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ON THE RIEMANN ZETA–FUNCTION AND THE DIVISOR PROBLEM III.

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Dedicated to Prof. Imre Kátai
on the occasion of his seventieth birthday

Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2}+it)|$. If $E^*(t) = E(t) - 2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)$ and we set

$$\int_0^T E^*(t)dt = 3\pi T/4 + R(T),$$

then we obtain

$$R(T) = O_\varepsilon\left(T^{593/912+\varepsilon}\right), \quad \int_0^T R^4(t)dt \ll_\varepsilon T^{3+\varepsilon}$$

and

$$\int_0^T R^2(t)dt = T^2 P_3(\log T) + O_\varepsilon\left(T^{11/6+\varepsilon}\right),$$

where $P_3(y)$ is a cubic polynomial in y with positive leading coefficient.

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1. Introduction and statement of results

This paper is a continuation of the author's works [5], [6], where the analogy between the Riemann zeta-function $\zeta(s)$ and the divisor problem was investigated. As usual, let the error term in the classical Dirichlet divisor problem be

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1), \tag{1.1}$$

and

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt - T\left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right), \tag{1.2}$$

where $d(n)$ is the number of divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215...$ is Euler's constant. In view of F.V. Atkinson's classical explicit formula for $E(T)$ (see [1] and [3, Chapter 15]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [7], [8] and T. Meurman [10]), where

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x) = \frac{1}{2}\sum_{n\le 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1), \tag{1.3}$$

which is a better analogue of $E(T)$ than $\Delta(x)$. M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right),$$

and in particular in [8] he proved that

$$\int_0^T (E^*(t))^2 dt \ll T^{4/3}\log^3 T. \tag{1.4}$$

In the first part of the author's work [5] the bound in (1.4) was complemented with the new bound

$$\int_0^T (E^*(t))^4 dt \ll_\varepsilon T^{16/9+\varepsilon}; \tag{1.5}$$

neither (1.4) or (1.5) seem to imply each other. Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ (same as $a = O_\varepsilon(b)$) means that the $\ll$-constant depends on ε. In the second part of the same work (op. cit.) it was proved that

$$\int_0^T |E^*(t)|^5 dt \ll_\varepsilon T^{2+\varepsilon}, \tag{1.6}$$

and some further results on higher moments of $|E^*(t)|$ were obtained as well. In [6] the author sharpened (1.4) to

$$\int_0^T (E^*(t))^2 dt = T^{4/3} P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{1.7}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be evaluated explicitly.

The aim of the present work is to investigate the integral of $E^*(t)$. More precisely, we define the error-term function $R(T)$ by the relation

$$\int_0^T E^*(t) dt = \frac{3\pi}{4} T + R(T). \tag{1.8}$$

We have (see [2], [4] for the first formula and [14] for the second one)

$$\int_0^T E(t) dt = \pi T + G(T), \qquad \int_0^T \Delta(t) dt = \frac{T}{4} + H(T), \tag{1.9}$$

where both $G(T)$, $H(T)$ are $O(T^{3/4})$ and also $\Omega_\pm(T^{3/4})$ (for $g(x) > 0$ ($x > > x_0$) $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$,

$f(x) = \Omega_\pm(g(x)))$ means that there are unbounded sequences $\{x_n\}$, $\{y_n\}$, and constants $A, B > 0$ such that $f(x_n) > Ag(x_n)$ $f(y_n) < -Bg(y_n)$. Since

$$\int_0^T \Delta(at)dt = \frac{1}{a}\int_0^{aT} \Delta(x)dx \quad (a > 0, \quad T > 0) \tag{1.10}$$

holds, it is obvious from (1.3), (1.9) and (1.10) that $\frac{3\pi}{4}$ is the "correct" constant in (1.8), and that trivially one has the bound $R(T) = O(T^{3/4})$, so that the problem is to improve it. We shall prove

Theorem 1. *We have*

$$R(T) = O_\varepsilon(T^{593/912+\varepsilon}), \quad \frac{593}{912} = 0.6502129\ldots. \tag{1.11}$$

Theorem 2. *We have*

$$\int_0^T R^2(t)dt = T^2 P_3(\log T) + O_\varepsilon(T^{11/6+\varepsilon}), \tag{1.12}$$

where $P_3(y)$ is a cubic polynomial in y with positive leading coefficient, whose all coefficients may be explicitly evaluated.

