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On the Reversion of an Asymptotic Expansion and the Zeros of the Airy Functions*

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Abstract. The general theories of the derivation of inverses of functions from their power series and asymptotic expansions are discussed and compared. The asymptotic theory is applied to obtain asymptotic expansions of the zeros of the Airy functions and their derivatives, and also of the associated values of the functions or derivatives. A Maple code is constructed to generate exactly the coefficients in these expansions. The only limits on the number of coefficients are those imposed by the capacity of the computer being used and the execution time that is available.

The sign patterns of the coefficients suggest open problems pertaining to error bounds for the asymptotic expansions of the zeros and stationary values of the Airy functions.

Key words. Airy functions, asymptotic expansions, error bounds, inversion theorems, Lagrange's reversion theorem, phase principle, principle of the argument, symbolic computation, zeros

AMS subject classifications. 33C10, 41-04, 41A60

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1. Introduction and Summary.

The work described in this paper originated in a project at the National Institute of Standards and Technology to update and extend the well-known *Handbook of Mathematical Functions* published by the National Bureau of Standards in 1964 [5]. Almost all of the special functions covered in this handbook have real and/or complex zeros, and one of the tools for the computation of these zeros is the reversion of asymptotic expansions for the functions.

The construction of inverses of analytic functions is a well-understood branch of complex analysis, and we begin in section 2 by stating the relevant theorems. Less well known is the corresponding theory for functions represented by asymptotic expansions in the neighborhood of a singularity: section 2 continues with a description of available theory for these problems.

In section 3 we show how to apply the theory of section 2 to construct the asymptotic expansions of the real and complex zeros and stationary values of the Airy functions, together with the associated values of the functions or their first derivatives.

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In section 4 we describe results that have been obtained for the Airy functions with the aid of a computer algebra system. The number of coefficients in the asymptotic expansions greatly exceeds the number already known; indeed, the only restraints on the number of coefficients that we are able to generate exactly are the capacity of the computer being used and the time available for computations. At the end of section 4 we discuss the problem of constructing error bounds for the asymptotic expansions, and state open problems for these bounds together with a conjecture.

Our computer algebra code for generating the coefficients in the asymptotic expansions described in sections 3 and 4 is available in the appendix.

2. Inversion and Enumeration Theorems.

THEOREM 2.1. *Let $f(z)$ be analytic at $z = z_0$, $f(z_0) = w_0$, and $f'(z_0) \neq 0$.*[1] *Then the equation $w = f(z)$ has a unique solution $z = F(w)$ such that $F(w_0) = z_0$ and $F(w)$ is analytic at $w = w_0$.*

This is a standard theorem of complex analysis, proofs of which will be found, for example, in [2, Chapter 6] and [12, Chapter 7]. Since $f(z)$ and $F(w)$ are analytic at $z = z_0$ and $w = w_0$ they can be expanded in power series

$$f(z) = f_0 + f_1(z - z_0) + f_2(z - z_0)^2 + \cdots, \tag{2.1}$$

$$F(w) = F_0 + F_1(w - w_0) + F_2(w - w_0)^2 + \cdots \tag{2.2}$$

that converge in the neighborhoods of z_0 and w_0, respectively. Clearly $F_0 = F(w_0) = z_0$, $F_1 = F'(w_0) = 1/f'(z_0) = 1/f_1$, and higher coefficients F_j can be found by equating coefficients in the identity

$$z - z_0 = \sum_{j=1}^{\infty} F_j \Big\{ \sum_{k=1}^{\infty} f_k (z - z_0)^k \Big\}^j. \tag{2.3}$$

For example,

$$F_2 = -\frac{f_2}{f_1^3}, \quad F_3 = \frac{2f_2^2 - f_1 f_3}{f_1^5}, \quad F_4 = \frac{-5f_2^3 + 5f_1 f_2 f_3 - f_1^2 f_4}{f_1^7}. \tag{2.4}$$

A general formula for F_j was found by Lagrange in 1768 [3] by ingenious manipulation of series, as shown in the following theorem.

THEOREM 2.2. *For $j = 1, 2, \ldots,$*

$$F_j = \frac{1}{j!} \Big[\frac{d^{j-1}}{dz^{j-1}} \Big\{ \frac{z - z_0}{f(z) - f_0} \Big\}^j \Big]_{z=z_0}. \tag{2.5}$$

Proofs of Theorem 2.2 in standard complex analysis texts are based on residue theory which did not appear until the early 19th century.

For asymptotic expansions the distinguished point z_0 is most commonly located at infinity, and the typical problem is to determine the inverse of a function $f(z)$ having the expansion

$$f(z) \sim z + f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + \frac{f_3}{z^3} + \cdots \tag{2.6}$$

as $z \to \infty$ in a certain sector in the complex plane. An existence theorem that corresponds to Theorem 2.1 is as follows.

[1] By "$f(z)$ be analytic at $z = z_0$" we mean that $f(z)$ is differentiable in a neighborhood of z_0.

