping if necessary) that M is a martingale of class (D). Then we can write

$$E^X[Y] = E[M_\infty \textstyle\int_0^\infty Y_s\, dA_s]$$

for any previsible process $Y = (Y_s) \geqq 0$, since it is clearly true for processes of the form $Y_s I_{(s,t]}$ with some $\mathscr{F}_s$-measurable function Y_s. Our assumption implies that X is of class (D) and so we can extend P^X to the product field $\bar{\mathscr{F}}$ via previsible projection; cf. (2.6). Now take $Y = Y_t I_{(0,t]}$ where $Y_t \geqq 0$ is $\mathscr{F}_t$-measurable, and denote by (Y_{s-}) the previsible projection of Y. We have

$$\begin{aligned} E^X[Y] = E[M_t \textstyle\int_0^t Y_{s-}\, dA_s] &= E[\textstyle\int_0^t M_{s-} Y_{s-}\, dA_s] \\ &= E[Y_t \textstyle\int_0^t M_{s-}\, dA_s] \end{aligned}$$

due to [5] VII T16, T19. But the proof of (2.8) shows that the left side coincides with $E[Y_t B_t]$ if $B = (B_t)$ is the increasing process in the Doob–Meyer decomposition of X. Thus we obtain

$$B_t = \textstyle\int_0^t M_{s-}\, dA_s = \int_0^t X_{s-}(1 - A_{s-})^{-1}\, dA_s \qquad (t < T_X)\,.$$

This implies the uniqueness of B up to T_X and the Itô–Watanabe–Meyer formula for A in terms of B:

$$1 - A_t = \exp(\textstyle\int_0^t X_{s-}^{-1}\, dB_s^c) \prod_{s \leqq t} (1 - X_{s-}^{-1}\, \Delta B_s) \qquad (t < T_X)$$

where B^c is the continuous part of B and where we set $\Delta B_s \equiv B_s - B_{s-}$. The use of the measure P^X does not seem to simplify the work which has still to be done for a complete proof of (2.9), and so we refer to [12] for the remaining arguments.

Let us now return to the general case where $X = (X_t)$ is a right-continuous process of bounded variation. In view of the Rao decomposition (2.1) we can immediately write down the analogues of the additive decompositions (2.3), (2.4), (2.8). In particular we have the following counterpart of the Doob–Meyer decompostion (2.8). $\mathscr{V}$ denotes the class of all processes $V = (V_t)$ which can be written as the difference of two integrable increasing processes.

(2.11) THEOREM (Fisk–Orey–Rao). *Any right-continuous process of bounded variaton can be written in the form $X_t = M_t + V_t$ where $M = (M_t)$ is a local martingale and $V = (V_t)$ is a previsible process in $\mathscr{V}$.*

This representation of X as a semimartingale in the sense of [6] is unique (cf. [6] page 107).

(2.12) REMARK. Let us say that a right-continuous process $X = (X_t)$ adapted to $(\mathscr{F}_t)$ is *locally of bounded variation* if there is an increasing sequence of stopping times T_n with $\sup T_n = \infty$ P-a.s. such that the stopped processes $(X_{T_n \wedge t})_{t \geq 0}$ are all of bounded variation. The quasi-martingales in [2] and [11], the F-processes in [9] and the local semimartingales in [6] are all locally of bounded variation. Conversely, any process which is locally of bounded variation is a local semimartingale in the sense of [6], i.e. can be written in the form $X_t = M_t + V_t$ where (M_t) is a local martingale and (V_t) is a previsible process which is locally in class $\mathscr{V}$ (just patch together the unique decompositions (2.1) of the stopped processes $(X_{T_n \wedge t})$).

[1] *Note added in proof.* This is not literally true. To be correct, define Q_n on $\tilde{\Omega} \equiv \Omega \times ((0, \infty] + D)$ where D is an extra copy of the dyadic rationals, using $\tilde{J}_{n,k} \equiv J_{n,k} + \{d\}$ instead of $J_{n,k}$ in the construction above, where $d \in D$ represents $k2^{-n}$. The corresponding system $(\mathscr{P}_n)$ is indeed standard. Now use right-continuity of X and Th. 2.3 of [9] to conclude that the resulting measure Q on $\tilde{\Omega}$ is in fact concentrated on $\bar{\Omega}$.

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