On the Representation of Timed Polyhedra*

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Abstract. In this paper we investigate *timed polyhedra*, i.e. polyhedra which are finite unions of full dimensional simplices of a special kind. Such polyhedra form the basis of timing analysis and in particular of verification tools based on timed automata. We define a representation scheme for these polyhedra based on their extreme vertices, and show that this compact representation scheme is canonical for all (*convex and non-convex*) polyhedra in *any* dimension. We then develop relatively efficient algorithms for membership, boolean operations, projection and passage of time for this representation.

1 Introduction and Motivation

Timed automata, automata augmented with clock variables [AD94], has proven to be a very useful formalism for modeling phenomena which involve both discrete transitions and quantitative timing information. Although their state-space is non-countable, the reachability problem, as well as other verification, synthesis and optimizations problems for timed automata are solvable. This is due to the fact that the clock space admits an equivalence relation (time-abstract bisimulation) of finite index, and it is hence sufficient to manipulate these equivalence classes, which form a restricted class of polyhedra that we call *timed polyhedra*.

Several verification tools for timed automata have been built during the last decade, e.g. Kronos [DOTY96,Y97], Timed Cospan [AK96] and Uppaal [LPY97], and the manipulation of timed polyhedra is the computational core of such tools. Difference bound matrices (DBM) are a well-known data-structure for representing *convex* timed polyhedra, but usually, in the course of verification, the accumulated reachable states can form a highly non-convex set whose representation and manipulation pose serious performance problems to these tools.

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Consequently, the search for new representation schemes for timed polyhedra is a very active domain of research [ABK$^+$97], [BMPY97], [S99], [MLAH99b], [BLP$^+$99], [W00].

In this paper, we propose a new representation scheme for non-convex timed polyhedra based on a reformulation and extension of our previous work in [BMP99] where we proposed a canonical representation for non-convex *orthogonal* polyhedra. As in [BMP99] the representation is based on a certain subset of the vertices of the polyhedron. The size of our canonical representation is $O(nd!)$ where n is the number of vertices and d is the dimension. Based on this representation we develop relatively-efficient algorithms for membership, Boolean operations, projection and passage of time on arbitrary timed polyhedra of any dimension. In order to simplify the presentation we restrict the discussion in this paper to *full-dimensional* timed polyhedra, but the results can be extended to treat unions of polyhedra of varying dimension.

The rest of the paper is organized as follows: in section 2 we define orthogonal polyhedra and give new proofs of the main results from [BMP99] concerning their representation by extreme vertices. In section 3 we introduce timed polyhedra and prove that they can be represented canonically by their extreme vertices. In section 4 we discuss Boolean operations, projections, and the calculation of the effect of time passage. Finally we mention some related work and future research directions.

2 Griddy Polyhedra and Their Representation

Throughout the paper we assume a d-dimensional $\mathbb{R}$-vector space. In this section we also assume a fixed basis for this space so that points and subsets can be identified through their coordinates by points and subset of $\mathbb{R}^d$. The results of this section are invariant under change of basis,[1] and this fact will be exploited in the next section.

We assume that all our polyhedra live inside a bounded subset $X = [0, m]^d \subseteq \mathbb{R}^d$ (in fact, the results hold also for $\mathbb{R}^d_+$). We denote elements of X as $\mathbf{x} = (x_1, \ldots, x_d)$, the zero vector by $\mathbf{0}$ and the vector $(1, 1, \ldots, 1)$ by $\mathbf{1}$. The *elementary grid* associated with X is $\mathbf{G} = \{0, 1 \ldots, m-1\}^d \subseteq \mathbb{N}^d$. For every point $\mathbf{x} \in X$, $\lfloor \mathbf{x} \rfloor$ is the grid point corresponding to the integer part of the components of $\mathbf{x}$. The grid admits a natural partial order defined as $(x_1, \ldots, x_d) \leq (x'_1, \ldots, x'_d)$ if for every i, $x_i \leq x'_i$. The set of subsets of the elementary grid forms a Boolean algebra $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations.

Definition 1 (Griddy Polyhedra). *Let $\mathbf{x} = (x_1, \ldots, x_d)$ be a grid point. The elementary box associated with $\mathbf{x}$ is the closed subset of X of the form $B(\mathbf{x}) = [x_1, x_1+1] \times [x_2, x_2+1] \times \ldots [x_d, x_d+1]$. The point $\mathbf{x}$ is called the leftmost corner of $B(\mathbf{x})$. The set of boxes is denoted by $\mathbf{B}$. A griddy polyhedron P is a union of elementary boxes, i.e. an element of $2^{\mathbf{B}}$.*

[1] Griddy polyhedra were called orthogonal polyhedra in [AA98,BMP99]. We prefer here the term griddy since the results do not depend on an orthogonal basis.

Although $2^{\mathbf{B}}$ is not closed under usual complementation and intersection, it is closed under the following operations:

$$A \sqcup B = A \cup B \qquad A \sqcap B = cl(int(A) \cap int(B)) \qquad \neg A = cl(\sim A)$$

(where *cl* and *int* are the topological closure and interior operations[2]). The bijection B between $\mathbf{G}$ and $\mathbf{B}$ which associates every box with its leftmost corner clearly induces an isomorphism between $(2^{\mathbf{G}}, \cap, \cup, \sim)$ and $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$. In the sequel we will switch between point-based and box-based terminology according to what serves better the intuition.

