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On the restriction of representations of $\mathrm{GL}_2(F)$ to a Borel subgroup

Vytautas Paskunas

Abstract

Let F be a non-Archimedean local field and let p be the residual characteristic of F. Let $G = \mathrm{GL}_2(F)$ and let P be a Borel subgroup of G. In this paper we study the restriction of irreducible smooth representations of G on $\overline{\mathbf{F}}_p$-vector spaces to P. We show that in a certain sense P controls the representation theory of G. We then extend our results to smooth $\mathcal{O}[G]$-modules of finite length and unitary K-Banach space representations of G, where $\mathcal{O}$ is the ring of integers of a complete discretely valued field K with residue field $\overline{\mathbf{F}}_p$.

1. Introduction

Let F be a non-Archimedean local field and let p be the residual characteristic of F. Let $G = \mathrm{GL}_2(F)$ and let P be a Borel subgroup of G. In this paper we study the restriction of smooth irreducible $\overline{\mathbf{F}}_p$-representations of G to P. We show that in a certain sense P controls the representation theory of G. We then extend our results to smooth $\mathcal{O}[G]$-modules of finite length and unitary K-Banach space representations of G, where $\mathcal{O}$ is the ring of integers of a complete discretely valued field K, with residue field $\overline{\mathbf{F}}_p$ and uniformizer ϖ_K.

The study of smooth irreducible $\overline{\mathbf{F}}_p$-representations of G have been initiated by Barthel and Livne in [BL94]. They have shown that smooth irreducible $\overline{\mathbf{F}}_p$-representations of G with central character fall into four classes:

(1) one-dimensional representations $\chi \circ \det$;
(2) (irreducible) principal series $\mathrm{Ind}_P^G(\chi_1 \otimes \chi_2)$, with $\chi_1 \neq \chi_2$;
(3) special series $\mathrm{Sp} \otimes \chi \circ \det$;
(4) supersingular.

Here, Sp is defined by an exact sequence

$$0 \to \mathbf{1} \to \mathrm{Ind}_P^G \mathbf{1} \to \mathrm{Sp} \to 0,$$

and the supersingular representations can be characterised by the fact that they are not subquotients of $\mathrm{Ind}_P^G \chi$ for any smooth character $\chi : P \to \overline{\mathbf{F}}_p^{\times}$. Such representations have only been classified in the case when $F = \mathbf{Q}_p$, by Breuil [Bre03]. If $F \neq \mathbf{Q}_p$ no such classification is known so far, although in a joint work with Breuil [BP07] we can show that there are 'a lot more' supersingular representations than in the case $F = \mathbf{Q}_p$.

The main result of this paper can be summed as follows.

Theorem 1.1. *Let π and π' be smooth $\overline{\mathbf{F}}_p$-representations of G, such that π is irreducible with a central character, then the following hold:*

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(i) *if π is in the principal series, then $\pi|_P$ is of length 2; otherwise $\pi|_P$ is an irreducible representation of P;*

(ii) *we have*

$$\mathrm{Hom}_P(\mathrm{Sp}, \pi') \cong \mathrm{Hom}_G(\mathrm{Ind}_P^G \mathbf{1}, \pi'),$$

and if π is not in the special series, then

$$\mathrm{Hom}_P(\pi, \pi') \cong \mathrm{Hom}_G(\pi, \pi').$$

The first part of this theorem and the second part with π' irreducible are due to Berger [Ber05] in the case $F = \mathbf{Q}_p$. Berger uses the theory of (ϕ, Γ)-modules and the classification of supersingular representations. Our proof is completely different and purely representation theoretic. In fact, this paper grew out of trying to find a simple representation theoretic reason to explain Berger's results. Vigneras [Vig06] has studied the restriction of principal series representation of split reductive p-adic groups to a Borel subgroup. Her results contain the first part of the theorem in the case where π is not supersingular and F arbitrary.

Using the theorem, we extend the result to smooth $\mathcal{O}[G]$ modules of finite length.

Theorem 1.2. *Let π and π' be smooth $\mathcal{O}[G]$ modules, and suppose that π is of finite length and that the irreducible subquotients of π admit a central character. Let $\phi \in \mathrm{Hom}_{\mathcal{O}[P]}(\pi, \pi')$ and suppose that ϕ is not G-equivariant. Let τ be the maximal submodule of π, such that $\phi|_\tau$ is G-equivariant, and let σ be an irreducible G-submodule of π/τ, then*

$$\sigma \cong \mathrm{Sp} \otimes \delta \circ \det,$$

for some smooth character $\delta : F^\times \to \overline{\mathbf{F}}_p^\times$. Moreover, choose $v \in \pi$ such that the image $\overline{v}$ in σ spans σ^{I_1}, then $\Pi\phi(v) - \phi(\Pi v) \neq 0$, $\varpi_K(\Pi\phi(v) - \phi(\Pi v)) = 0$, and

$$g(\Pi\phi(v) - \phi(\Pi v)) = \delta(\det g)(\Pi\phi(v) - \phi(\Pi v)), \quad \forall g \in G,$$

where Π and I_1 are defined in § 2.

This criterion implies the following.

Corollary 1.3. *Let Π_1 and Π_2 be unitary K-Banach space representations of G. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be G-invariant norms defining the topology on Π_1 and Π_2. Set*

$$L_1 = \{v \in \Pi_1 : \|v\|_1 \leqslant 1\}, \quad L_2 = \{v \in \Pi_2 : \|v\|_2 \leqslant 1\}.$$

Suppose that $L_1 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$ is of finite length as an $\mathcal{O}[G]$ module and that the irreducible subquotients admit a central character. Moreover, suppose that if $\mathrm{Sp} \otimes \delta \circ \det$ is a subquotient of $L_1 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$, then $\delta \circ \det$ is not a subobject of $L_2 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$, then

$$\mathcal{L}_G(\Pi_1, \Pi_2) \cong \mathcal{L}_P(\Pi_1, \Pi_2),$$

where $\mathcal{L}(\Pi_1, \Pi_2)$ denotes continuous K-linear maps.