THEOREM 2.3. *Let $f(z)$ be analytic in a domain that includes a closed annular sector $\boldsymbol{S}$ with vertex at the origin*[2] *and angle less than 2π, and let $f(z)$ satisfy* (2.6) *as $z \to \infty$ in $\boldsymbol{S}$ uniformly with respect to* $\operatorname{ph} z$. *Also, let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be closed annular sectors with vertices at the origin, $\boldsymbol{S}_1$ being properly interior to $\boldsymbol{S}$ and $\boldsymbol{S}_2$ being properly interior to $\boldsymbol{S}_1$.*

(i) *If the boundary arcs of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are of sufficiently large radius, then the equation $w = f(z)$ has exactly one root $z = F(w)$, say, in $\boldsymbol{S}_1$ for each $w \in \boldsymbol{S}_2$.*

(ii) *$F(w)$ is analytic within $\boldsymbol{S}_2$.*

(iii) *As $w \to \infty$ in $\boldsymbol{S}_2$,*

$$z \sim w - F_0 - \frac{F_1}{w} - \frac{F_2}{w^2} - \frac{F_3}{w^3} - \cdots, \tag{2.7}$$

where the coefficients F_j are constants.

Part (i) is proved in [8, Chapter 1, section 6]. Part (ii) is a consequence of Theorem 2.1.[3] Part (iii) is proved in [8, Chapter 1, section 8.4].

As in the case of the series expansions (2.1) and (2.2) associated with Theorem 2.1, the coefficients $F_0, F_1, F_2, \ldots$ in (2.7) can be found by successive resubstitutions into (2.7) by means of (2.6) with $f(z) = w$. The first few may be verified to be

$$F_0 = f_0, \quad F_1 = f_1, \quad F_2 = f_0 f_1 + f_2, \quad F_3 = f_0^2 f_1 + f_1^2 + 2 f_0 f_2 + f_3. \tag{2.8}$$

Corresponding to Theorem 2.2 we have the following result.

THEOREM 2.4. *For $j = 1, 2, \ldots$, jF_j is the coefficient of z^{-1} in the asymptotic expansion of $\{f(z)\}^j$.*

Proof. This result is stated without proof in a footnote on p. 22 of [8]. For completeness we supply here an adaptation of Copson's proof of Theorem 2.2 [2, section 6.23].

The process of resubstitution shows that the formulas expressing the F_j in (2.7) in terms of the f_j in (2.6) are the same, whether the asymptotic expansion (2.6) diverges or converges. For the purpose of proving Theorem 2.4 we may therefore assume the latter to be the case for all sufficiently large $|z|$. Let us transform the point at infinity to the origin by means of the substitutions $z = 1/\zeta$, $w = 1/\omega$. Then (2.6) becomes

$$\frac{1}{\omega} = \frac{1}{\zeta} + f_0 + f_1 \zeta + f_2 \zeta^2 + \cdots.$$

Hence

$$\omega = \zeta(1 + f_0 \zeta + f_1 \zeta^2 + f_2 \zeta^3 + \cdots)^{-1} = \phi(\zeta),$$

say. Since the series within the parentheses converges for all sufficiently small values of $|\zeta|$, $\phi(\zeta)$ is analytic at $\zeta = 0$; moreover, $\phi(0) = 0$, $\phi'(0) = 1$.

By Theorem 2.1, when ω lies in a neighborhood $\mathcal{N}$ of the origin, the equation $\phi(t) = \omega$ has a single root $t = \zeta$ such that (a) $\zeta = 0$ when $\omega = 0$; (b) ζ is an analytic function of ω. Let $\mathcal{C}$ be a circle centered at the origin and within $\mathcal{N}$. If $|\omega|$ is sufficiently small, then both ζ and ω lie within $\mathcal{C}$. Hence, by the residue theorem,

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi'(t)}{t\{\phi(t) - \omega\}} dt = \frac{1}{\zeta} + \frac{\phi'(0)}{\phi(0) - \omega} = \frac{1}{\zeta} - \frac{1}{\omega}. \tag{2.9}$$

[2] By a "closed annular sector $\boldsymbol{S}$ with vertex at the origin" we mean a point set z in the complex plane such that $\alpha \le \operatorname{ph} z \le \beta$ and $\rho \le |z| < \infty$, where α, β, and ρ are constants such that $\alpha < \beta$ and $\rho \ge 0$.

[3] From Ritt's theorem [8, Chapter 1, section 4.3] it follows that $f'(z) \to 1$ as $z \to \infty$ in $\boldsymbol{S}_1$.

The last quantity has a removable singularity at $\omega = 0$ and can therefore be expanded in a convergent series of the form

$$\frac{1}{\zeta} - \frac{1}{\omega} = -\sum_{j=0}^{\infty} F_j \omega^j. \tag{2.10}$$

Moreover, since

$$\frac{1}{\zeta} - \frac{1}{\omega} = z - w$$

and asymptotic expansions are unique, the coefficients F_j in (2.10) are identical with those in (2.7).

