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ON THE RICCI CURVATURE OF SUBMANIFOLDS IN THE WARPED PRODUCT $L \times_f F$

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ABSTRACT. The warped product $L \times_f F$ of a line L and a Kaehler manifold F is a typical example of Kenmotsu manifold. In this paper we determine submanifolds of $L \times_f F$ which are tangent to the structure vector field and satisfy certain conditions concerning with Ricci curvature and mean curvature.

1. Fundamental equations on Kenmotsu manifold

A *Kenmotsu manifold* ([7]) is a $(2m+1)$-dimensional Riemannian manifold which has an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \tag{1.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \qquad g(\xi, X) = \eta(X), \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$(\widetilde{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{1.4}$$

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi \tag{1.5}$$

for any vector fields X and Y, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. A typical example of Kenmotsu manifold is the warped product $L \times_f F$, where F is a Kaehler manifold and $f(t) = ce^t$ (c is a nonzero constant) a function on a line L. In fact a Kenmotsu structure (ϕ, ξ, η, g) on $L \times_f F$ is given as follows. Denote by (J, G) the Kaehler structure of F and let $(t, x_1, \cdots, x_{2m})$ be a local coordinate of

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$L \times_f F$ where t and $(x_1, \cdots, x_{2m})$ are the local coordinates of L and F, respectively. We define a Riemannian metric tensor g, a vector field ξ and a 1-form η as follows.

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)G_{(x)} \end{pmatrix},$$

$$\xi = d/dt, \quad \eta(X) = g(X, \xi).$$

We also define a $(1,1)$-tensor field ϕ by

$$\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\phi}_{(t,x)} \end{pmatrix},$$

where

$$\tilde{\phi}_{(t,x)} = (exp(t\xi))_* J_x (exp(-t\xi))_*.$$

Then we can easily verify that the aggregate (ϕ, ξ, η, g) satisfies (1.1)-(1.5) (for more details, see [7]).

We notice that Kenmotsu structure is normal but not Sasakian in the sense of [1, 9, 11] and especially is not compact because of (1.5). Moreover, in order that a Kenmotsu manifold has (point wise) constant ϕ-holomorphic sectional curvature c, it is necessary and sufficient that its curvature tensor $\widetilde{R}$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z = & \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\end{aligned}$$

for any vector fields X, Y, Z ([7]). In the sequel we will denote such a manifold by $\widetilde{M}^{2m+1}(c)$.

REMARK. An example of Kenmotsu manifold with constant ϕ-holom orphic sectional curvature is the warped product $L \times_f F(k)$, where $F(k)$ denotes a Kaehler manifold with constant holomorphic sectional curvature k. Moreover, if a Kenmotsu manifold is a space of constant ϕ-holomorphic sectional curvature c, then it is a space of constant curvature $c = -1$ (for details, see [7]). As already shown in [7, 10] the warped product $L \times_f CE^m$ is a Kenmotsu manifold of constant curvature $c = -1$ whose automorphism group has the maximum dimension, where CE^m denotes the complex Euclidean space with $\dim_C = m$.

2. Fundamental properties on submanifolds of $L \times_f F$

Let M be an n-dimensional submanifold of a Kenmotsu manifold $\widetilde{M}$ in which the structure vector field ξ is tangent to M. Denoting by ∇ and $\nabla^{\perp}$ the induced connections on M and the normal bundle $T^{\perp}M$ of M respectively, we have the equations of Gauss and Weingarten

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.2}$$

for tangent vector fields X, Y and normal vector field N to M, where h and A_N denote the second fundamental form and the shape operator in the direction of N which are related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.3}$$

We first notice that (1.5) and (2.3) yield

$$A_N \xi = 0 \tag{2.4}$$

for any normal vector field N to M since the structure vector field ξ is tangent to M.

For a tangent vector field X and normal vector field N to M, we put

$$\phi X = PX + FX \quad \text{and} \quad \phi N = tN + t^{\perp}N, \tag{2.5}$$

where PX and tN denote the tangential component of ϕX and ϕN, respectively. Then we can easily see that P and $t^{\perp}$ are skew-symmetric endomorphisms acting on T_pM and $T_p^{\perp}M$, respectively. If ϕ maps T_pM into T_pM for each $p \in M$ and the structure vector field ξ is tangent to M, then M is said to be *invariant* in $\widetilde{M}$. On the other side, if ϕ maps T_pM into $T_p^{\perp}M$ for each point $p \in M$ and ξ is tangent to M, then M is said to be *totally real* (or *anti-invariant*) in $\widetilde{M}$ (cf. [11]).