Definition 2 (Color Function). *Let P be an griddy polyhedron. The color function $c : X \to \{0, 1\}$ is defined as follows: if $\mathbf{x}$ is a grid point then $c(\mathbf{x}) = 1$ iff $B(\mathbf{x}) \subseteq P$; otherwise, $c(\mathbf{x}) = c(\lfloor \mathbf{x} \rfloor)$.*

Note that c almost coincides with the characteristic function of P as a subset of X. It differs from it only on right-boundary points (see Figure 1-(a)).

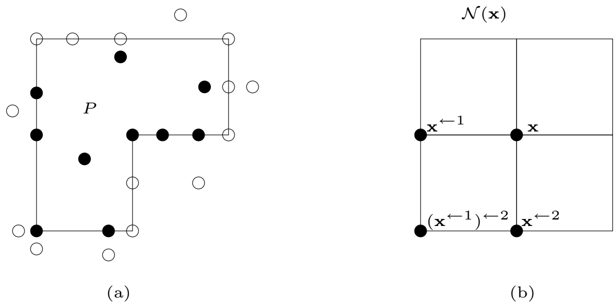

Fig. 1. (a) A griddy polyhedron and a sample of the values of the color function it induces on X. (b) The neighborhood and predecessors of a grid point $\mathbf{x}$.

Definition 3 (Predecessors, Neighborhoods and Cones). *In the following we consider $\mathbf{x}$ to be a grid point $\mathbf{x} = (x_1, \ldots, x_d)$.*

- *The i-predecessor of $\mathbf{x}$ is $\mathbf{x}^{\leftarrow i} = (x_1, \ldots, x_i - 1, \ldots, x_d)$.* We use $\mathbf{x}^{\leftarrow ij}$ as a shorthand for $(\mathbf{x}^{\leftarrow i})^{\leftarrow j}$.
- *The neighborhood of $\mathbf{x}$ is the set $\mathcal{N}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \ldots \{x_d - 1, x_d\}$,* i.e. the vertices of a box lying between $\mathbf{x} - \mathbf{1}$ and $\mathbf{x}$, (Figure 1-(b)).
- *The backward cone based at $\mathbf{x}$ is* ${}_{\triangleright}\mathbf{x} = \{\mathbf{y} \in \mathbf{G} : \mathbf{y} \leq \mathbf{x}\}$ (Figure 2-(a)).
- *The forward cone based at $\mathbf{x}$ is* $\mathbf{x}^{\triangleleft} = \{\mathbf{y} \in \mathbf{G} : \mathbf{x} \leq \mathbf{y}\}$ (Figure 2-(b)).

[2] See [B83] for definitions.

Every grid point $\mathbf{x}$ is contained in all the forward cones $\mathbf{y}^{\triangleleft}$ such that $\mathbf{y} \in {}_{\triangleright}\mathbf{x}$.

Let $\oplus$ denote addition modulo 2, known also as the exclusive-or (XOR) operation in Boolean algebra, $p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$. We will use the same notation for the symmetric set difference operation on sets, defined as $A \oplus B = \{\mathbf{x} : (\mathbf{x} \in A) \oplus (\mathbf{x} \in B)\}$.[3] This is an associative and commutative operation, satisfying $A \oplus A = 0$ on numbers, and $A \oplus A = \emptyset$ on sets. We will show that every griddy polyhedron admits a canonical decomposition into a XOR of cones.

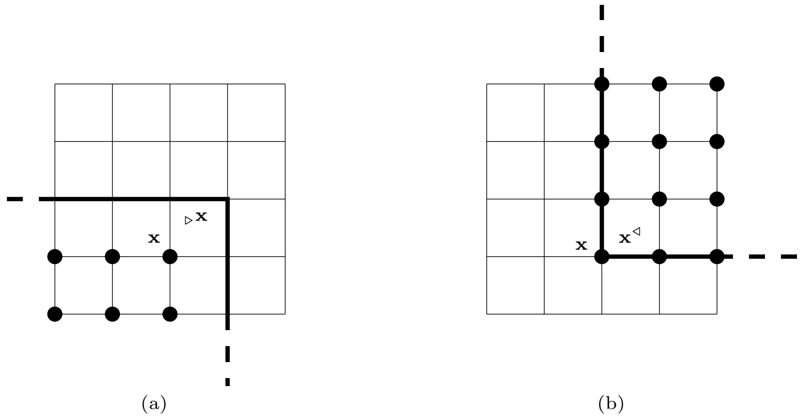

Fig. 2. (a) The backward cone based at $\mathbf{x}$. (b) The forward cone based at $\mathbf{x}$.

Theorem 1 ($\oplus$-representation). *For any griddy polyhedron P there is a unique finite set V of grid points such that* $P = \bigoplus_{\mathbf{v} \in V} \mathbf{v}^{\triangleleft}$.