Moreover, Theorem 1.1 implies the following.

Corollary 1.4. *Let Π be a unitary K-Banach space representation of G, let $\|\cdot\|$ be a G-invariant norm defining the topology on Π. Set*

$$L = \{v \in \Pi : \|v\| \leqslant 1\}.$$

Suppose that $L \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$ is a finite length $\mathcal{O}[G]$ module and that the irreducible subquotients are either supersingular or characters, then every closed P-invariant subspace of Π is also G-invariant.

According to Breuil's p-adic Langlands philosophy a two-dimensional p-adic representation of the absolute Galois group of F should be related to a unitary K-Banach space representation of G; see a forthcoming work of Colmez [Col07] for the case $F = \mathbf{Q}_p$, where the restriction to a Borel subgroup plays a prominent role. However, if $F \neq \mathbf{Q}_p$ it is an open problem to construct such unitary K-Banach space representations of G. We hope that our results will help to understand this.

2. Notation

Let $\mathfrak{o}$ be the ring of integers of F, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and let q be the number of elements in the residue field $\mathfrak{o}/\mathfrak{p}$. We fix a uniformiser ϖ and an embedding $\mathfrak{o}/\mathfrak{p} \hookrightarrow \overline{\mathbf{F}}_p$. For $\lambda \in \mathbf{F}_q$ we denote the Teichmüller lift of λ to $\mathfrak{o}$ by $[\lambda]$. Set

$$\Pi = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

Let P be subgroup of upper-triangular matrices in G, T the subgroup of diagonal matrices, $K = \mathrm{GL}_2(\mathfrak{o})$ and

$$I = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}.$$

All of the representations in this paper are on $\overline{\mathbf{F}}_p$-vector spaces, except for § 6.

3. Key

In this section we show how to control the action of s on a supersingular representation π in terms of the action of P. All of the hard work here is done by Barthel and Livne in [BL94], we just record a consequence of their proof of [BL94, Theorem 33].

Let σ be an irreducible representation of K. Let $\tilde{\sigma}$ be a representation of $F^\times K$ such that ϖ acts trivially on $\tilde{\sigma}$ and $\tilde{\sigma}|_K = \sigma$. Set $\mathcal{F}_\sigma = \text{c-Ind}_{F^\times K}^G \tilde{\sigma}$ and $\mathcal{H}_\sigma = \mathrm{End}_G(\mathcal{F}_\sigma)$. It is shown in [BL94, Proposition 8] that as an algebra $\mathcal{H}_\sigma \cong \overline{\mathbf{F}}_p[T]$, for a certain $T \in \mathcal{H}_\sigma$, defined in [BL94, § 3]. Fix $\varphi \in \mathcal{F}_\sigma$ such that $\mathrm{Supp}\,\varphi = F^\times K$ and $\varphi(1)$ spans σ^{I_1}. Since φ generates $\mathcal{F}_\sigma$ as a G-representation, T is determined by $T\varphi$.

Lemma 3.1. *We have the following.*

(i) *If* $\sigma \cong \psi \circ \det$, *for some character* $\psi : \mathfrak{o}^\times \to \overline{\mathbf{F}}_p^\times$, *then*

$$T\varphi = \Pi\varphi + \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi.$$

(ii) *Otherwise,*

$$T\varphi = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi.$$

Proof. In the notation of [BL94] this is a calculation of $T([1, e_{\vec{0}}])$. The claim follows from [BL94, (19)]. □

Let π be a supersingular representation of G, such that ϖ acts trivially. Let $v \in \pi^{I_1}$ and suppose that $\langle K \cdot v \rangle \cong \sigma$. The Frobenius reciprocity gives $\alpha \in \mathrm{Hom}_G(\mathcal{F}_\sigma, \pi)$, such that $\alpha(\varphi) = v$.

Lemma 3.2. *There exists an* $n \geqslant 1$ *such that* $\alpha \circ T^n = 0$.

Proof. Now $\mathrm{Hom}_G(\mathcal{F}_\sigma, \pi)$ is naturally a right $\mathcal{H}_\sigma$-module; let $M = \langle \alpha \cdot \mathcal{H}_\sigma \rangle$ be an $\mathcal{H}_\sigma$-submodule of $\mathrm{Hom}_G(\mathcal{F}_\sigma, \pi)$ generated by α. The proof of [BL94, Proposition 32] implies that $\dim_{\overline{\mathbf{F}}_p} M$ is finite. Let $\overline{T}$ be the image of T in $\mathrm{End}_{\overline{\mathbf{F}}_p}(M)$ and let $m(X)$ be the minimal polynomial of $\overline{T}$. Let $\lambda \in \overline{\mathbf{F}}_p$ be such that $m(\lambda) = 0$, then we may write $m(X) = (X - \lambda)h(X)$. Since $m(X)$ is minimal the composition

$$h(T)(\mathcal{F}_\sigma) \to \mathcal{F}_\sigma \to \pi$$

is non-zero. According to [BL94, Theorem 19], $\mathcal{F}_\sigma$ is a free $\mathcal{H}_\sigma$ module, hence $h(T)$ is an injection and so $h(T)(\mathcal{F}_\sigma)$ is isomorphic to $\mathcal{F}_\sigma$. This implies that π is a quotient of $\mathcal{F}_\sigma/(T - \lambda)$. Since π is supersingular [BL94, Corollary 36] implies that $\lambda = 0$, and hence $m(X) = X^n$, for some $n \geqslant 1$. □

Corollary 3.3. *Let π be a supersingular representation, such that ϖ acts trivially. Let $v \in \pi^{I_1}$ be such that $\langle K \cdot v \rangle$ is an irreducible representation of K. Set $v_0 = v$ and for $i \geqslant 0$ set*

$$v_{i+1} = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_i.$$

Then $v_i \in \pi^{I_1}$ for all $i \geqslant 1$ and there exists an $n \geqslant 1$, such that $v_n = 0$.