On differentiating (2.9) and (2.10), then integrating by parts, we find that

$$\sum_{j=1}^{\infty} jF_j \omega^{j-1} = -\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi'(t)}{t\{\phi(t)-\omega\}^2}dt = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{t^2\{\phi(t)-\omega\}}dt.$$

Expanding the last integrand in ascending powers of ω, integrating term by term, and then equating coefficients, we conclude that jF_j is the coefficient of t in the Laurent expansion of $\{\phi(t)\}^{-j}$ in powers of t. Transforming back to the original variables we arrive at Theorem 2.4. $\square$

After an asymptotic series has been reverted, a resulting sequence of zeros may need to be enumerated. The standard analytical tool for this problem is the phase principle, or principle of the argument, proofs of which will be found in complex analysis texts, including those already cited.

THEOREM 2.5 (phase principle). *Let $f(z)$ be analytic on a simply connected domain that contains a simple closed contour $\mathcal{C}$. Assume that the zeros of $f(z)$ are counted according to their multiplicity and that none are on $\mathcal{C}$. Then the number of zeros within $\mathcal{C}$ is $(2\pi)^{-1}$ times the increase in any continuous branch of* $\mathrm{ph}\{f(z)\}$ *as z goes once round $\mathcal{C}$ in the positive sense.*

3. Application to Airy Functions. The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are solutions of the differential equation

$$\frac{d^2w}{dz^2} = zw.$$

Each is an entire function of z. Many properties of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are derived in [4], [7, Appendix], and [8, Chapters 2 and 11]. Each of the functions $\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$, $\mathrm{Bi}(z)$, and $\mathrm{Bi}'(z)$ is known to have an infinite number of zeros on the negative real axis, all of which are simple. In ascending order of absolute value they are denoted by a_s, a'_s, b_s, and b'_s, respectively, with $s = 1, 2, 3, \ldots$.

To find the asymptotic expansions of a_s, a'_s, b_s, and b'_s for large s, we need to revert the compound expansions

$$\mathrm{Ai}(-x) = \frac{1}{\sqrt{\pi}x^{1/4}}\Big\{\cos(\xi - \tfrac{1}{4}\pi)P(\xi) + \sin(\xi - \tfrac{1}{4}\pi)Q(\xi)\Big\}, \tag{3.1}$$

$$\mathrm{Ai}'(-x) = \frac{x^{1/4}}{\sqrt{\pi}}\Big\{\sin(\xi - \tfrac{1}{4}\pi)R(\xi) - \cos(\xi - \tfrac{1}{4}\pi)S(\xi)\Big\}, \tag{3.2}$$

$$\mathrm{Bi}(-x) = \frac{1}{\sqrt{\pi}x^{1/4}}\Big\{-\sin(\xi - \tfrac{1}{4}\pi)P(\xi) + \cos(\xi - \tfrac{1}{4}\pi)Q(\xi)\Big\}, \tag{3.3}$$

$$\mathrm{Bi}'(-x) = \frac{x^{1/4}}{\sqrt{\pi}}\Big\{\cos(\xi - \tfrac{1}{4}\pi)R(\xi) + \sin(\xi - \tfrac{1}{4}\pi)S(\xi)\Big\}, \tag{3.4}$$

where the variable x is real and positive, $\xi = \frac{2}{3}x^{3/2}$, and

$$P(\xi) \sim \sum_{j=0}^{\infty}(-)^j \frac{u_{2j}}{\xi^{2j}}, \qquad Q(\xi) \sim \sum_{j=0}^{\infty}(-)^j \frac{u_{2j+1}}{\xi^{2j+1}}, \tag{3.5}$$

$$R(\xi) \sim \sum_{j=0}^{\infty}(-)^j \frac{v_{2j}}{\xi^{2j}}, \qquad S(\xi) \sim \sum_{j=0}^{\infty}(-)^j \frac{v_{2j+1}}{\xi^{2j+1}} \tag{3.6}$$

as $x \to \infty$. The coefficients u_j, v_j are defined by

$$u_j = \frac{(2j+1)(2j+3)\cdots(6j-1)}{(216)^j j!}, \qquad v_j = -\frac{6j+1}{6j-1}u_j, \qquad j \geq 1, \tag{3.7}$$

with $u_0 = v_0 = 1$.