If the Kenmotsu manifold $\widetilde{M}$ has (point wise) constant ϕ-holomorphic sectional curvature c, then the equation of Gauss for M is given by

(2.6)

$$\begin{aligned}
g(R(X,Y)Z,W) &= \frac{c-3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
&+ \frac{c+1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) \\
&- g(Y,Z)\eta(X)\eta(W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
&+ 2g(X,\phi Y)g(\phi Z,W)\} + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W))
\end{aligned}$$

for tangent vector fields X, Y, Z, W to M.

The mean curvature vector field H of M in $\widetilde{M}$ is defined by $H = \frac{1}{n}\text{trace } h$. The Ricci tensor S and the scalar curvature ρ at a point $p \in M$ are given respectively by $S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$ and $\rho = \sum_{i=1}^{n} S(e_i, e_i)$, where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of the tangent space T_pM. For a submanifold M of $\widetilde{M}(c)$, by taking contracting on (2.6) we have the following basic formula:

$$(2.7)\ \ \rho = \frac{(n-1)}{4}\{c(n-2) - 3n - 2\} + \frac{3(c+1)}{4}\|P\|^2 + n^2\|H\|^2 - |h|^2,$$

where $|h|^2$ denotes the squared norm of the second fundamental form.

3. Ricci tensor of submanifolds in Kenmotsu manifold

In his paper [5], Chen proved that there exists a basic inequality on Ricci tensor S for an n-dimensional submanifold M in a real space form $R^m(c)$; namely,

$$S \leq ((n-1)c + \frac{n^2}{4}\|H\|^2)g$$

with the equality holding if and only if either M is a totally geodesic submanifold or $n = 2$ and M is a totally umbilical submanifold.

In this section we will investigate the inequality for an n-dimensional submanifold M of $\widetilde{M}^{2m+1}(c)$ whose structure vector field ξ is tangent to M. In order to do that we need a lemma due to Chen ([2, 3, 4]).

LEMMA C. ([2, 3, 4]) *Let $a_1, \cdots, a_n, d$ be $n+1$ $(n \geq 2)$ real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i{}^2 + d)$$

then $2a_1a_2 \geq d$ with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For a submanifolds M in $\widetilde{M}^{2m+1}(c)$, we have the following.

THEOREM 3.1. *Let M be a submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field ξ is tangent to M. Then the Ricci tensor S of M*

satisfies

(3.1)

$$S(X,X) \leq \frac{n^2\|H\|^2}{4} + \frac{-3n+2}{4} + \frac{3(c+1)}{2}\|PX\|^2 + \frac{(n-2)c}{4} - \frac{(n-2)(c+1)}{4}\eta^2(X)$$

for any unit vector $X \in T_pM$. *The equality holds identically if and only if* M *is totally geodesic in* $\widetilde{M}^{2m+1}(c)$.

Proof. Let M be a submanifold of $\widetilde{M}^{2m+1}(c)$. Then it follows from (2.7) that

(3.2)

$$\rho = \frac{(n-1)(n-2)c}{4} - \frac{(3n+2)(n-1)}{4} + \frac{3(c+1)}{4}\|P\|^2 + n^2\|H\|^2 - |h|^2.$$

We put

(3.3)

$$\delta = \rho - \frac{(n-1)(n-2)c}{4} + \frac{(3n+2)(n-1)}{4} - \frac{3(c+1)}{4}\|P\|^2 - \frac{n^2\|H\|^2}{2}.$$

Then from (3.2) and (3.3) we find

$$n^2\|H\|^2 = 2(\delta + |h|^2). \tag{3.4}$$

Assume that $H \neq 0$. Let $\{e_1, e_2, \cdots, e_{2m+1}\}$ be an orthonormal basis of $T_p\widetilde{M}$ such that

(1) $e_1, \cdots, e_n$ are tangent to M,
(2) $e_{n+1} = \frac{H}{\|H\|}$.