Proof. Set $\sigma = \langle K \cdot v \rangle$. If σ is not a character then Lemma 3.1(ii) implies that $v_i = (\alpha \circ T^i)(\varphi)$, for all $i \geqslant 0$ in particular I_1 acts trivially on v_i and the statement follows from Lemma 3.2. If σ is a character, then after twisting we may assume that $\sigma = \mathbf{1}$. Since I acts trivially on Πv_0 the space $\langle K \cdot (\Pi v_0) \rangle$ is a quotient of $\mathrm{Ind}_I^K \mathbf{1}$. Now

$$v_1 = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} s(\Pi v_0).$$

If $v_1 = 0$, then we are done. If $v_1 \neq 0$, then [Pas04, (3.1.7) and (3.1.8)] imply that $\langle K \cdot v_1 \rangle \cong \mathrm{St}$, where St is the inflation of the Steinberg representation of $\mathrm{GL}_2(\mathbf{F}_q)$. We may apply the previous part to v_1. □

Lemma 3.4. *Let π be a smooth representation of G and let $v \in \pi^{I_1}$. Suppose that*

$$\sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v = 0.$$

Then

$$sv = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & \varpi^{-1}[\lambda] \end{pmatrix} v.$$

Proof. Since

$$tv = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} tv$$

we obtain

$$v = -\sum_{\lambda \in \mathbf{F}_q^\times} t^{-1} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} tv = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} 1 & \varpi^{-1}[\lambda] \\ 0 & 1 \end{pmatrix} v.$$

If $\beta \in F^\times$, then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\beta^{-1} & 1 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta^{-1} & 1 \end{pmatrix}. \tag{1}$$

Since $v \in \pi^{I_1}$ and

$$\begin{pmatrix} 1 & 0 \\ \varpi[\lambda] & 1 \end{pmatrix} \in I_1 \quad \forall \lambda \in \mathbf{F}_q^\times$$

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we obtain

$$sv = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & \varpi^{-1}[\lambda] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi[\lambda^{-1}] & 1 \end{pmatrix} v = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & \varpi^{-1}[\lambda] \end{pmatrix} v. \qquad \square$$

Since $G = PI_1 \cup PsI_1$, we use Lemma 3.4 to show that the action of P on π already 'contains all the information' about the action of G on π.

4. Supersingular representations

In this section we study the restriction of supersingular representations of G to a Borel subgroup.

LEMMA 4.1. *Let π be a smooth representation of G and let $v \in \pi^{I_1}$ be non-zero and such that I acts on v via a character χ, then there exists $j \in \{0, \dots, q-1\}$ (usually non-unique) such that*

$$w := \sum_{\lambda \in \mathbf{F}_q} \lambda^j \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} tv$$

is in π^{I_1} and $\langle K \cdot w \rangle$ is an irreducible representation of K.

Proof. Set $\tau = \langle K \cdot (\Pi v) \rangle$. For $0 \leqslant j \leqslant q-1$ set

$$w_j = \sum_{\lambda \in \mathbf{F}_q} \lambda^j \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} s(\Pi v) = \sum_{\lambda \in \mathbf{F}_q} \lambda^j \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} tv.$$

The set $\{\Pi v, w_j : 0 \leqslant j \leqslant q-1\}$ spans τ.

If $w_0 = 0$ then Lemma 3.4 implies that

$$\begin{aligned} \Pi v &= \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} sv = -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & [\lambda] \end{pmatrix} v \\ &= -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} \varpi & [\lambda] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -[\lambda] & 0 \\ 0 & [\lambda^{-1}] \end{pmatrix} v = -\sum_{\lambda \in \mathbf{F}_q^\times} \chi\left(\begin{pmatrix} -[\lambda] & 0 \\ 0 & [\lambda^{-1}] \end{pmatrix}\right) \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} tv. \end{aligned}$$

Since

$$\chi\left(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\lambda^{-1}] \end{pmatrix}\right) = \lambda^r, \quad \forall \lambda \in \mathbf{F}_q^\times$$

for some $0 \leqslant r < q-1$, we obtain that τ is spanned by the set $\{w_j : 1 \leqslant j \leqslant q-1\}$. Let σ be a K-irreducible subrepresentation of τ. The space σ^{I_1} is one dimensional, so I acts on σ^{I_1} by a character. However, one may verify that the group

$$\left\{ \begin{pmatrix} [\lambda] & 0 \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbf{F}_q^\times \right\}$$

acts on the set w_j for $1 \leqslant j \leqslant q-1$ by distinct characters, hence σ^{I_1} is spanned by w_j for some $1 \leqslant j \leqslant q-1$.