Proof. First we show existence of V. Let $\prec$ be a total order relation on $\mathbf{G}$, $\mathbf{x}_1 \prec \mathbf{x}_2 \prec \ldots \mathbf{x}_{m^d}$, which is consistent with the partial order $\leq$ (any lexicographic or diagonal order will do). By definition, $\mathbf{x} \prec \mathbf{y}$ implies $\mathbf{y} \notin {}_{\triangleright}\mathbf{x}$. The following iterative algorithm constructs V. We denote by V_j the value of V after j steps and by c_j the color function of the associated polyhedron $P_j = \bigoplus_{\mathbf{v} \in V_j} \mathbf{v}^{\triangleleft}$.

```
V_0:=∅
for j = 1 to m^d do
  if c(x_j) ≠ c_{j-1}(x_j)
    then V:=V ∪ {x_j}
end
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This algorithm is correct because at the end of every step j in the loop we have $c_j(\mathbf{x}_k) = c(\mathbf{x}_k)$ for every $k \leq j$. It holds trivially at the beginning and in

[3] Note that on $2^{\mathbf{B}}$, $A \oplus B$ should be modified into $cl(int(A) \oplus int(B))$.

every step, the color of $\mathbf{x}_k$ for all $k < j$ is preserved (because their color is not influenced by the cone $\mathbf{x}_j^{\triangleleft}$) and updated correctly for $\mathbf{x}_j$ if needed.

For uniqueness we show that if there are two sets of points U and V such that

$$P = \bigoplus_{\mathbf{v}\in V} \mathbf{v}^{\triangleleft} = \bigoplus_{\mathbf{u}\in U} \mathbf{u}^{\triangleleft}$$

then $U = V$. Indeed, by the properties of $\oplus$, we have

$$\emptyset = P \oplus P = \bigoplus_{\mathbf{v}\in V} \mathbf{v}^{\triangleleft} \oplus \bigoplus_{\mathbf{u}\in U} \mathbf{u}^{\triangleleft} = \bigoplus_{\mathbf{y}\in U\oplus V} \mathbf{y}^{\triangleleft}$$

(if a point $\mathbf{v}$ appears in both U and V it can be eliminated because $\mathbf{v}^{\triangleleft} \oplus \mathbf{v}^{\triangleleft} = \emptyset$). The set denoted by the rightmost expression is empty only if $U \oplus V = \emptyset$, otherwise any minimal vertex $\mathbf{v}$ in $U \oplus V$ belongs to it. □

Remark: Since every set of points defines a $\oplus$-formula and a polyhedron, we have an interesting non-trivial bijection on subsets of $\mathbf{G}$. Note that for the chess board $V = \mathbf{G}$.

Let $\mathcal{V} : \mathbf{G} \to \{0, 1\}$ be the characteristic function of the set $V \subseteq \mathbf{G}$.

Observation 1. *For every point* $\mathbf{x}$, $c(\mathbf{x}) = \bigoplus_{\mathbf{y}\in {}_{\triangleright}\mathbf{x}} \mathcal{V}(\mathbf{y})$

This gives us immediately a decision procedure for the membership problem $\mathbf{x} \in P$: just check the parity of $V \cap {}_{\triangleright}\mathbf{x}$. In [BMP99] we have shown that the elements of V are those vertices of P that satisfy some local conditions. In the following we give an alternative proof of this fact, which requires some additional definitions to facilitate induction over the dimensions.

Definition 4 (k-Neighborhood and k-Backward Cone). *Let* $\mathbf{x} = (x_1, \ldots, x_d)$.

- *The k-neighborhood of* $\mathbf{x}$ *is the set*

$$N^k(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_k - 1, x_k\} \times \{x_{k+1}\} \times \ldots \times \{x_d\}.$$

- *The k-backward cone of* $\mathbf{x}$ *is the set*

$$M^k(\mathbf{x}) = \{x_1\} \times \ldots \times \{x_k\} \times \{0, \ldots, x_{k+1}\} \times \ldots \times \{0, \ldots, x_d\}.$$

Note that $N^0(\mathbf{x}) = \{\mathbf{x}\}$, $N^d(\mathbf{x}) = \mathcal{N}(\mathbf{x})$, $M^0(\mathbf{x}) = {}_{\triangleright}\mathbf{x}$ and $M^d(\mathbf{x}) = \{\mathbf{x}\}$ (see Figure 3).

Observation 2. *For every* $k \geq 0$*,*

$$N^k(\mathbf{x}) = N^{k-1}(\mathbf{x}) \oplus N^{k-1}(\mathbf{x}^{\leftarrow k}) \quad \textit{and} \quad M^k(\mathbf{x}) = M^{k-1}(\mathbf{x}) \oplus M^{k-1}(\mathbf{x}^{\leftarrow k}).$$