The asymptotic formula (1.12) bears resemblance to (1.7), and it is proved by a similar technique. The exponents in the error terms are, in both cases, less than the exponent of T in the main term by 1/6. This comes from the use of (2.9) of Lemma 2.5, and in both cases the exponent of the error term is the limit of the method. From (1.7) one obtains that $E^*(T) = \Omega(T^{1/6}(\log T)^{3/2})$, which shows that $E^*(T)$ cannot be too small. Likewise, (1.7) yields the following

Corollary. *We have*

$$R(T) = \Omega\left(T^{1/2}(\log T)^{3/2}\right). \tag{1.13}$$

Theorem 3. *We have*

$$\int_0^T R^4(t)dt \ll_\varepsilon T^{3+\varepsilon}. \tag{1.14}$$

It is rather difficult to ascertain the true maximum order of magnitude of $R(T)$, but the omega-result (1.13) makes it reasonable to believe that maybe it is $T^{1/2+o(1)}$ $(T \to \infty)$. It also seems reasonable to conjecture that

$$R(T) = O_\varepsilon(T^{1/2+\varepsilon}) \tag{1.15}$$

holds. If (1.15) is true, then from Lemma 3, taking $H = T^{1/4}$, it would follow that

$$E^*(T) \ll_\varepsilon T^{1/4+\varepsilon} \tag{1.16}$$

or equivalently

$$E(T) = 2\pi\Delta^*\left(\frac{T}{2\pi}\right) + O_\varepsilon(T^{1/4+\varepsilon}). \tag{1.17}$$

By [4, Theorem 1.2] and (1.17) we have

$$\left|\zeta\left(\frac{1}{2}+iT\right)\right|^2 \ll \log T \int_{T-1}^{T+1} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt + 1 \ll$$
$$\ll \log T\,(\log T + E(T+1) - E(T-1)) \ll_\varepsilon$$
$$\ll_\varepsilon \log T\left(\log T + 2\pi\Delta^*\left(\frac{T+1}{2\pi}\right) - 2\pi\Delta^*\left(\frac{T-1}{2\pi}\right)\right) + T^{1/4+\varepsilon} \ll_\varepsilon T^{1/4+\varepsilon},$$

since, from (1.3) and $d(n) \ll_\varepsilon n^\varepsilon$,

$$\Delta^*(T+H) - \Delta^*(T) = O(H\log T) + \frac{1}{2}\sum_{4T<n\le 4(T+H)} (-1)^n d(n) \ll_\varepsilon HT^\varepsilon$$

holds for $1 \ll H \ll T$. Therefore the conjectural (1.17) implies the hitherto unproved bound

$$\zeta\left(\frac{1}{2}+iT\right) \ll_\varepsilon T^{1/8+\varepsilon}. \tag{1.18}$$

This significance of (1.18) shows the strength of the conjecture (1.15), and the importance of the estimation of $R(T)$ and its mean values.

Furthermore we note that if (1.17) is true, then $\theta = p$, where

$$\theta = \inf\{c > 0 : E(T) = O(T^c)\}, \quad \rho = \inf\left\{d > 0 : \Delta(T) = O(T^d)\right\}.$$

Namely as $\theta \ge 1/4$ and $\rho \ge 1/4$ are known to hold (this follows e.g. from mean square results, see [4]) $\theta = \rho$ follows from (1.17) and $\rho = \sigma$, proved recently by Lau-Tsang [9], where

$$\sigma = \inf\{s > 0 : \Delta^*(T) = O(T^s)\}\,.$$

The reader is also referred to M. Jutila [7] for a discussion on some related implications. In any case our unconditional results on $R(T)$ show, as is to be expected, that there is a lot of cancellation in the mean sense between $E(T)$ and $2\pi\Delta^*(T/(2\pi))$, or in other words that the function $E^*(T)$ is on the average much smaller than either $E(T)$ or $2\pi\Delta^*(T/(2\pi))$.

2. The necessary lemmas

In this section we shall state the lemmas which are necessary for the proof of our theorems. The first one brings forth a formula for $\int_0^T E(t)dt$, which is closely related to F.V. Atkinson's classical explicit formula for $E(T)$ (see [1] or e.g. Chapter 15 of [3] or Chapter 2 of [4]).