Suppose that $w_0 \neq 0$. If w_0 and Πv are linearly independent, then the natural map $\mathrm{Ind}_I^K \chi^s \to \tau$ is an injection, because it induces an injection on $(\mathrm{Ind}_I^K \chi^s)^{I_1}$. It follows from [Pas04, (3.1.5)] that $\langle K \cdot w_0 \rangle$ is an irreducible representation of K. If w_0 and Πv are not linearly independent, then $\chi = \chi^s$. It follows from [Pas04, (3.1.8)] that $\langle K \cdot w_0 \rangle$ is isomorphic to a twist of the Steinberg representation by a character. $\square$

PROPOSITION 4.2. *Let π be a smooth representation of G and let w be a non-zero vector in π. Then there exists a non-zero $v \in \langle P \cdot w \rangle \cap \pi^{I_1}$ such that $\langle K \cdot v \rangle$ is an irreducible representation of K.*

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Proof. Since π is smooth there exists $k \geqslant 0$ such that w is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^{k+1} & 1 \end{smallmatrix}\right)$. Then $w_1 := t^k w$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{smallmatrix}\right)$. Iwahori decomposition gives us

$$I_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ 0 & 1+\mathfrak{p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{pmatrix}.$$

Hence, $\tau := \langle I_1 \cdot w_1 \rangle = \langle (I_1 \cap P) \cdot w_1 \rangle \subseteq \langle P \cdot w \rangle$. Since I_1 is a pro-p group, we have $\tau^{I_1} \neq 0$, and hence $\langle P \cdot w \rangle \cap \pi^{I_1} \neq 0$. Let $w_2 \in \langle P \cdot w \rangle \cap \pi^{I_1} \neq 0$ be non-zero. Since $|I/I_1|$ is prime to p, there exists a smooth character $\chi : I \to \overline{\mathbf{F}}_p^{\times}$ such that

$$w_3 := \sum_{\lambda, \mu \in \mathbf{F}_q^{\times}} \chi\left(\begin{pmatrix} [\lambda^{-1}] & 0 \\ 0 & [\mu^{-1}] \end{pmatrix}\right) \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} w_2$$

is non-zero. As I acts now on w_3 by a character χ we may apply Lemma 4.1 to w_3 to obtain the required vector. □

Theorem 4.3. *Let π be supersingular, then $\pi|_P$ is an irreducible representation of P.*

Proof. Let $w \in \pi$ be non-zero. According to Proposition 4.2 there exists a non-zero $v \in \langle P \cdot w \rangle \cap \pi^{I_1}$, such that $\sigma := \langle K \cdot v \rangle$ is an irreducible representation of K. Corollary 3.3 implies that there exists a non-zero $v' \in \pi^{I_1} \cap \langle P \cdot v \rangle$ such that

$$\sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v' = 0.$$

According to Lemma 3.4 $sv' \in \langle P \cdot v' \rangle$. Since $G = PI_1 \cup PsI_1$ and π is an irreducible G-representation we have

$$\pi = \langle G \cdot v' \rangle = \langle P \cdot v' \rangle \subseteq \langle P \cdot w \rangle.$$

Hence, $\pi = \langle P \cdot w \rangle$ for all $w \in \pi$ and so $\pi|_P$ is irreducible. □

Theorem 4.4. *Let π and π' be smooth representations of G, such that π is supersingular, then*

$$\mathrm{Hom}_P(\pi, \pi') \cong \mathrm{Hom}_G(\pi, \pi').$$

Proof. As $\mathrm{Hom}_G(\pi, \pi') \hookrightarrow \mathrm{Hom}_P(\pi, \pi')$ we only have to prove surjectivity. Let $\phi \in \mathrm{Hom}_P(\pi, \pi')$ be non-zero. We are going to find $v' \in \pi^{I_1}$ such that $\phi(v') \in (\pi')^{I_1}$ and

$$\sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v' = 0, \quad \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t \phi(v') = 0.$$

Choose $v \in \pi^{I_1}$ such that $\langle K \cdot v \rangle$ is an irreducible representation of K. Since $\pi|_P$ is irreducible by Theorem 4.3, ϕ is an injection and hence $\phi(v) \neq 0$. Since v is fixed by I_1 and ϕ is P-equivariant, we have that $\phi(v)$ is fixed by $I_1 \cap P$. Since π' is smooth there exists an integer $k \geqslant 1$ such that $\phi(v)$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^k & 1 \end{smallmatrix}\right)$. Suppose that $k > 1$. Lemma 4.1 implies that there exists j, such that $0 \leqslant j \leqslant q-1$ and if we set

$$v_1 = \sum_{\lambda \in \mathbf{F}_q} \lambda^j \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v,$$

then $v_1 \in \pi^{I_1}$ and $\langle K \cdot v_1 \rangle$ is an irreducible representation of K. Since ϕ is P-equivariant, $\phi(v_1)$ is fixed by $I_1 \cap P$ and

$$\phi(v_1) = \sum_{\lambda \in \mathbf{F}_q} \lambda^j \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t \phi(v).$$

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If $\alpha \in \mathfrak{o}$ and $\beta \in \mathfrak{p}$, then

$$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha(1+\alpha\beta)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1+\alpha\beta)^{-1} & 0 \\ \beta & 1+\alpha\beta \end{pmatrix}.$$

This matrix identity coupled with

$$\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^{k-1} & 1 \end{pmatrix} t = t \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^{k} & 1 \end{pmatrix},$$

implies that $\phi(v_1)$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^{k-1} & 1 \end{smallmatrix}\right)$. By repeating the argument we obtain $w \in \pi^{I_1}$ such that $\langle K \cdot w\rangle$ is an irreducible representation of K and $\phi(w)$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{smallmatrix}\right)$. Iwahori decomposition implies that $\phi(w)$ is fixed by I_1. Set $v_0 = w$ and for $i \geqslant 0$,

$$v_{i+1} = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_i.$$

Since v_i are fixed by I_1, $\phi(v_i)$ are fixed by $I_1 \cap P$. Moreover,

$$\phi(v_{i+1}) = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t \phi(v_i).$$

Since $\phi(v_0)$ is fixed by I_1, the argument used above implies that $\phi(v_{i+1})$ are fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{smallmatrix}\right)$ and hence fixed by I_1. Corollary 3.3 implies that $v_n = 0$ for some $n \geqslant 1$. Let m be the smallest integer such that $v_m = 0$ and set $v' = v_{m-1}$. Then $v' \in \pi^{I_1}$, $\phi(v') \in (\pi')^{I_1}$ and

$$\sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v' = 0, \quad \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t \phi(v') = 0.$$