Theorem 2 (Extreme Vertices). *A point* $\mathbf{x}$ *is a basis for a cone in the canonical decomposition of a griddy polyhedron* P *if and only if* $\mathbf{x}$ *is a vertex of* P *satisfying*

$$\bigoplus_{\mathbf{y}\in\mathcal{N}(\mathbf{x})} c(\mathbf{y}) = 1 \tag{1}$$

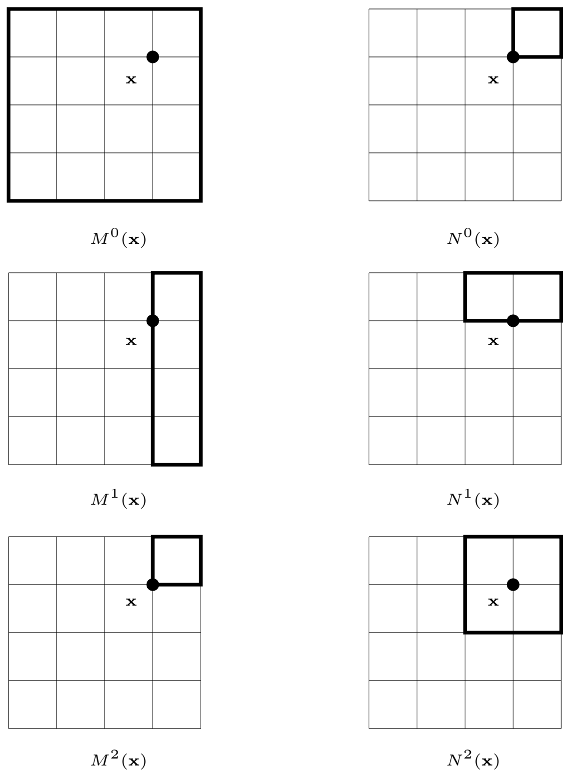


Fig. 3. The k-backward cone $M^k(\mathbf{x})$ and the k-neighborhood $N^k(\mathbf{x})$ for $k = 0, 1, 2$.

Proof. First we prove that for every $\mathbf{x} \in \mathbf{G}$ and every k, $0 \leq k \leq d$

$$\bigoplus_{\mathbf{y} \in N^k(\mathbf{x})} c(\mathbf{y}) = \bigoplus_{\mathbf{y} \in M^k(\mathbf{x})} \mathcal{V}(\mathbf{y}). \tag{2}$$

The proof is done by induction on k. For $k = 0$ it is just a rephrasing of observation 1. Suppose it holds for $k - 1$. By summing up the claims for $\mathbf{x}$ and for $\mathbf{x}^{\leftarrow k}$ we obtain

$$\bigoplus_{\mathbf{y} \in N^{k-1}(\mathbf{x})} c(\mathbf{y}) \;\oplus \bigoplus_{\mathbf{y} \in N^{k-1}(\mathbf{x}^{\leftarrow k})} c(\mathbf{y}) \;= \bigoplus_{\mathbf{y} \in M^{k-1}(\mathbf{x})} \mathcal{V}(\mathbf{y}) \;\oplus \bigoplus_{\mathbf{y} \in M^{k-1}(\mathbf{x}^{\leftarrow k})} \mathcal{V}(\mathbf{y})$$

which, by virtue of Observation 2 and the inductive hypothesis, gives the result for k. Substituting $k = d$ in (2), we characterize the elements of V as those satisfying condition (1). In [BMP99] we have proved that these points constitute a subset of the vertices of P which we call "extreme" vertices, following a geometrical definition in [AA98] for $d \leq 3$. □

We review our main algorithmic results from [BMP99], assuming a fixed dimension d and denoting by n_P the number of extreme vertices of a polyhedron P, which we assume to be sorted in some fixed order.

Observation 3 (Boolean Operations). *Let A and B be two griddy polyhedra.*

- *The symmetric difference of A and B, $A \oplus B$, can be computed in time $O(n_A + n_B)$.*
- *The complement of A, $X - A$, can be obtained in time $O(1)$.*
- *The union, $A \cup B$, and the intersection, $A \cap B$, can be computed in time $O(n_A n_b)$.*

Proof. Computing $A \oplus B$ is trivial and so is complementation using $X - A = \mathbf{0}^{\triangleleft} \oplus A$. Observing that $A \cup B = A \oplus B \oplus A \cap B$, computing union reduces to computing intersection. Now, using recursively the identity $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$, write

$$(\bigoplus_{\mathbf{x}_i} \mathbf{x}_i^{\triangleleft}) \cap (\bigoplus_{\mathbf{y}_j} \mathbf{y}_j{}^{\triangleleft}) = \bigoplus_{\mathbf{x}_i, \mathbf{y}_j} (\mathbf{x}_i{}^{\triangleleft} \cap \mathbf{y}_j{}^{\triangleleft}) = \bigoplus_{\mathbf{x}_i, \mathbf{y}_j} \max(\mathbf{x}_i, \mathbf{y}_j)^{\triangleleft}$$

where max denotes the maximum component-wise. □

We use the following terminology from [BMP99]:

Definition 5 (Slices, Sections and Cones). *Let P be a griddy polyhedron, $i \in \{1, \ldots, d\}$ and $z \in \{0, \ldots, m-1\}$.*

- *The (i, z)-slice of P is the d-dimensional polyhedron* $J_{i,z}(P) = P \sqcap \{\mathbf{x} : z \leq x_i \leq z + 1\}$.
- *The (i, z)-section of P is the $(d-1)$-dimensional polyhedron* $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap \{\mathbf{x} : x_i = z\}$.
- *The i-projection of P is the $(d-1)$-dimensional polyhedron* $\pi_{\downarrow i}(P) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_d) : \exists z \; (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots x_d) \in P\}$.

Observation 4 (Computation of Slice, Section and Projection). *The computation of $J_{i,z}$ and of $\mathcal{J}_{i,z}$ can be performed in time $O(n)$. The projection $\pi_{\downarrow i}(P)$ of P is computable in time $O(n^3)$.*

Proof. Using identity $(A \oplus B) \sqcap C = (A \sqcap C) \oplus (B \sqcap C)$, we have