Lemma 1. *We have*

(2.1)
$$\begin{aligned}\int_0^T E(t)dt = \pi T + \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4}\sum_{n\le T}(-1)^n d(n)n^{-5/4}e_2(T,n)\sin f(T,n) - \\ - 2\sum_{n\le c_0T} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin\left(T\log\left(\frac{T}{2\pi n}\right) - T + \frac{1}{4}\pi\right) + \\ + O(T^{1/4}),\end{aligned}$$

where $c_0 = \frac{1}{2\pi} + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2\pi}}$, $\operatorname{ar\,sinh} x = \log(x + \sqrt{1+x^2})$, *and for* $1 \le n \ll T$,

$$\begin{aligned}e_2(T,n) = \left(1 + \frac{\pi n}{T}\right)^{-1/4}\left\{\left(\frac{2T}{\pi n}\right)^{1/2}\operatorname{ar\,sinh}\left(\frac{\pi n}{2T}\right)^{1/2}\right\}^{-1/2} = \\ = 1 + b_1\frac{n}{T} + b_2\left(\frac{n}{T}\right)^2 + \dots,\end{aligned} \tag{2.2}$$

$$f(T,n) = 2T\mathrm{ar\,sinh}\left(\sqrt{\pi n/(2T)}\right) + \sqrt{2\pi nT + \pi^2 n^2} - \frac{1}{4}\pi =$$
$$= -\frac{1}{4}\pi + 2\sqrt{2\pi nT} + a_3 n^{3/2} T^{-1/2} + a_5 n^{5/2} T^{-3/2} + a_7 n^{7/2} T^{-5/2} + \ldots$$

We also need a formula for the integral of $\Delta^*(x)$. From a classical result of G.F. Voronoï [14] (this also easily follows from pp. 90-91 of [3]) we have

$$\int_0^X \Delta(x)dx = \frac{X}{4} + \frac{X^{3/4}}{2\sqrt{2}\pi^2} \sum_{n=1}^{\infty} d(n) n^{-5/4} \sin\left(4\pi\sqrt{nX} - \frac{1}{4}\pi\right) + O(1).$$

To relate the above integral to the one of $\Delta^*(x)$ we proceed as on pp. 472-473 of [3], using (1.3) and (1.10). In this way we are led to

Lemma 2. *We have*

$$\int_0^T \Delta^*(t)dt = \frac{T^{3/4}}{2\sqrt{2}\pi^2} \sum_{n\le T^2} (-1)^n d(n) n^{-5/4} \sin\left(4\pi\sqrt{nT} - \frac{1}{4}\pi\right) + \tag{2.3}$$
$$+ O(T^{\frac{1}{4}}).$$

We need also a result which relates $E^*(T)$ to its integral over a short interval. This is

Lemma 3. *For $T^\varepsilon \le H \ll T$ we have, for some constant $C > 0$,*

$$\begin{aligned} E^*(T) &\le \frac{1}{H}\int_T^{T+H} E^*(t)dt + CH\log T, \\ E^*(T) &\ge \frac{1}{H}\int_{T-H}^{T} E^*(t)dt - CH\log T. \end{aligned} \tag{2.4}$$

Proof. From (1.2) we have, for $0 \le u \ll T$,

$$0 \le \int_T^{T+u} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = (T+u)\left(\log\left(\frac{T+u}{2\pi}\right) + 2\gamma - 1\right) -$$

$$-T\left(\log\left(\frac{T}{2\pi}\right)+2\gamma-1\right)+E(T+u)-E(T).$$

By the mean-value theorem this implies

$$E(T)\le E(T+u)+O(u\log T),$$

giving by integration

$$E(T)\le \frac{1}{H}\int_T^{T+H} E(t)dt + CH\log T \qquad (1\ll H\ll T,\quad C>0). \tag{2.5}$$

From (1.3) we have ($T^\varepsilon\le H\ll T$)

$$\Delta^*(T)-\frac{1}{H}\int_T^{T+H}\Delta^*(t)dt \ll H\log T+\frac{1}{H}\int_T^{T+H}\sum_{4T<n\le 4t} d(n)dt \ll H\log T \tag{2.6}$$

on applying a result of P. Shiu [13] on the values of multiplicative functions in short intervals. It follows that

$$\Delta^*(T)=\frac{1}{H}\int_T^{T+H}\Delta^*(t)dt+O(H\log T) \qquad (T^\varepsilon\le H\ll T).$$

Hence

$$\begin{aligned}2\pi\Delta^*\left(\frac{T}{2\pi}\right) &= \frac{2\pi}{H}\int_{T/2\pi}^{T/2\pi+H}\Delta^*(x)dx+O(H\log T)=\\ &=\frac{1}{H}\int_{T/2\pi}^{T+2\pi H}\Delta^*\left(\frac{t}{2\pi}\right)dx+O(H\log T)=\\ &=\frac{2\pi}{H}\int_T^{T+H}\Delta^*\left(\frac{t}{2\pi}\right)dx+O(H\log T),\end{aligned} \tag{2.7}$$

on replacing H by $H/2\pi$ in the last step. On combining (2.5) and (2.7) we obtain the first inequality in (2.4), and the second one follows analogously.