Lemma 3.4 applied to v' and $\phi(v')$ implies that

$$\begin{aligned} \phi(sv') &= -\phi\bigg(\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & \varpi^{-1}[\lambda] \end{pmatrix} v'\bigg) \\ &= -\sum_{\lambda \in \mathbf{F}_q^\times} \begin{pmatrix} -\varpi[\lambda^{-1}] & 1 \\ 0 & \varpi^{-1}[\lambda] \end{pmatrix} \phi(v') = s\phi(v'). \end{aligned}$$

Since $G = PI_1 \cup PsI_1$ this implies that $\phi(\pi(g)v') = \pi'(g)\phi(v')$, for all $g \in G$. Since π is irreducible $\pi = \langle G \cdot v'\rangle$ and this implies that ϕ is G-equivariant. □

5. Non-supersingular representations

Let $\chi : T \to \overline{\mathbf{F}}_p^\times$ be a smooth character. We consider it as a character of P, via $P \to P/U \cong T$. We define a smooth representation κ_χ of P by the short exact sequence

$$0 \to \kappa_\chi \to \mathrm{Ind}_P^G \chi \to \chi \to 0 \tag{2}$$

where the map on the right is given by the evaluation at the identity. The representation κ_χ is absolutely irreducible by [Vig06, Théorème 5]. If $\chi = \psi \circ \det$ for some smooth character $\psi : F^\times \to \overline{\mathbf{F}}_p^\times$, then the sequence splits as a P-representation and we obtain

$$\mathrm{Sp} \otimes \psi \circ \det|_P \cong \kappa_\chi.$$

Lemma 5.1. *Let π be a smooth representation of G. Suppose that $\mathrm{Hom}_P(\chi, \pi) \neq 0$, then χ extends uniquely to a character of G, and*

$$\mathrm{Hom}_P(\chi, \pi) \cong \mathrm{Hom}_G(\chi, \pi).$$

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Proof. Let $\phi \in \mathrm{Hom}_P(\chi, \pi)$ be non-zero and let v be a basis vector of the underlying vector space of χ. Since π is a smooth representation of G, there exists $k \geqslant 1$ such that $\phi(v)$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^k & 1 \end{smallmatrix}\right)$. Since $t\phi(v) = \phi(tv) = \chi(t)\phi(v)$, we obtain that $\phi(v)$ is fixed by $\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^{k-1} & 1 \end{smallmatrix}\right)$, and by repeating this we obtain that $\phi(v)$ is fixed by sUs. Now sUs and P generate G. This implies the claim. □

Corollary 5.2. *Let π' be a smooth representation of G. Suppose that $\chi \neq \chi^s$ and let $\phi \in \mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \pi')$ be non-zero, then ϕ is an injection.*

Proof. Lemma 5.1 implies that $\mathrm{Hom}_P(\chi, \mathrm{Ind}_P^G \chi) = 0$. Hence, the sequence (2) cannot split. So if $\mathrm{Ker}\,\phi \neq 0$, then $\mathrm{Ker}\,\phi$ contains κ_χ. Hence, ϕ induces a homomorphism $\bar{\phi} \in \mathrm{Hom}_P(\chi, \pi')$. Lemma 5.1 implies that $\bar{\phi} = 0$ and hence $\phi = 0$. □

Corollary 5.3. *Suppose that $\chi \neq \chi^s$, then*

$$\mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \mathrm{Ind}_P^G \chi) \cong \mathrm{Hom}_G(\mathrm{Ind}_P^G \chi, \mathrm{Ind}_P^G \chi).$$

Proof. Suppose that $\phi_1, \phi_2 \in \mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \mathrm{Ind}_P^G \chi)$ are non-zero, then by Corollary 5.2 the restriction of ϕ_1 and ϕ_2 to κ_χ induces non-zero homomorphisms in $\mathrm{Hom}_P(\kappa_\chi, \kappa_\chi)$. Since κ_χ is absolutely irreducible this implies that there exists a scalar $\lambda \in \overline{\mathbf{F}}_p^\times$ such that the restriction of $\phi_1 - \lambda\phi_2$ to κ_χ is zero. Now $\phi_1 - \lambda\phi_2 \in \mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \mathrm{Ind}_P^G \chi)$ and is not an injection, hence by Corollary 5.2 it must be equal to zero. □

Theorem 5.4. *Let π be a smooth representation of G, then the restriction to κ_χ induces an isomorphism*

$$\iota : \mathrm{Hom}_G(\mathrm{Ind}_P^G \chi, \pi) \cong \mathrm{Hom}_P(\kappa_\chi, \pi).$$

Proof. If $\chi \neq \chi^s$, then the injectivity of ι is given by Corollary 5.2. If $\chi = \chi^s$, then the injectivity follows from Lemma 5.1 and [BL94, Theorem 30(1)(b)]. We are going to show that ι is surjective.