$$J_{i,z}(\bigoplus_{\mathbf{x}\in V} \mathbf{x}^{\triangleleft}) = \bigoplus_{\mathbf{x}\in V} (J_{i,z}(\mathbf{x}^{\triangleleft})).$$

This gives an immediate algorithm for computing $J_{i,z}(P)$ and, hence for computing $\mathcal{J}_{i,z}(P)$. For projection write $\pi_{\downarrow i}(P) = \bigsqcup_z \pi_{\downarrow i}(J_{i,z}(P))$. □

3 Timed Polyhedra and their Representation

In this section we extend our results to *timed polyhedra* whose building blocks are certain types of simplices. Let Π denote the set of permutations on $\{1, \ldots, d\}$. We write permutations $\sigma \in \Pi$ as $(\sigma(1)\ \sigma(2)\ \cdots\ \sigma(d))$. There is a natural bijection between Π and the set of simplices occurring in a specific triangulation of the d-unit hypercube (sometimes called the Kuhn triangulation [L97]). Hence the correspondence between grid points and elementary boxes from the previous section can be extended into a correspondence between $\mathbf{G} \times \Pi$ and the set of such triangulations of the elementary boxes in X.

Definition 6 (Elementary Simplices, Timed Polyhedra and Cones). *Let $\mathbf{v}$ be a grid point and let σ be a permutation.*

- *The elementary simplex associated with $(\mathbf{v}, \sigma)$ is*

$$B(\mathbf{v}, \sigma) = \{\mathbf{x} : 0 \leq x_{\sigma(1)} - v_{\sigma(1)} \leq x_{\sigma(2)} - v_{\sigma(2)} \leq \ldots \leq x_{\sigma(d)} - v_{\sigma(d)} \leq 1\}.$$

- *A timed polyhedron is a union of elementary simplices. It can be expressed as a color function $c : \mathbf{G} \times \Pi \to \{0, 1\}$.*
- *The σ-forward cone based on a point $\mathbf{v}$ is*

$$(\mathbf{v}, \sigma)^{\triangleleft} = \{\mathbf{x} : 0 \leq x_{\sigma(1)} - v_{\sigma(1)} \leq x_{\sigma(2)} - v_{\sigma(2)} \leq \ldots \leq x_{\sigma(d)} - v_{\sigma(d)}\}.$$

These notions are illustrated in Figure 4 for $d = 2$. In the sequel we will use the word simplex both for a pair $(\mathbf{v}, \sigma)$ and for the set $B(\mathbf{v}, \sigma)$. Note that for every $\mathbf{v}$,

$$\bigoplus_{\sigma\in\Pi} B(\mathbf{v}, \sigma) = B(\mathbf{v}) \quad \text{and} \quad \bigoplus_{\sigma\in\Pi} (\mathbf{v}, \sigma)^{\triangleleft} = \mathbf{v}^{\triangleleft}.$$

Theorem 3 ($\oplus$-Representation of Timed Polyhedra). *For any timed polyhedron P, there is a unique finite set $V \subseteq \mathbf{G} \times \Pi$ such that*

$$P = \bigoplus_{(\mathbf{v},\sigma)\in V} (\mathbf{v}, \sigma)^{\triangleleft}$$

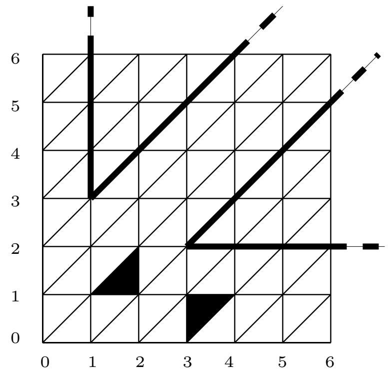


Fig. 4. The set of basic simplices in dimension 2, the simplices $B((1,1),(2\ 1))$ and $B((3,0),(1\ 2))$ and the forward-cones $((3,2),(2\ 1))^{\triangleleft}$ and $((1,3),(1\ 2))^{\triangleleft}$.

Proof. The proof is similar to the proof of Theorem 1. The only additional detail is the extension of the order on $\mathbf{G}$ into a lexicographic order on $\mathbf{G} \times \Pi$. Since the cones ${}_{\triangleright}(\mathbf{x}, \sigma)$ and ${}_{\triangleright}(\mathbf{x}, \tau)$ are disjoint for $\sigma \neq \tau$, the properties used in the proof are preserved. □

Our goal for the rest of the section is to find a *local characterization* of the elements $(\mathbf{v}, \sigma)$ of V, similar to property (1) in Theorem 2, based on the parity of the colors on some suitably-defined *neighborhood* of $(\mathbf{v}, \sigma)$.

Observation 5 (Decomposition of Timed Polyhedra). *Any timed polyhedron P can be decomposed into*

$$P = \bigoplus_{\sigma \in \Pi} P_\sigma.$$

where each P_σ is a griddy polyhedron in some basis of $\mathbb{R}^d$.

Proof. By letting $V_\sigma = \{\mathbf{v} : (\mathbf{v}, \sigma) \in V\}$ we can rewrite P as

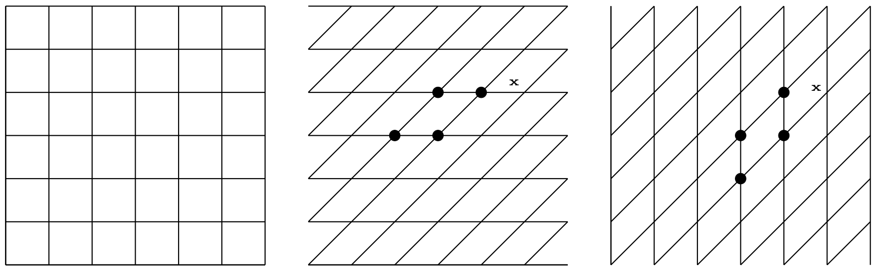
$$P = \bigoplus_{\sigma \in \Pi} \bigoplus_{\mathbf{v} \in V_\sigma} (\mathbf{v}, \sigma)^{\triangleleft} = \bigoplus_{\sigma \in \Pi} P_\sigma.$$