At a zero of $\mathrm{Ai}(-x)$ we have from (3.1) that

$$\tan(\xi + \tfrac{1}{4}\pi) = \frac{Q(\xi)}{P(\xi)} \sim \frac{5}{72}\frac{1}{\xi} - \frac{39655}{1119744}\frac{1}{\xi^3} + \cdots. \tag{3.8}$$

Hence

$$\xi + \tfrac{1}{4}\pi - s\pi = \arctan\frac{Q(\xi)}{P(\xi)} \sim \frac{5}{72}\frac{1}{\xi} - \frac{1105}{31104}\frac{1}{\xi^3} + \cdots, \tag{3.9}$$

where s now denotes an arbitrary integer. When s is large and positive, we may apply Theorem 2.3,[4] the roles of z and w being played here by ξ and $(s - \frac{1}{4})\pi$. With the aid of (2.8), we derive

$$\xi \sim (s - \tfrac{1}{4})\pi + \frac{5}{72}\frac{1}{(s-\frac{1}{4})\pi} - \frac{1255}{31104}\frac{1}{\{(s-\frac{1}{4})\pi\}^3} + \cdots. \tag{3.10}$$

Since $x = (\frac{3}{2}\xi)^{2/3}$ we conclude that for large s the equation $\mathrm{Ai}(-x) = 0$ is satisfied by

$$x = T(t), \tag{3.11}$$

where

$$t = \tfrac{3}{8}\pi(4s - 1) \tag{3.12}$$

and

$$T(t) \sim t^{2/3}\Big(1 + \frac{5}{48}\frac{1}{t^2} - \frac{5}{36}\frac{1}{t^4} + \cdots\Big), \quad t \to \infty. \tag{3.13}$$

In the foregoing analysis we have used the symbol s in two senses, and it is important to observe that we cannot assume that the expansion given by (3.11)–(3.13) pertains to $x = -a_s$ and not, for example, to $x = -a_{s+1}$. To settle this question, we apply Theorem 2.5.

A suitable choice for the contour $\mathcal{C}$ of Theorem 2.5 is shown in Figure 3.1. The rays OA and OE are given by $\operatorname{ph} z = 2\pi/3$ and $\operatorname{ph} z = -2\pi/3$, respectively. AB

[4]The conditions of Theorem 2.3 are satisfied since $P(\xi)$ and $Q(\xi)$, as defined by (3.1) and (3.3), are analytic in the sector $|\operatorname{ph}\xi| \leq \pi - \delta\,(< \pi)$, and the expansions (3.5) and (3.8) are uniformly valid as $\xi \to \infty$ in this sector.

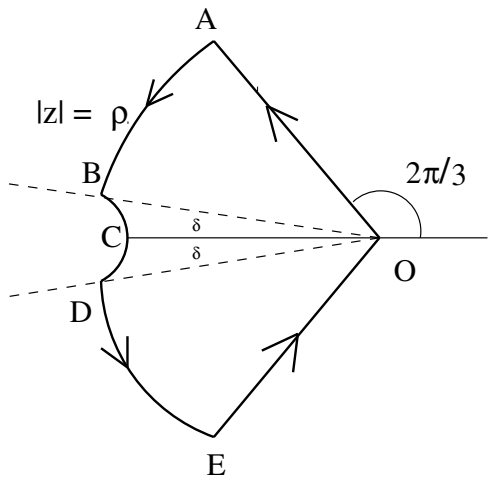

Fig. 3.1 *z-plane. Contour $\mathcal{C}$ for enumerating the zeros of* $\mathrm{Ai}(z)$.

and DE are arcs of a circle $|z| = \rho$, B and D being the points $\rho e^{i(\pi-\delta)}$ and $\rho e^{i(\pi+\delta)}$, respectively, where $\delta(< \pi/3)$ is an arbitrary positive constant. BCD is the curve through B and D on which $\Re\{\frac{2}{3}(-z)^{3/2}\} = \text{const}$. In consequence C is the point $z = -\rho(\cos\frac{3}{2}\delta)^{2/3}$. If we choose ρ to satisfy

$$\tfrac{2}{3}\rho^{3/2}\cos\tfrac{3}{2}\delta = (s + \tfrac{1}{4})\pi,$$

where s is a large positive integer, then by use of the asymptotic and other properties of $\mathrm{Ai}(z)$ on the contour, we can show that as we pass once round $\mathcal{C}$ the change in $\mathrm{ph}\{\mathrm{Ai}(z)\}$ is $2s\pi + o(1)$. (We do not reproduce details of this analysis here since they are similar to those used in an enumeration of the complex zeros of $\mathrm{Bi}(z)$ given in [7, Appendix].) Theorem 2.5 shows immediately that $\mathcal{C}$ contains exactly s zeros of $\mathrm{Ai}(z)$ for all sufficiently large values of s. Since all the zeros of $\mathrm{Ai}(z)$ are known to be real and negative [7, Appendix], we may now set $x = -a_s$ in (3.11)–(3.13).

In reaching the expansion (3.13) we used the binomial theorem for dividing the series in (3.8), the expansion

$$\arctan t = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \cdots$$

for the principal value of the inverse tangent function in (3.9), and the binomial theorem again in forming the two-thirds power of the series (3.10). For deriving the higher coefficients, however, more efficient methods are needed. For example, if

$$p(t) = p_1 t + p_3 t^3 + p_5 t^5 + \cdots,$$

with $p_1 \neq 0$, then

$$\arctan p(t) = q(t) = q_1 t + q_3 t^3 + q_5 t^5 + \cdots,$$

where $q_1 = p_1$ and higher q's can be generated from the recurrence relation obtained by equating the coefficients in the identity

$$p'(t) = \Big[1 + \{p(t)\}^2\Big] q'(t).$$

In practice, however, we take advantage of the availability of comprehensive computer algebra software packages to perform the necessary manipulations of series and to store the wanted coefficients directly in electronic form. We employ the interactive system Maple V [1] for our present purpose.[5] However, since Maple (like other computer packages) is essentially a "black box," we guard against possible programming or algorithmic errors in the package by using inverse operations. For example, after finding the inverse tangent of a series we then compute the tangent of the result to reproduce the original series as a check. Similarly, after reverting a series with the aid of Theorems 2.3 and 2.4, we apply these theorems to the reverted series to recover the original series.