Let $\varphi_1 \in \mathrm{Ind}_P^G \chi$ be an I_1 invariant function such that $\mathrm{Supp}\,\varphi_1 = PI_1$ and $\varphi_1(1) = 1$. Set

$$\varphi_2 = \sum_{\lambda \in \mathbf{F}_q} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} s\varphi_1.$$

Then $\{\varphi_1, \varphi_2\}$ is a basis of $(\mathrm{Ind}_P^G \chi)^{I_1}$ and I acts on φ_1 by a character χ and on φ_2 by a character χ^s. Since $G = PK$ we have

$$(\mathrm{Ind}_P^G \chi)^{K_1} \cong \mathrm{Ind}_I^K \chi,$$

as a representation of K, and hence $\sigma = \langle K \cdot \varphi_2 \rangle$ is an irreducible representation of K, which is not a character. We let $F^\times$ act on σ via χ. Frobenius reciprocity gives us a map

$$\alpha : \text{c-}\mathrm{Ind}_{F^\times K}^G \sigma \to \mathrm{Ind}_P^G \chi.$$

It follow from [BL94, Theorem 30(3)] that there exists $\lambda \in \overline{\mathbf{F}}_p^\times$, determined by χ, such that α induces an isomorphism

$$\text{c-}\mathrm{Ind}_{F^\times K}^G \sigma/(T - \lambda) \cong \mathrm{Ind}_P^G \chi,$$

where $T \in \mathrm{End}_G(\text{c-}\mathrm{Ind}_{F^\times K}^G \sigma)$ is as in § 3. Lemma 3.1 implies that

$$\varphi_2 = \lambda^{-1}\biggl(\sum_{\mu \in \mathbf{F}_q} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} t\varphi_2\biggr).$$

Let $\psi \in \mathrm{Hom}_P(\kappa_\chi, \pi)$ be non-zero. Since $\mathrm{Supp}\,\varphi_2 = PsI_1$ we have $\varphi_2(1) = 0$ and hence $\varphi_2 \in \kappa_\chi$. Since κ_χ is irreducible $\psi(\varphi_2) \neq 0$ and the P-equivariance of ψ gives:

$$\psi(\varphi_2) = \lambda^{-1}\biggl(\sum_{\mu \in \mathbf{F}_q} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} t\psi(\varphi_2)\biggr). \quad (3)$$

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This equality coupled with the argument used in the proof of Theorem 4.4 implies that $\psi(\varphi_2)$ is fixed by $\left(\begin{smallmatrix}1 & 0\\ \mathfrak{p} & 1\end{smallmatrix}\right)$. Since ψ is P-equivariant, $\psi(\varphi_2)$ is fixed by $I_1 \cap P$. The Iwahori decomposition implies that $\psi(\varphi_2)$ is fixed by I_1.

So I_1 fixes $\Pi\psi(\varphi_2)$ and I acts on $\Pi\psi(\varphi_2)$ via the character χ. Hence, $\langle K \cdot \Pi\psi(\varphi_2)\rangle$ is a quotient of $\mathrm{Ind}_I^K \chi$. Now

$$\sum_{\mu \in \mathbf{F}_q} \begin{pmatrix}1 & [\mu]\\ 0 & 1\end{pmatrix} s(\Pi\psi(\varphi_2)) = \psi\bigg(\sum_{\mu \in \mathbf{F}_q} \begin{pmatrix}1 & [\mu]\\ 0 & 1\end{pmatrix} t\varphi_2\bigg) = \lambda\psi(\varphi_2) \neq 0. \tag{4}$$

If $\chi|_{T\cap K} \neq \chi^s|_{T\cap K}$, then this implies that $\langle K \cdot \Pi\psi(\varphi_2)\rangle \cong \mathrm{Ind}_I^K \chi$. Equation (4) and [Pas04, (3.1.5)] imply that $\langle K \cdot \psi(\varphi_2)\rangle \cong \sigma$. If $\chi|_{T\cap K} = \psi \circ \det$ for some $\psi : \mathfrak{o}^\times \to \overline{\mathbf{F}}_p^\times$, then the above equality implies that if $\Pi\psi(\varphi_2)$ and $\psi(\varphi_2)$ are linearly independent, then

$$\langle K \cdot \Pi\psi(\varphi_2)\rangle \cong \mathrm{Ind}_I^K \chi,$$

otherwise

$$\langle K \cdot \Pi\psi(\varphi_2)\rangle \cong \mathrm{St} \otimes \psi \circ \det,$$

where St is the lift to K of Steinberg representation of $\mathrm{GL}_2(\mathbf{F}_q)$. In both cases we obtain that $\langle K \cdot \psi(\varphi_2)\rangle \cong \mathrm{St} \otimes \psi \circ \det \cong \sigma$. Hence, $\langle G \cdot \psi(\varphi_2)\rangle$ is a quotient of $\text{c-Ind}_{F^\times K}^G \sigma$. Equation (3) and Lemma 3.1 imply that $\langle G \cdot \psi(\varphi_2)\rangle$ is a quotient of

$$\text{c-Ind}_{F^\times K}^G \sigma/(T-\lambda) \cong \mathrm{Ind}_P^G \chi.$$

Hence, ι is also surjective. $\square$

COROLLARY 5.5. *Suppose that $\chi \neq \chi^s$ and let π be a smooth representation of G, then*

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \chi, \pi) \cong \mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \pi).$$

Proof. Let $\psi \in \mathrm{Hom}_P(\mathrm{Ind}_P^G \chi, \pi)$ be non-zero. It follows from Corollary 5.2 that the composition

$$\mathrm{Ind}_P^G \chi \to \pi \to \pi/\langle G \cdot \psi(\kappa_\chi)\rangle$$

is zero. Hence, the image of ψ is contained in $\langle G \cdot \psi(\kappa_\chi)\rangle$. It follows from Theorem 5.4 applied to $\pi = \langle G \cdot \psi(\kappa_\chi)\rangle$ and the irreducibility of $\mathrm{Ind}_P^G \chi$ that $\mathrm{Ind}_P^G \chi$ is isomorphic to $\langle G \cdot \psi(\kappa_\chi)\rangle$ as a G-representation. The G-equivariance of ψ follows from Corollary 5.3. $\square$

COROLLARY 5.6. *Let π be a smooth representation of G, then*

$$\mathrm{Hom}_P(\mathrm{Sp}, \pi) \cong \mathrm{Hom}_G(\mathrm{Ind}_P^G \mathbf{1}, \pi).$$

Note that $\mathrm{Hom}_G(\mathrm{Sp}, \mathrm{Ind}_P^G \mathbf{1}) = 0$, but $\mathrm{Hom}_G(\mathrm{Ind}_P^G \mathbf{1}, \mathrm{Ind}_P^G \mathbf{1}) \neq 0$, so the above result cannot be improved.