Each P_σ is a XOR of σ-cones, and hence it is griddy in the coordinate system corresponding to σ which is related to the orthogonal system by the transformation $y_{\sigma(1)} = x_{\sigma(1)}$ and $y_{\sigma(i)} = x_{\sigma(i)} - x_{\sigma(i-1)}$ for $i \geq 2$. □

We denote the color function of P_σ by c_σ. We call the corresponding grid the σ-*grid.* In this grid the i-predecessor of $\mathbf{x}$ is denoted by $\mathbf{x}^{\overset{\sigma}{\leftarrow} i}$, and the neighborhood of $\mathbf{x}$ by $N_\sigma(\mathbf{x})$ (see Figure 5). By definition $(\mathbf{x}, \sigma) \in V$ iff if $\mathbf{x} \in V_\sigma$, and, since Theorem 2 does not depend on orthogonality, we have:

Observation 6. *A simplex* $(\mathbf{x}, \sigma)$ *occurs in* V *iff* $\bigoplus_{\mathbf{y} \in N_\sigma(\mathbf{x})} c_\sigma(\mathbf{y}, \sigma) = 1.$

This definition is based on P_σ, not on P itself. We will show that this sum can be reduced to the sum of the values of c on a certain set of simplices $S(\mathbf{x}, \sigma)$, to be defined below.



(a) (b) (c)

Fig. 5. (a) The orthogonal grid. (b) The (2 1)-grid and $N_{(2\ 1)}(\mathbf{x})$. (c) The (1 2)-grid and $N_{(1\ 2)}(\mathbf{x})$.

Definition 7 (Permutation and Simplex Predecessors). *With every* $i \in \{1, \ldots, d\}$ *define the* i*-predecessor function* ${}^{\leftarrow i} : \Pi \to \Pi$ *such that for every* σ *such that* $\sigma(k) = i$

$$\sigma^{\leftarrow i}(j) = \begin{cases} \sigma(j) & \text{if } j < k \\ \sigma(j+1) & \text{if } k < j < d \\ i & \text{if } j = d \end{cases}$$

The i*-predecessor of a simplex* $(\mathbf{x}, \sigma)$*, for* $i \in \{1, \ldots, d\}$ *is defined as* $(\mathbf{x}, \sigma)^{\leftarrow i} = (\mathbf{x}^{\leftarrow i}, \sigma^{\leftarrow i})$.

In other words, $\sigma^{\leftarrow i}$ is the permutation obtained from σ by picking x_i and putting it at the end of the ordering. We denote by $\sigma^{\leftarrow ij}$ the successive application of the operator $(\sigma^{\leftarrow i})^{\leftarrow j}$. Note that unlike the analogue definition on the grid, the permutation predecessors is not commutative, i.e. $\sigma^{\leftarrow ij} \neq \sigma^{\leftarrow ji}$.

The fact that $(\mathbf{y}, \tau) = (\mathbf{x}, \sigma)^{\leftarrow i}$ has the following geometrical interpretation: $B(\mathbf{y}, \tau)$ is the first simplex outside $B(\mathbf{x})$ encountered while going backward in direction i from $B(\mathbf{x}, \sigma)$. We can lift these functions into ${}^{\leftarrow i} : 2^{\mathbf{G} \times \Pi} \to 2^{\mathbf{G} \times \Pi}$ in the natural way. The following definition is crucial for this paper. It specifies which neighborhood of a simplex determines its membership in V.

Definition 8 (Simplex k-Neighborhood). *The simplex* k*-neighborhood of a simplex* $(\mathbf{x}, \sigma)$ *is the set of simplices defined recursively as:* $S^0(\mathbf{x}, \sigma) = \{(\mathbf{x}, \sigma)\}$ *and*

$$S^{k+1}(\mathbf{x}, \sigma) = S^k(\mathbf{x}, \sigma) \cup (S^k(\mathbf{x}, \sigma))^{\leftarrow \sigma(d-k)}.$$

This notion is depicted in Figure 6. We write S for S^d. Note that unlike the definition of neighborhood for griddy polyhedra (Definition 4), where the recursion over dimensions can be done in any order, here the order is important and depends on the permutation.

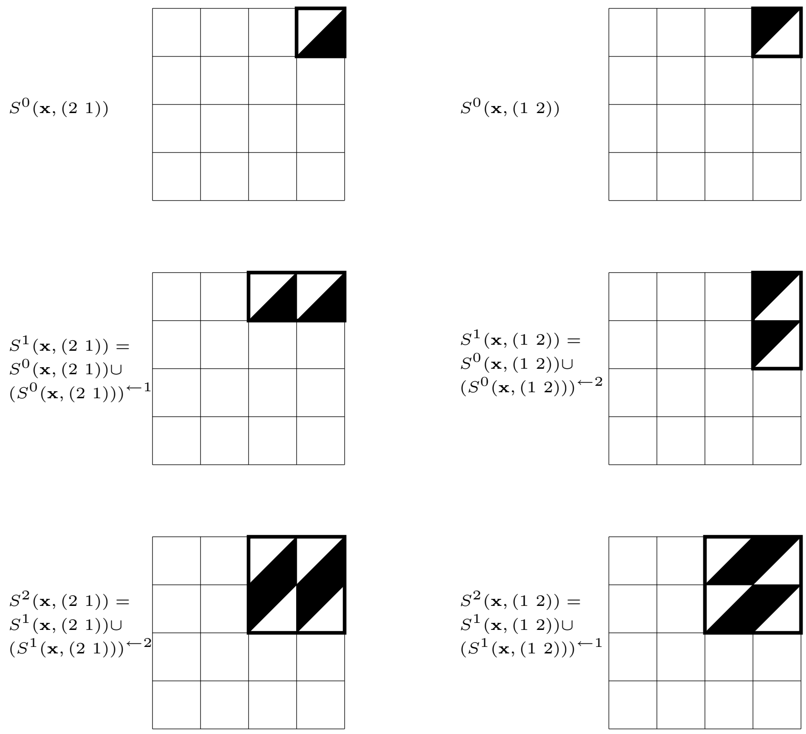


Fig. 6. The simplex k-neighborhoods of two simplices.

We intend to show that

$$\bigoplus_{\mathbf{y}\in N_\sigma(\mathbf{x})} c_\sigma(\mathbf{y},\sigma) = \bigoplus_{(\mathbf{y},\tau)\in S(\mathbf{x},\sigma)} c(\mathbf{y},\tau).$$

Definition 9 (Close Permutations). *With every $\sigma \in \Pi$ and every i, $0 \leq i \leq d$, we associate a set of permutations defined recursively as $\Pi_\sigma^0 = \Pi$ and*

$$\Pi_\sigma^{i+1} = \Pi_\sigma^i \cap \{\tau : \sigma(i) = \tau(i)\}.$$

In other words, Π_σ^i is the set of permutation that agree with σ on the first i coordinates.