The analysis for determining asymptotic expansions, again for large s, of the zeros a'_s, b_s, and b'_s of the functions $\mathrm{Ai}'(z)$, $\mathrm{Bi}(z)$, and $\mathrm{Bi}'(z)$ is similar and the results are presented in the next section. There are no zeros of $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ in the complex plane other than those on the negative real axis. However, in the sector $\pi/3 < \operatorname{ph} z < \pi/2$ the functions $\mathrm{Bi}(z)$ and $\mathrm{Bi}'(z)$ have infinite sets of zeros β_s and β'_s, respectively, with $s = 1, 2, 3, \ldots$, and conjugate sets of zeros $\overline{\beta}_s$ and $\overline{\beta}'_s$ in the sector $-\pi/2 < \operatorname{ph} z < -\pi/3$. Asymptotic expansions for these zeros are also included in section 4.

In addition to the zeros the associated values $\mathrm{Ai}'(a_s)$, $\mathrm{Ai}(a'_s)$, $\mathrm{Bi}'(b_s)$, $\mathrm{Bi}(b'_s)$, $\mathrm{Bi}'(\beta_s)$, and $\mathrm{Bi}(\beta'_s)$ are of importance in applications. As in the case of the expansion for a_s given by (3.11)–(3.13), the first few terms in the corresponding expansions of $\mathrm{Ai}'(a_s)$ and $\mathrm{Ai}(a'_s)$ can be found by substituting the expansions for a_s and a'_s into (3.2) and (3.1), respectively. One way to generate the higher terms is to use the properties [4, p. B48]

$$\mathrm{Ai}'(a_s) = \frac{(-1)^{s-1}}{\pi M(a_s)}, \qquad \mathrm{Ai}(a'_s) = \frac{(-1)^{s-1}}{\pi N(a'_s)}, \tag{3.14}$$

in which

$$M(x) = \Big\{\mathrm{Ai}^2(x) + \mathrm{Bi}^2(x)\Big\}^{1/2}, \qquad N(x) = \Big\{\mathrm{Ai}'^2(x) + \mathrm{Bi}'^2(x)\Big\}^{1/2}, \tag{3.15}$$

and as $x \to -\infty$,

$$M^2(x) = \frac{1}{\pi(-x)^{1/2}} \sum_{j=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (6j-1)}{(96)^j j!} \frac{1}{x^{3j}}, \tag{3.16}$$

$$N^2(x) = \frac{(-x)^{1/2}}{\pi} \sum_{j=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (6j-1)}{(96)^j j!} \frac{1+6j}{1-6j} \frac{1}{x^{3j}}. \tag{3.17}$$

A second method is to use the properties [6], [7, Appendix]

$$\mathrm{Ai}'(a_s) = (-)^{s-1}\Big(-\frac{da_s}{ds}\Big)^{-1/2}, \qquad \mathrm{Ai}(a'_s) = (-)^{s-1}\Big(a'_s \frac{da'_s}{ds}\Big)^{-1/2}, \tag{3.18}$$

in which s is regarded as a continuous variable. Our Maple code uses the second method, but we also applied the first method as an overall check on the coefficients in all the asymptotic expansions.

[5] Certain commercial equipment, instruments, or materials are identified in this paper to foster understanding. Such identification does not imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the materials or equipment identified are necessarily the best available for the purpose.

4. Results for Zeros and Associated Values of the Airy Functions. Following the notation of [4] and [7, Appendix], we present the desired asymptotic expansions in the forms

$$a_s = -T\{\tfrac{3}{8}\pi(4s-1)\}, \qquad \mathrm{Ai}'(a_s) = (-)^{s-1}V\{\tfrac{3}{8}\pi(4s-1)\},$$
$$a'_s = -U\{\tfrac{3}{8}\pi(4s-3)\}, \qquad \mathrm{Ai}(a'_s) = (-)^{s-1}W\{\tfrac{3}{8}\pi(4s-3)\},$$

$$b_s = -T\{\tfrac{3}{8}\pi(4s-3)\}, \qquad \mathrm{Bi}'(b_s) = (-)^{s-1}V\{\tfrac{3}{8}\pi(4s-3)\},$$
$$b'_s = -U\{\tfrac{3}{8}\pi(4s-1)\}, \qquad \mathrm{Bi}(b'_s) = (-)^{s}W\{\tfrac{3}{8}\pi(4s-1)\},$$