Putting $a_i = h_{ii}^{n+1}, i = 1, \cdots, n$ and using (3.4), we get

(3.5)

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = 2\left\{\delta + \sum_{i=1}^{n}(a_i)^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i,j\leq n}(h_{ij}^r)^2\right\}.$$

Equation (3.5) is equivalent to

(3.6)

$$\Big(\sum_{i=1}^{3}\bar{a}_i\Big)^2 = 2\left\{\delta + \sum_{i=1}^{3}(\bar{a}_i)^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i,j\leq n}(h_{ij}^r)^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta\right\},$$

where $\bar{a}_1 = a_1, \bar{a}_2 = a_2 + a_3 + \cdots + a_{n-1}, \bar{a}_3 = a_n$. Applying Lemma C to (3.6) (for $n = 3$), we have $2\bar{a}_1\bar{a}_2 \geq d$ with equality holding if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$, where we put

$$d = \delta + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1 \leq i,j \leq n} (h_{ij}^r)^2 - \sum_{2 \leq \alpha \neq \beta \leq n-1} a_\alpha a_\beta.$$

This inequality is equivalent to

$$\sum_{1 \leq \alpha \neq \beta \leq n-1} a_\alpha a_\beta \geq \delta + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1 \leq i,j \leq n} (h_{ij}^r)^2,$$

which yields, by (3.3)

(3.7)
$$\begin{aligned} &\frac{(n-1)(n-2)c}{4} - \frac{(3n+2)(n-1)}{4} + \frac{3(c+1)}{4}\|P\|^2 + \frac{n^2\|H\|^2}{2} \\ &\geq \rho - \sum_{1 \leq \alpha \neq \beta \leq n-1} a_\alpha a_\beta + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1 \leq i,j \leq n} (h_{ij}^r)^2. \end{aligned}$$

Using (2.6), we have

(3.8)
$$\begin{aligned} &\rho - \sum_{1 \leq \alpha \neq \beta \leq n-1} a_\alpha a_\beta + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1 \leq i,j \leq n} (h_{ij}^r)^2 \\ = &\sum_{1 \leq i \neq j \leq n-1} R(e_i, e_j, e_j, e_i) + 2S(e_n, e_n) - \sum_{1 \leq \alpha \neq \beta \leq n-1} a_\alpha a_\beta \\ &+ 2 \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1 \leq i,j \leq n} (h_{ij}^r)^2 \\ = &\frac{(n-1)(n-2)c}{4} - \frac{3(n-1)(n-2)}{4} - \frac{(c+1)(2n-4)}{4} + 2S(e_n, e_n) \\ &+ \frac{(c+1)(2n-4)}{4}\eta^2(e_n) + \frac{3(c+1)}{4}\|P\|^2 - \frac{3(c+1)}{2}\|Pe_n\|^2 \\ &+ 2\sum_{i=1}^{n-1} (h_{in}^{n+1})^2 + \sum_{r=n+2} \Big\{ (h_{nn}^r)^2 + 2\sum_{i=1}^{n-1} (h_{in}^r)^2 + (\sum_{\alpha=1}^{n-1} h_{\alpha\alpha}^r)^2 \Big\}. \end{aligned}$$

Combining (3.7) and (3.8) yields

(3.9)

$$S(e_n,e_n)+\sum_{1\le i<n}(h_{in}^{n+1})^2+\sum_{r=n+2}\Big[\sum_{i=1}^{n-1}(h_{in}^r)^2+\frac{1}{2}\{(h_{nn}^r)^2+(\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r)^2\}\Big]$$
$$\le \frac{n^2\|H\|^2}{4}+\frac{-6n+4}{8}+\frac{3(c+1)}{4}\|Pe_n\|^2+\frac{(2n-4)}{8}\{c-(c+1)\eta^2(e_n)\}$$

and consequently

(3.10)

$$\begin{aligned}S(e_n,e_n)\ \le\ &\frac{n^2\|H\|^2}{4}+\frac{-3n+2}{4}+\frac{3(c+1)}{4}\|Pe_n\|^2\\&+\frac{(n-2)c}{4}-\frac{(n-2)(c+1)}{4}\eta^2(e_n).\end{aligned}$$

Moreover, it is clear from (3.9) that the equality holds if and only if

(3.11)

$$h_{\alpha n}^{n+1}=0,\quad h_{in}^r=0,\quad \sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r=0$$
$$\text{for } 1\le\alpha\le n-1,\ 1\le i\le n,\ n+2\le r\le 2m+1.$$