6. Applications

Let K be a complete discrete valuation field, $\mathcal{O}$ the ring of integers and ϖ_K a uniformizer, and we assume that $\mathcal{O}/\varpi_K\mathcal{O} \cong \overline{\mathbf{F}}_p$. We extend the results of previous sections to smooth $\mathcal{O}[G]$ modules of finite length and, after passing to the limit, to unitary K-Banach space representations of G.

THEOREM 6.1. *Let π and π' be smooth $\mathcal{O}[G]$ modules and suppose that π is of finite length and let the irreducible subquotients of π admit a central character. Let $\phi \in \mathrm{Hom}_{\mathcal{O}[P]}(\pi, \pi')$ and suppose that ϕ is not G-equivariant. Let τ be the maximal submodule of π, such that $\phi|_\tau$ is G-equivariant, and let σ be an irreducible G-submodule of π/τ, then*

$$\sigma \cong \mathrm{Sp} \otimes \delta \circ \det,$$

for some smooth character $\delta : F^\times \to \overline{\mathbf{F}}_p^\times$. *Moreover, choose* $v \in \pi$ *such that the image* $\overline{v}$ *in* σ *spans* σ^{I_1}; *then* $\Pi\phi(v) - \phi(\Pi v) \neq 0$, $\varpi_K(\Pi\phi(v) - \phi(\Pi v)) = 0$, *and*

$$g(\Pi\phi(v) - \phi(\Pi v)) = \delta(\det g)(\Pi\phi(v) - \phi(\Pi v)), \quad \forall g \in G.$$

Proof. We denote by $\operatorname{Ind}_1^G \pi'$ the space of smooth functions from G to the underlying $\mathcal{O}$ module of π', equipped with the G action via right translations. Let $\alpha : \pi \to \operatorname{Ind}_1^G \pi'$ be a P-equivariant map, given by

$$[\alpha(w)](g) = g\phi(w) - \phi(gw), \quad \forall w \in \pi, \ \forall g \in G.$$

Then $\tau = \operatorname{Ker} \alpha$. Hence, α induces a P-equivariant map

$$\overline{\alpha} : \sigma \to \operatorname{Ind}_1^G \pi'.$$

Suppose that $\overline{\alpha}$ is G-equivariant, then

$$[g^{-1}\alpha(gv)](1) = [g^{-1}\overline{\alpha}(g\overline{v})](1) = [\overline{\alpha}(\overline{v})](1) = [\alpha(v)](1) = 0.$$

Hence, $g\phi(v) = \phi(gv)$, for all $g \in G$. So the maximality of τ implies that $\overline{\alpha}$ is not G-equivariant. Hence, Theorem 4.4, Lemma 5.1, Corollaries 5.5 and 5.6 imply that

$$\sigma \cong \operatorname{Sp} \otimes \delta \circ \det$$

for some smooth character $\delta : F^\times \to \overline{\mathbf{F}}_p^\times$, and

$$\langle G \cdot \alpha(v) \rangle \cong \operatorname{Ind}_P^G \mathbf{1} \otimes \delta \circ \det.$$

After twisting we may assume that δ is the trivial character. It follows from [BL94, Theorem 30(1)(b)] that

$$\operatorname{Hom}_G(\operatorname{Ind}_P^G \mathbf{1}, \operatorname{Ind}_P^G \mathbf{1}) \cong \overline{\mathbf{F}}_p.$$

Corollary 5.6 applied to $\pi = \operatorname{Ind}_P^G \mathbf{1}$ implies that $\overline{\alpha}(\overline{v})$ is a scalar multiple of the function denoted by φ_2 in the proof of Theorem 5.4. By construction $\alpha(v) = \overline{\alpha}(\overline{v})$. Hence, $\alpha(v)$ is fixed by I_1 and $\Pi\alpha(v) + \alpha(v)$ spans the trivial subrepresentation of G. In particular,

$$[\Pi\alpha(v)](1) + [\alpha(v)](1) = [h\Pi\alpha(v)](1) + [h\alpha(v)](1), \quad \forall h \in P.$$

Since ϕ is P-equivariant, we obtain

$$\Pi\phi(v) - \phi(\Pi v) = h(\Pi\phi(v) - \phi(\Pi v)), \quad \forall h \in P.$$

Suppose that $\Pi\phi(v) = \phi(\Pi v)$. Since $\alpha(v)$ is I_1-invariant we obtain

$$h\Pi u\phi(v) - \phi(h\Pi uv) = [u\alpha(v)](h\Pi) = [\alpha(v)](h\Pi) = h(\Pi\phi(v) - \phi(\Pi v)) = 0,$$

for all $h \in P$ and $u \in I_1$. Also

$$hu\phi(v) - \phi(huv) = [u\alpha(v)](h) = [\alpha(v)](h) = 0, \quad \forall u \in I_1, \ \forall h \in P.$$

Since $G = PI_1 \cup P\Pi I_1$, we obtain that $g\phi(v) = \phi(gv)$, for all $g \in G$, but this contradicts the maximality of τ. So $\Pi\phi(v) - \phi(\Pi v) \neq 0$. Since σ is irreducible $\varpi_K \overline{v} = 0$, and hence

$$[\varpi_K \alpha(v)](\Pi) = \varpi_K(\Pi\phi(v) - \phi(\Pi v)) = 0,$$

so $\mathcal{O}(\Pi\phi(v) - \phi(\Pi v)) = \overline{\mathbf{F}}_p(\Pi\phi(v) - \phi(\Pi v))$. Lemma 5.1 implies that G acts trivially on $\Pi\phi(v) - \phi(\Pi v)$. ∎