$$\beta_s = e^{i\pi/3}T\{\tfrac{3}{8}\pi(4s-1) + \tfrac{3}{4}i\ln 2\},$$
$$\mathrm{Bi}'(\beta_s) = (-)^s\sqrt{2}e^{-i\pi/6}V\{\tfrac{3}{8}\pi(4s-1) + \tfrac{3}{4}i\ln 2\},$$
$$\beta'_s = e^{i\pi/3}U\{\tfrac{3}{8}\pi(4s-3) + \tfrac{3}{4}i\ln 2\},$$
$$\mathrm{Bi}(\beta'_s) = (-)^{s-1}\sqrt{2}e^{i\pi/6}W\{\tfrac{3}{8}\pi(4s-3) + \tfrac{3}{4}i\ln 2\},$$

in which

$$T(t) \sim t^{2/3}\sum_{j=0}^{\infty}\frac{T_j}{t^{2j}}, \qquad U(t) \sim t^{2/3}\sum_{j=0}^{\infty}\frac{U_j}{t^{2j}},$$
$$V(t) \sim \frac{t^{1/6}}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{V_j}{t^{2j}}, \qquad W(t) \sim \frac{t^{-1/6}}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{W_j}{t^{2j}}.$$

The coefficients T_j, U_j, V_j, and W_j are generated as rational numbers by use of the Maple code given in the appendix. In theory, there is no limit to the number that can be found. In practice, however, the number is limited by the capacity of the computer being used and the execution time. We generated the coefficients T_j, U_j, V_j, and W_j for $j = 0, 1, \ldots, 19$ on a Sun Sparc Station 5 in 25.5 seconds, and for $j = 0, 1, \ldots, 99$ on an SGI Origin 2000 in 3.5 hours. Higher coefficients grow rapidly in magnitude with j. For example, at $j = 100$ the number of decimal digits in the numerator and denominator are 717 and 372, respectively, for T_j, and 788 and 441, respectively, for V_j. For illustration, we record here the first 10 members of each set:

$$T_0 = 1, \quad T_1 = \tfrac{5}{48}, \quad T_2 = -\tfrac{5}{36}, \quad T_3 = \tfrac{77125}{82944}, \quad T_4 = -\tfrac{108056875}{6967296}, \quad T_5 = \tfrac{162375596875}{334430208},$$

$$T_6 = -\tfrac{1622671914671875}{66217181184}, \quad T_7 = \tfrac{150126478779573265625}{82639042117632}, \quad T_8 = -\tfrac{6449327269279398894531 25}{3470839768940544},$$

$$T_9 = \tfrac{1304211699744558907504492187 5}{520200964553048064},$$

$$U_0 = 1, \quad U_1 = -\tfrac{7}{48}, \quad U_2 = \tfrac{35}{288}, \quad U_3 = -\tfrac{181223}{207360}, \quad U_4 = \tfrac{18683371}{1244160},$$

$$U_5 = -\tfrac{91145884361}{191102976}, \quad U_6 = \tfrac{91725210265629647}{3783838924800}, \quad U_7 = -\tfrac{8517284704344771067699}{4722230978150400},$$

$$U_8 = \tfrac{1309491636954247277596 31}{708334646722560}, \quad U_9 = -\tfrac{2078786418470107087898077264845 53}{832321543284876902400 0},$$

Table 4.1 *Terms needed in $T(t)$ for a_s for various relative precisions.*

	Relative precision			
s	10^{-8}	10^{-16}	10^{-32}	10^{-64}
5	3	$*$	$*$	$*$
10	3	7	$*$	$*$
25	2	5	12	46
50	2	4	9	25
75	2	4	8	21
100	2	3	8	19
1000	1	3	5	11
10000	1	2	4	8

Table 4.2 *Terms needed in $V(t)$ for $\mathrm{Ai}'(a_s)$ for various relative precisions.*

	Relative precision			
s	10^{-8}	10^{-16}	10^{-32}	10^{-64}
5	4	$*$	$*$	$*$
10	3	7	$*$	$*$
25	2	5	13	50
50	2	4	10	26
75	2	4	9	22
100	2	4	8	20
1000	1	3	5	11
10000	1	2	4	8

$$V_0 = 1, \quad V_1 = \tfrac{5}{48}, \quad V_2 = -\tfrac{1525}{4608}, \quad V_3 = \tfrac{2397875}{663552}, \quad V_4 = -\tfrac{74898940625}{891813888},$$

$$V_5 = \tfrac{144198303734375}{42807066624}, \quad V_6 = -\tfrac{28089789994850703125}{135612787064832}, \quad V_7 = \tfrac{19888767068290223828125}{1098991936733184},$$

$$V_8 = -\tfrac{484715181260975902241960546875}{227464955097287491584}, \quad V_9 = \tfrac{5433120902332048531433784592597656 25}{1670502630234479338192896},$$

$$W_0 = 1, \quad W_1 = -\tfrac{7}{96}, \quad W_2 = \tfrac{1673}{6144}, \quad W_3 = -\tfrac{84394709}{26542080}, \quad W_4 = \tfrac{780277135421}{10192158720},$$

$$W_5 = -\tfrac{204449051051945}{65229815808}, \quad W_6 = \tfrac{60526598528984532760 69}{30997208471961600},$$

$$W_7 = -\tfrac{66594937359786266452955770 9}{38684516173008076800}, \quad W_8 = \tfrac{10362584000313005713369580323 7}{50785826360461885440},$$

$$W_9 = -\tfrac{68292764972326753800978321499638197424 7}{2181880986428707707027456000}.$$