Since Lemma C yields that $2\bar{a}_1\bar{a}_2=d$ if and only if $\bar{a}_1+\bar{a}_2=\bar{a}_3$, (3.6) also implies that the equality holds if and only if $\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^{n+1}=h_{nn}^{n+1}$. Since e_n can be any unit tangent vector of M^n, (3.10) implies the inequality (3.1). Now, assume that for all unit tangent vector e_i the equality sign of (3.1) holds identically. Then we have

$$\begin{aligned}&h_{ij}^{n+1}=0\quad(1\le i\ne j\le n),\\&h_{ij}^r=0\quad(1\le i,j\le n,n+2\le r\le 2m+1),\\&\sum_{k\ne i}h_{kk}^{n+1}=h_{ii}^{n+1},\end{aligned}$$

from which together with (2.4), we conclude that M is totally geodesic.□

COROLLARY 3.2. *Let M be a totally real submanifold of $\widetilde{M}^{2m+1}(c)$. Then the Ricci tensor S of M satisfies*

$$S(X,X) \leq \frac{n^2\|H\|^2}{4} + \frac{-3n+2}{4} + \frac{(n-2)c}{4} - \frac{(n-2)(c+1)}{4}\eta^2(X)$$

for any unit vector $X \in T_pM$. The equality holds identically if and only if M is totally geodesic in $\widetilde{M}^{2m+1}(c)$.

COROLLARY 3.3. *Let M be a submanifold of the warped product $L \times_f CE^m$ whose structure vector field ξ is tangent to M. Then the Ricci tensor S of M satisfies*

$$S \leq (-n+1+\frac{n^2\|H\|^2}{4})g. \tag{3.12}$$

The equality holds identically if and only if M is totally geodesic in $L \times_f CE^m$.

4. Ricci curvature and squared mean curvature

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \cdots, e_k\}$ of L such that $e_1=X$. Define the Ricci curvature Ric_L of L at X by

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. Such a curvature is simply called a *k-Ricci curvature* ([5]). The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}.$$

For each integer k, $2 \leq k \leq n$, two Riemannian invariants θ_k, $\bar{\theta}_k$ on the n-dimensional Riemannian manifold M is defined by

$$\theta_k(p) = \frac{1}{k-1}\inf_{L,X} Ric_L(X), \ p \in M, \tag{4.1}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

$$\bar{\theta}_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \ p \in M, \tag{4.2}$$

where L runs over all k-plane sections in T_pM which is orthogonal to ξ and X runs over all unit vectors in L.

For a submanifold M in a Riemannian manifold the relative null space of M at p is defined by

$$N_p = \{X \in T_pM | h(X, Y) = 0 \text{ for all } Y \in T_pM\}. \tag{4.3}$$

Recently Chen ([5]) established a relationship between *k-Ricci curvature* and the squared mean curvature for submanifold in a real space form. In this section we investigate *k-Ricci curvature* for submanifold of Kenmotsu manifold with constant ϕ-holomorphic sectional curvature whose structure vector field ξ is tangent to the submanifold.

THEOREM 4.1. *Let M be an n-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ $(c \le -1)$ whose structure vector field ξ is tangent to M. Then*

(1) *For each unit vector $X \in T_pM$, we have*

$$\begin{aligned} \|H\|^2 \ge & \frac{4}{n^2}\{Ric(X) + (n-1) - \frac{3(c+1)}{4}\|PX\|^2 \\ & - \frac{(n-2)(c+1)}{4} + \frac{(n-2)(c+1)}{4}\eta^2(X)\}. \end{aligned} \tag{4.4}$$

(2) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (4.4) if and only if $X \in N_p$.*

(3) *The equality case of (4.4) holds identically for all unit tangent vector at p if and only if p is a totally geodesic point.*

Proof. (1) Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for T_pM and $\{e_{n+1}, \cdots, e_{2m+1}\}$ for $T_p{}^{\perp}M$ with $e_1 = X$. Then, from (2.7), we have