Note, of course, that $\Pi_\sigma^d = \{\sigma\}$. With every such set of permutations we associate the polyhedron $P_\sigma^i = \bigoplus_{\tau\in\Pi_\sigma^i} P_\tau$ and let c_σ^i denote its corresponding color function.

For every k, $0 \leq k \leq d$, let $\rho_k(\mathbf{x}, \sigma)$ denote the quantity

$$\rho_k(\mathbf{x}, \sigma) = \bigoplus_{s \in S^k(\mathbf{x},\sigma)} c_\sigma^{d-k}(s)$$

Observation 7 (Fundamental Property). *Let $(\mathbf{v}, \sigma)$ be a simplex, and let τ be a permutation in Π_σ^{i-1} for some $i \in \{i, \ldots, d\}$. Then:*

- $c_\tau((\mathbf{v}, \sigma)^{\leftarrow\sigma(i)}) = c_\tau(\mathbf{v}, \sigma)$ *when* $\tau \notin \Pi_\sigma^i$.
- $c_\tau((\mathbf{v}, \sigma)^{\leftarrow\sigma(i)}) = c_\tau(\mathbf{v}^{\overset{\sigma}{\leftarrow}\sigma(i)}, \sigma)$ *when* $\tau \in \Pi_\sigma^i$.

Proof. Working in the coordinate system on which P_σ is griddy, i.e. $y_{\sigma(1)} = x_{\sigma(1)}$ and $y_{\sigma(i)} = x_{\sigma(i)} - x_{\sigma(i-1)}$ for $i \geq 2$, it is easy to see that $(\mathbf{v}, \sigma)$ and $(\mathbf{v}, \sigma)^{\leftarrow\sigma(i)}$ are both included in a same elementary box when $\tau \notin \Pi_\sigma^i$, and are included in two different consecutive boxes in direction $\sigma(i)$ when $\tau \in \Pi_\sigma^i$. □

Observing that any simplex $(\mathbf{v}, \tau) \in S^k(\mathbf{x}, \sigma)$ satifies $\tau \in \Pi_\sigma^{d-k}$, we obtain, for every k:

$$\rho_{k+1}(\mathbf{x}, \sigma) = \rho_k(\mathbf{x}, \sigma) \oplus \rho_k(\mathbf{x}^{\overset{\sigma}{\leftarrow}\sigma(d-k)}, \sigma) \tag{3}$$

Theorem 4 (Main Result). *A cone $(\mathbf{x}, \sigma)^{\triangleleft}$ occurs in the canonical decomposition of a timed polyhedron P iff $\mathbf{x}$ is a vertex of P satisfying condition*

$$\bigoplus_{(\mathbf{y},\tau) \in S^d(\mathbf{x},\sigma)} c(\mathbf{y}, \tau) = 1 \tag{4}$$

Proof. From (3) we have that $\rho_d(\mathbf{x}, \sigma)$ corresponds to $\bigoplus_{\mathbf{y} \in N_\sigma(\mathbf{x})} c_\sigma(\mathbf{y}, \sigma)$, and hence it indicates the extremity of $(\mathbf{x}, \sigma)$.

It remains to prove that an extreme point is a vertex by induction over d. Case $d = 1$ is immediate by a systematic inspection. Now, for $d \geq 1$, $S(\mathbf{x}, \sigma)$ is the union of $S^+ = S^{d-1}(\mathbf{x}, \sigma)$ and $S^- = (S^{d-1}(\mathbf{x}, \sigma))^{\leftarrow\sigma(1)}$. Observing that these two sets are separated by hyper-plane H of equation $x_{\sigma(1)} = \mathbf{x}_{\sigma(1)}$ and that the sum of c on S^+ and S^- must differ, $\mathbf{x}$ must belong to a facet included in H. Interchanging, if necessary, S^+ and S^-, assume that sum of c on S^+ is 1 (and 0 on S^-). Pushing away coordinate $x_{\sigma(1)}$, the sum of c on S^+ reduces to the extremity condition in dimension $d - 1$. The induction hypothesis says that $\mathbf{x}$ must belong to some edge linearly independent of H. As a consequence $\mathbf{x}$ must be a vertex. □

4 Algorithms on Timed Polyhedra

In this section we describe operations on timed polyhedra using the extreme vertices representation, assuming dimension d fixed, and using n_P to denote represent the number of $(\mathbf{x}, \sigma)$ pairs in the canonical decomposition of P, stored in some fixed order.

Theorem 5 (Boolean operations). *Complementation, symmetric difference and union (or intersection) of timed polyhedra A and B can be computed in time $O(1)$, $O(n_A + n_B)$ and $O(n_A n_B)$ respectively.*

Proof. Exactly as in Theorem 3. □

The two other operations needed for the verification of timed automata are the projection of a timed polyhedron on $x_i = 0$, which corresponds to resetting a clock variable to zero, and the forward time cone which corresponds to the effect of time passage which increases all clock variables uniformly. As in griddy polyhedra we will use the slicing operation for the computation.