Previous calculations [4], [7, Appendix] stop at the first six coefficients for $T(t)$ and $U(t)$, and the first four coefficients for $V(t)$ and $W(t)$.

In Tables 4.1 and 4.2 we indicate the number of terms needed in the expansions of a_s and $\mathrm{Ai}'(a_s)$ for $s = 5, 10, 25, 50, 75, 100, 1000, 10000$ to achieve relative precisions[6]

[6] By "relative precision" of two nonzero real or complex numbers a and $\hat{a}$, say, we mean $|\ln(a/\hat{a})|$. When a and $\hat{a}$ are nearly equal this quantity is almost indistinguishable numerically from the relative errors $|(a/\hat{a}) - 1|$ and $|(\hat{a}/a) - 1|$. One advantage of relative precision is that it is a metric. See [9].

of 10^{-8}, 10^{-16}, 10^{-32}, 10^{-64}. Each entry in the tables is the number of terms that ensure the relative contribution of the first neglected term in each series is less than the specified relative precision. An asterisk ($*$) indicates that no term in the series is sufficiently small to yield the desired relative precision. Tables 4.1 and 4.2 were generated in Fortran 90 using the multiple-precision package of [11].

It needs to be stressed that the criterion stated in the preceding paragraph serves only as a guide to the actual accuracy yielded by each expansion. Only if strict and realistic bounds are known for the remainder terms, as opposed to neglected terms, can we be quite certain that the specified precision is attained. At present the only bounds that appear to be available in the literature are those of Pittaluga and Sacripante [10]. For the expansions of a_s and b_s they showed that the Jth error term (that is, the error on stopping the expansion at $j = J - 1$) is bounded by the first neglected term and has the same sign as this term when $J = 1, 2, 3, 4, 5$, and also that the sixth error term has the opposite sign to the fifth term. For the expansions of a'_s and b'_s, they showed that the first terms in each expansion furnish lower bounds.

Our calculations establish that each set of coefficients alternates in sign (except for the second coefficient in the expansions of $T(t)$ and $V(t)$) at least as far as $j = 99$. Consequently, we conjecture that in the expansions of a_s, b_s, a'_s, b'_s, $\mathrm{Ai}'(a_s)$, $\mathrm{Bi}'(b_s)$, $\mathrm{Ai}(a'_s)$, and $\mathrm{Bi}(b'_s)$, the Jth error term is bounded by the first neglected term and has the same sign for all values of $J \geq 1$. This is an open problem, as is the problem of finding error bounds for the asymptotic expansions of the complex zeros and associated values of $\mathrm{Bi}(z)$ and $\mathrm{Bi}'(z)$.