(4.5)

$$n^2\|H\|^2 = \rho + |h|^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

It follows from (4.5) that

(4.6)

$$
\begin{aligned}
n^2\|H\|^2 &= \rho + \sum_{r=n+1}^{2m+1} \{(h_{11}^r)^2 + (h_{22}{}^r + \cdots + h_{nn}^r)^2 + 2\sum_{1\le i<j\le n}(h_{ij}^r)^2\} \\
&\quad - \frac{3(c+1)}{4}\|P\|^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r \\
&\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4} \\
&= \rho + \frac{1}{2}\sum_{r=n+1}^{2m+1}\{(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\} \\
&\quad + 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{3(c+1)}{4}\|P\|^2 \\
&\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.
\end{aligned}
$$

On the other hand, (2.6) implies

(4.7)

$$
\begin{aligned}
K_{ij} &= \sum_{r=n+1}^{2m+1}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\} + \frac{c-3}{4} \\
&\quad - \frac{c+1}{4}\{\eta^2(e_i) + \eta^2(e_j)\} + \frac{3(c+1)}{4} g^2(e_i, \phi e_j)
\end{aligned}
$$

and consequently

(4.8)

$$
\begin{aligned}
\sum_{2\le i<j\le n} K_{ij} &= \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\} + \frac{(n-1)(n-2)(c-3)}{8} \\
&\quad + \frac{3(c+1)}{8}\|P\|^2 - \frac{3(c+1)}{4}\|Pe_1\|^2 \\
&\quad - \frac{(c+1)(n-2)}{4}\{1 - \eta^2(e_1)\}.
\end{aligned}
$$

Taking account of (4.8) into (4.6), we get

(4.9)

$$\begin{aligned} n^2\|H\|^2 \geq{}& \rho + \frac{1}{2}\sum_{r=n+1}^{2m+1}(h_{11}{}^r + h_{22}{}^r + \ldots + h^r_{nn})^2 - 2\sum_{2\leq i<j\leq n} K_{ij} \\ &+ \frac{(n-1)(n-2)(c-3)}{4} + \frac{3(c+1)}{4}\|P\|^2 - \frac{3(c+1)}{2}\|Pe_1\|^2 \\ &- \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-1)(n-2)c}{4} + \frac{(3n+2)(n-1)}{4} \\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(h^r_{1j})^2 - \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\} \\ \geq{}& \rho + \frac{1}{2}n^2\|H\|^2 - 2\sum_{2\leq i<j\leq n} K_{ij} + 2(n-1) - \frac{3(c+1)}{2}\|Pe_1\|^2 \\ &- \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{1}{2}n^2\|H\|^2 \geq{}& \rho - 2\sum_{2\leq i<j\leq n} K_{ij} + 2(n-1) - \frac{3(c+1)}{2}\|Pe_1\|^2 \\ &- \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\}, \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{2}n^2\|H\|^2 \geq{}& 2Ric(X) - \frac{3(c+1)}{2}\|PX\|^2 + 2(n-1) \\ &- \frac{(c+1)(n-2)}{2}\{1-\eta^2(X)\}. \end{aligned}$$

(2) Assume that $H(p) = 0$. The equality holds in (4.4) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, \quad r = n+1, \cdots, 2m+1. \end{cases}$$

Then $h^r_{1j} = 0$, for all $j = 1, 2, \cdots, n$ and $r = n+1, \cdots, 2m+1$, which means that $X \in N_p$.

(3) The equality case of (4.4) holds for all unit tangent vector at p if and only if

$$\begin{cases} h^r_{ij} = 0,\ i \neq j \quad \text{and} \quad r = n+1, \cdots, 2m+1, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0,\ i = 1, \cdots, n \text{ and } r = n+1, \cdots, 2m+1, \end{cases}$$

which implies by (2.4) that p is a totally geodesic point. □

THEOREM 4.2. *Let M be an n-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field ξ is tangent to M. Then*

$$\|H\|^2 \geq \frac{\rho}{n(n-1)} - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n+2}{4n}. \tag{4.10}$$

Proof. Let $p \in M$ and let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for T_pM. From (2.6) we have

(4.11)

$$n^2\|H\|^2 = \rho + |h|^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

We choose an orthonormal basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots e_{2m+1}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \cdots, e_n$ diagonalize the shape operator A_{n+1}, then

$$A_{n+1} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

$$A_r = (h^r_{ij}) \quad \text{with} \quad \text{trace} A_r = 0, \quad r = n+2, \cdots, 2m+1,$$

which and (4.11) imply

(4.12)

$$n^2\|H\|^2 = \rho + \sum_{i=1}^{n} {a_i}^2 + \sum_{r=n+1}^{2m+1} \sum_{1 \leq i \neq j \leq n} (h^r_{ij})^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