Corollary 6.2. *Let* π *and* π' *be as above and suppose that if* $\operatorname{Sp} \otimes \delta \circ \det$ *is a subquotient of* π, *then* $\delta \circ \det$ *is not a subobject of* π'. *Then*

$$\operatorname{Hom}_G(\pi, \pi') \cong \operatorname{Hom}_P(\pi, \pi').$$

DEFINITION 6.3. A unitary K-Banach space representation Π of G is a K-Banach space Π equipped with a K-linear action of G, such that the map $G \times \Pi \to \Pi$, $(g, v) \mapsto gv$ is continuous and such that the topology on Π is given by a G-invariant norm.

COROLLARY 6.4. *Let Π_1 and Π_2 be unitary K-Banach space representations of G. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be G-invariant norms defining the topology on Π_1 and Π_2. Set*

$$L_1 = \{v \in \Pi_1 : \|v\|_1 \leqslant 1\}, \quad L_2 = \{v \in \Pi_2 : \|v\|_2 \leqslant 1\}.$$

Suppose that $L_1 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$ is of finite length as an $\mathcal{O}[G]$ module and the irreducible subquotients admit a central character. Moreover, suppose that if $\mathrm{Sp} \otimes \delta \circ \det$ is a subquotient of $L_1 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$, then $\delta \circ \det$ is not a subobject of $L_2 \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$, then

$$\mathcal{L}_G(\Pi_1, \Pi_2) \cong \mathcal{L}_P(\Pi_1, \Pi_2),$$

where $\mathcal{L}(\Pi_1, \Pi_2)$ denotes continuous K-linear maps.

Proof. Corollary 6.2 implies that for all $k \geqslant 1$ we have

$$\mathrm{Hom}_G(L_1/\varpi_K^k L_1, L_2/\varpi_K^k L_2) \cong \mathrm{Hom}_P(L_1/\varpi_K^k L_1, L_2/\varpi_K^k L_2).$$

Since $\mathrm{Hom}_{\mathcal{O}}(L_1/\varpi_K^k L_1, L_2/\varpi_K^k L_2) \cong \mathrm{Hom}_{\mathcal{O}}(L_1, L_2/\varpi_K^k L_2)$ by passing to the limit we obtain

$$\mathrm{Hom}_G(L_1, L_2) \cong \mathrm{Hom}_P(L_1, L_2).$$

It follows from [Sch01, Proposition 3.1] that

$$\mathcal{L}(\Pi_1, \Pi_2) \cong \mathrm{Hom}_{\mathcal{O}}(L_1, L_2) \otimes_{\mathcal{O}} K.$$

Hence,

$$\mathcal{L}_G(\Pi_1, \Pi_2) \cong \mathrm{Hom}_G(L_1, L_2) \otimes_{\mathcal{O}} K \cong \mathrm{Hom}_P(L_1, L_2) \otimes_{\mathcal{O}} K \cong \mathcal{L}_P(\Pi_1, \Pi_2).$$ □

PROPOSITION 6.5. *Let π be a smooth $\mathcal{O}[G]$ module of finite length and suppose that the irreducible subquotients of π are either supersingular or characters, then every P-invariant $\mathcal{O}$-submodule of π is also G-invariant.*

Proof. Let π' be $\mathcal{O}[P]$ submodule of π. If σ is an irreducible subquotient of π, then by Theorem 4.3 $\sigma|_P$ is also irreducible, hence π and π' are $\mathcal{O}[P]$ submodules of finite length.

Let τ be an irreducible $\mathcal{O}[P]$-submodule of π'. Since π is a finite length $\mathcal{O}[G]$ module, the submodule $\langle G \cdot \tau \rangle$ is of finite length. Let σ be a G-irreducible quotient of $\langle G \cdot \tau \rangle$. Since τ generates $\langle G \cdot \tau \rangle$ as a G-representation, the P-equivariant composition

$$\tau \to \langle G \cdot \tau \rangle \to \sigma$$

is non-zero, and since τ is irreducible, it is an injection. Now $\sigma|_P$ is irreducible, so the above composition is an isomorphism. Theorem 4.4 and Lemma 5.1 imply that τ is G-invariant and isomorphic to σ. By induction on the length of π' as an $\mathcal{O}[P]$-module, π'/τ is a G-invariant $\mathcal{O}$-submodule of π/τ. Since π' is the set of elements of π whose image in π/τ lies in π'/τ, π' is G-invariant. □

COROLLARY 6.6. *Let Π be a unitary K-Banach space representation of G, let $\|\cdot\|$ be a G-invariant norm defining the topology on Π. Set*

$$L = \{v \in \Pi : \|v\| \leqslant 1\}.$$

Suppose that $L \otimes_{\mathcal{O}} \overline{\mathbf{F}}_p$ is a finite length $\mathcal{O}[G]$ module and the irreducible subquotients are either supersingular or characters, then every closed P-invariant subspace of Π is also G-invariant.

Proof. Let Π_1 be a closed P-invariant subspace of Π. Set $M = \Pi_1 \cap L$, then M is an open P-invariant lattice in Π_1. Proposition 6.5 implies that for all $k \geqslant 1$, $M/\varpi_K^k M$ is a G-invariant $\mathcal{O}$-submodule of $L/\varpi_K^k L$. By passing to the limit we obtain that M is a G-invariant $\mathcal{O}$-submodule of L. Since $\Pi_1 = M \otimes_{\mathcal{O}} K$ we obtain the claim. □

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Vytautas Paskunas paskunas@mathematik.uni-bielefeld.de
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

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