Appendix A. Maple Code.

```
# We define the necessary variables.  N is the number of terms
#    in the asymptotic expansions desired.
#  Must be used with Maple V release 4 and later.
restart:   N:=19:
N2:=2*N+2: u:=vector(N2):  v:=vector(N2):
F:=vector(N2):  T:=vector(N2):  U:=vector(N2):
u[1] := 5/72:  v[1]:=-7/72:
for i from 1 to N2-1 do
   u[i+1]:=(6*i+5)*(6*i+1)/(i+1)/72*u[i]:
   v[i+1]:=-(6*i+7)/(6*i+5)*u[i+1]:
od:
#
#****Construction of T(x)
# Build the polynomials P and Q in Eq. (3.5).
P:=asympt(1+sum('(-1)^i*u[2*i]*1/xi^(2*i)','i'=1..N),xi,N2):
Q:=asympt(sum('(-1)^i*u[2*i+1]*1/xi^(2*i+1)','i'=0..N-1),xi,N2):
# Now construct Eq. (3.9), then rewrite it as s-pi/4 = xi*eq3_9m(xi).
eq3_9:=asympt(arctan(Q/P),xi,N2):
eq3_9m:=asympt(1*xi^0-eq3_9/xi,xi,N2):
# Now we execute Theorem 2.4 on eq3_9m knowing that there is
#     a factor of \xi in front of the equation.
F[1]:=coeff(series(subs(xi=1/zeta,eq3_9m),zeta,N2),zeta,1):
F[2]:=coeff(series(subs(xi=1/zeta,eq3_9m),zeta,N2),zeta,2):
h7:=asympt(eq3_9m,xi,N2):
for j from 3 to N2 do
```

```
   h7:=asympt(h7*eq3_9m,xi,N2):
   F[j]:=coeff(series(subs(xi=1/zeta,h7),zeta,N2),zeta,j)/(j-1):
od:
# We now use the fact that T[0]=1 and that all the even F[j]
#     in Eq. (2.7) vanish.
for j from 1 to N+1 do   T[j]:=F[2*j]: od:
eq3_10:=
   asympt(lambda^0-sum('T[i]/lambda^(2*i)','i'=1..N+1),lambda,N2):
eq3_13:=asympt(subs(lambda=2/3*lambda,eq3_10)^(2/3),lambda,N2);
#
#****Construction of U(x)
# Build the polynomials R and S in Eq.(3.6).
R:=asympt(1+sum('(-1)^i*v[2*i]*1/xi^(2*i)','i'=1..N),xi,N2):
S:=asympt(sum('(-1)^i*v[2*i+1]*1/xi^(2*i+1)','i'=0..N-1),xi,N2):
# Now construct Eq. (3.9) and its modification as before.
eq3_9U:=asympt(arctan(S/R),xi,N2):
eq3_9mU:=asympt(1*xi^0-eq3_9U/xi,xi,N2):
# Now we execute Theorem 2.4 on eq3_9m knowing that there is
#     a factor of \xi in front of the equation.
F[1]:=coeff(series(subs(xi=1/zeta,eq3_9mU),zeta,N2),zeta,1):
F[2]:=coeff(series(subs(xi=1/zeta,eq3_9mU),zeta,N2),zeta,2):
h7:=asympt(eq3_9mU,xi,N2):
for j from 3 to N2 do
   h7:=asympt(h7*eq3_9mU,xi,N2):
   F[j]:=coeff(series(subs(xi=1/zeta,h7),zeta,N2),zeta,j)/(j-1):
od:
# We now use the fact that U[0]=1 and that all the even F[j]
#     in Eq. (2.7) vanish.
for j from 1 to N+1 do  U[j]:=F[2*j]: od:
eq3_10U:=
   asympt(lambda^0-sum('U[i]/lambda^(2*i)','i'=1..N+1),lambda,N2):
eq3_13U:=asympt(subs(lambda=2/3*lambda,eq3_10U)^(2/3),lambda,N2);
#
#****Construction of V(x)
# We implement Eq (3.18) appropriately using the chain rule and
#    factoring out a t^(1/6) in front
eq3_18:=
    asympt(3/2*diff(lambda^(2/3)*eq3_13,lambda),lambda,N2):
V:=asympt(1/sqrt(eq3_18)/lambda^(1/6),lambda,N2);
#
#****Construction of W(x)
# We implement Eq (3.18) appropriately using the chain rule and
#    factoring out a t^(-1/6) in front
eq3_18W:= asympt(3/2*lambda^(2/3)*eq3_13U*
              diff(lambda^(2/3)*eq3_13U,lambda),lambda,N2):
W:=asympt(1/sqrt(eq3_18W)*lambda^(1/6),lambda,N2);
```

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REFERENCES

[1] B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monaghan, and S. M. Watt, *MAPLE V Library Reference Manual*, Springer-Verlag, New York, 1991.

[2] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, London, 1935.

[3] J. L. Lagrange, *Nouvelle méthode pour résoudre les équations littérales par le moyen des séries*, Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin, 24 (1768), pp. 251–326. Reprinted, with changes, in *Oeuvres de Lagrange publiées par le soins de M.J.-A. Serret*, III, pp. 4–73, Gauthier-Villars, Paris, 1869, and Georg Olms Verlag, Hildesheime, 1973.

[4] J. C. P. Miller, *The Airy Integral*, British Assoc. Adv. Sci. Mathematical Tables, Part-Vol. B, Cambridge University Press, Cambridge, UK, 1946.

[5] National Bureau of Standards, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Appl. Math. Ser. 55, 10th printing, M. Abramowitz and I. A. Stegun, eds., U.S. Government Printing Office, Washington, DC, 1972.

[6] F. W. J. Olver, *A new method for the evaluation of zeros of Bessel functions and of other solutions of second-order differential equations*, Proc. Cambridge Philos. Soc., 46 (1950), pp. 570–580.

[7] F. W. J. Olver, *The asymptotic expansion of Bessel functions of large order*, Philos. Trans. Roy. Soc. London Ser. A, 247 (1954), pp. 328–368.

[8] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974; reprinted by A. K. Peters, Wellesley, MA, 1997.

[9] F. W. J. Olver, *A new approach to error arithmetic*, SIAM J. Numer. Anal., 15 (1978), pp. 368–393.

[10] G. Pittaluga and L. Sacripante, *Inequalities for the zeros of the Airy functions*, SIAM J. Math. Anal., 22 (1991), pp. 260–267.

[11] D. A. Smith, *Algorithm* 786: *Multiple-precision complex arithmetic and functions*, ACM Trans. Math. Software, 24 (1998), pp. 359–367.

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, UK, 1927.