On the other hand

$$0 \leq \sum_{1\leq i<j\leq n} (a_i - a_j)^2 = (n-1) \sum_{1\leq i\leq n} {a_i}^2 - 2 \sum_{1\leq i<j\leq n} a_i a_j,$$

which yields

$$n^2\|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} {a_i}^2 + 2 \sum_{1\leq i<j\leq n} a_i a_j \leq n \sum_{i=1}^{n} {a_i}^2,$$

which implies $\sum_{i=1}^{n} {a_i}^2 \geq n\|H\|^2$. Thus we have from(4.12)

$$\begin{aligned} n^2\|H\|^2 &\geq \rho + \sum_{i=1}^{n} {a_i}^2 - \frac{3(c+1)}{4}\|P\|^2 \\ &\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4} \\ &\geq \rho + n\|H\|^2 - \frac{3(c+1)}{4}\|P\|^2 \\ &\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}, \end{aligned}$$

or equivalently

$$\|H\|^2 \geq \frac{\rho}{n(n-1)} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 - \frac{(n-2)c}{4n} + \frac{(3n+2)}{4n}.$$

□

Corollary 4.3. *Let M be an $n(\geq 2)$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)(c \leq -1)$ whose structure vector field ξ is tangent to M. Then*

$$\|H\|^2 \geq \frac{1}{n^2}\{\rho + |h|^2 + n(n-1)\}.$$

The equality holds identically if and only if either $c = -1$ or $n = 2$ and M is totally real in the ambient manifold.

Proof. (4.11) says

$$\|H\|^2 = \frac{1}{n^2}\Big\{\rho+|h|^2-\frac{3(c+1)}{4}\|P\|^2-\frac{(n-2)(n-1)(c+1)}{4}+n(n-1)\Big\}$$

and consequently

$$\|H\|^2 \geq \frac{1}{n^2}\{\rho+|h|^2+n(n-1)\}$$

since $c \leq -1$. □

THEOREM 4.4. *Let M be an n-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field ξ is tangent to M. Then for any integer k, $2 \leq k \leq n$ and any point $p \in M$ we have*

$$\|H\|^2(p) \geq \theta_k(p) - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n+2}{4n}. \tag{4.13}$$

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for T_pM. Denoting by $L_{i_1,\cdots,i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$, we have

$$\tau(L_{i_1,\cdots,i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\cdots,i_k\}} Ric_{L_{i_1,\cdots,i_k}}(e_i), \tag{4.14}$$

$$\frac{1}{2}\rho(p) = \frac{1}{{}_{n-2}C_{k-2}}\sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1,\cdots,i_k}). \tag{4.15}$$

Combining (4.1), (4.14) and (4.15), we obtain

$$\frac{1}{2}\rho(p) \geq \frac{n(n-1)}{2}\theta_k(p),$$

which together with (4.10) yields (4.13). □

THEOREM 4.5. *Let M be an n-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field ξ is tangent to M. Then for any integer k, $2 \leq k \leq n$ and any point $p \in M$ we have*

$$\|H\|^2(p) \geq \frac{n-1}{n}\bar{\theta}_k(p) - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n-6}{4n}. \tag{4.16}$$

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for T_pM. Denoting by $L_{i_1,\cdots,i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$, we have

$$\tau(L_{i_1,\cdots,i_k}) = \frac{1}{2} \sum_{i\in\{i_1,\cdots,i_k\}} Ric_{L_{i_1,\cdots,i_k}}(e_i), \tag{4.17}$$

$$\frac{1}{2}\rho(p) = \frac{1}{{}_{n-2}C_{k-2}} \sum_{1\le i_1<\cdots<i_k\le n} \tau(L_{i_1,\cdots,i_k}). \tag{4.18}$$

Combining (4.2), (4.17) and (4.18), we find

$$\frac{1}{2}\rho(p) \geq -(n-1) + \frac{(n-1)^2}{2}\bar{\theta}_k(p),$$

which together with (4.10) implies (4.16). □

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