Definition 10 (Slice, Projection and Forward Time Cone). *Let P be a timed polyhedron, z an integer in $[0, m)$ and σ a permutation.*

- *The $(i, z) - \sigma$-slice of P is the d-dimensional polyhedron* $J^{\sigma}_{i,z}(P) = P \sqcap \bigsqcup_{\{\mathbf{x}:x_i=z\}} B(\mathbf{x}, \sigma)$.
- *The i-projection of P is the $(d-1)$-dimensional polyhedron* $\pi_{\downarrow i}(P) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_d) : \exists z \ (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots x_d) \in P\}$.
- *The forward time cone of P is the d-dimensional polyhedron* $P^{\Rightarrow} = \{\mathbf{x} : \exists t \geq 0 \ \mathbf{x} - t\mathbf{1} \in P\}$.

These operations are illustrated in Figure 7.

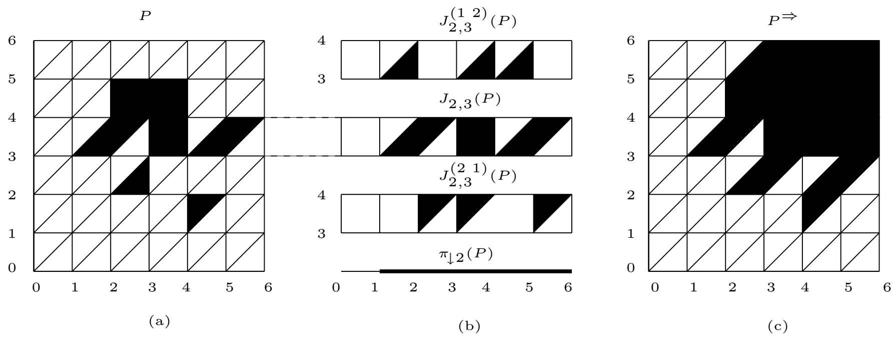


Fig. 7. A timed polyhedron P (a), some of its slices and projections (b) and its forward time cone (c).

Theorem 6 (Computation of Slice, Projection and Time Cone). *Given a timed polyhedron P,*

- *The computation of of $J^{\sigma}_{i,z}(P)$ can be done in time $O(n_P)$.*
- *The computation of $\pi_{\downarrow i}(P)$ can be done is time $O(n_P^3)$.*

– *The computation of $P^{\Rightarrow}$ can be done is time $O(n_P^3)$.*

Sketch of proof: The definition of $J^{\sigma}_{i,z}(P)$ in terms of vertex-permutation pairs is $J^{\sigma}_{i,z}(P) = P \cap \{(\mathbf{x}, \sigma) : x_i = z\}$. Hence the result follows immediately from the identity $(A \oplus B) \sqcap C = (A \sqcap C) \oplus (B \sqcap C)$ which reduces the problem into n_P intersections of cones with a hyper-plane. For every i

$$P = \bigcup_{z \in [0,m]} \bigcup_{\sigma \in \Pi} J^{\sigma}_{i,z}(P),$$

hence, since both projection and time cone distribute over union they can be written as

$$\pi_{\downarrow i}(P) = \bigcup_{z \in [0,m]} \bigcup_{\sigma \in \Pi} \pi_{\downarrow i}(J^{\sigma}_{i,z}(P))$$

and

$$P^{\Rightarrow} = \bigcup_{z \in [0,m]} \bigcup_{\sigma \in \Pi} (J^{\sigma}_{i,z}(P))^{\Rightarrow}.$$

Projection of a slices is trivial and time cone of a slice is obtained by replacing every elementary simplex by a forward cone. □

5 Past, and Future Work

In this paper, we introduced a new canonical representation scheme for timed polyhedra as well as algorithms for performing the operations required for reachability analysis of timed automata. To the best of our knowledge these results are original. The representation of polyhedra is a well-studied problem, but most of the computational geometry and solid modeling literature is concerned only with low dimensional polyhedra (motivated by computer graphics) or with convex polyhedra for which a very nice theory exists. No such theory exist for non-convex polyhedra (see, for example, [A98] and the references therein).

The closest work to ours was that of [AA97,AA98], which we strengthened and generalized to arbitrary dimension in [BMP99] and extended from orthogonal to timed polyhedra in the present paper. The fact that non-convex polyhedra of arbitrary dimension can be represented using a $\oplus$-formula is not new (see for example a recent result in [E95]) but so far only for griddy and timed polyhedra a natural canonical form has been found.

We intend to implement this representation scheme and its corresponding algorithms, as we did for griddy polyhedra, and to see how the performance compares with other existing methods. Although the reachability problem for timed automata is intractable, practically, the manipulated polyhedra might turn out to have few vertices. In addition to the potential practical value we believe that timed polyhedra are interesting mathematical objects whose study leads to a nice theory.

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