Lemma 4. *If* $1 \ll K \ll T^{3/4}, c_1 \ne 0, c_3, \ldots, c_{2L-1}$ *are real constants,* $L \ge 1$ *is fixed, and*

$$F(T,n) = c_1(Tn)^{1/2} + c_3 n^{3/2} T^{-12} + \ldots + c_{2L-1} n^{L-1/2} T^{3/2-L},$$

then for (κ, λ) *an exponent pair we have*

$$\sum_{K<k\le K'\le 2K} (-1)^k d(k) e^{F(T,k)i} \ll T^{\kappa/2} K^{(1+\lambda)/2} \log T. \tag{2.8}$$

Proof. The factor $(-1)^k$ is innocuous, and in fact can be omitted, as was done in Chapter 7 of [3]. It suffices thus to consider

$$S := \sum_{k\le K} d(k) e^{F(T,k)i} = 2 \sum_{m\le\sqrt{K}} \sum_{n\le K/m} e^{F(T,mn)i} - \sum_{m\le\sqrt{K}} \sum_{n\le\sqrt{K}} e^{F(T,mn)i},$$

where the familiar hyperbola method was applied. The sums over n are split into $\ll \log T$ subsums over the ranges $N < n \le N' \le 2N$. In view of ($C_{a,b} \ne 0$)

$$\frac{\partial^{a+b} F(T,mn)}{(\partial m)^a (\partial n)^b} \sim C_{a,b} T^{1/2} m^{1/2-a} n^{1/2-b} \qquad (a,b = 0,1,2\ldots, mn \ll T^{3/4})$$

it follows that (see Chapter 2 of [3] for the definition and properties of exponent pairs)

$$\sum_{N<n\le N'\le 2N} e^{F(T,mn)i} \ll \left(\frac{mT}{N}\right)^{\kappa/2} N^\lambda.$$

Hence we obtain

$$S \ll \log T \sum_{m\le\sqrt{K}} (mT)^{\kappa/2} \left\{ \left(\frac{K}{m}\right)^{\lambda-\kappa/2} + K^{(2\lambda-\kappa)/4} \right\} \ll T^{\kappa/2} K^{(1+\lambda)/2} \log T.$$

Lemma 5 (cf. Lemma 3 of [6]). *For* $a > -\frac{1}{2}$ *a constant we have*

$$\sum_{n\le x} d^2(n)n^a = x^{a+1}P_3(\log x; a) + O_\varepsilon(x^{a+1/2+\varepsilon}), \tag{2.9}$$

where $P_3(y;a)$ is a polynomial of degree three in y whose coefficients depend on a, and whose leading coefficient equals $1/(\pi^2(a+1))$. All the coefficients of $P_3(y;a)$ may be explicitly evaluated.

The last lemma is a new result of O. Robert - P. Sargos [12] which will be needed in the proof of Theorem 3. This is

Lemma 6. *Let $k \ge 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers n_1, n_2, n_3, n_4 such that $N < n_1, n_2, n_3, n_4 \le 2N$ and*

$$\left|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}\right| < \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon N^\varepsilon(N^4\delta + N^2). \tag{2.10}$$

3. Proof of Theorem 1

From Lemma 1 and Lemma 2 we deduce that

$$R(T) = O(T^{1/2}\log T) + \tag{3.1}$$

$$+\frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4}\sum_{n\le T}(-1)^n d(n)n^{-5/4}\left\{e_2(T,n)\sin f(T,n) - \sin\left(2\sqrt{2\pi nT} - \frac{\pi}{4}\right)\right\}.$$

The sum over n is written as

$$\sum_{n\le T^{3/4}} = \sum_{n\le T^{1/3}} + \sum_{T^{1/3}<n\le T} = \sum\nolimits_1 + \sum\nolimits_2,$$

say. In $\sum_1$ we use the asymptotic expansion (2.2) (actually $a_3 = \frac{1}{6}\sqrt{2\pi^3}$) to infer that

(3.2)

$$\sum\nolimits_1 = a_3 \sum_{n\le T^{1/3}} (-1)^n d(n) n^{-5/4}\left\{n^{3/2}T^{-1/2}\cos f(T,n) + O(n^{5/2}T^{-3/2})\right\} =$$

$$= a_3 T^{-1/2} \sum_{n\le T^{1/3}} (-1)^n d(n) n^{1/4}\cos f(T,n) + O(T^{-3/4}) =$$

$$= a_3 T^{-1/2}\sum\nolimits_3 + O(T^{-3/4}\log T),$$

say. The important thing is that in $\sum_3$ we have the increasing function $n^{1/4}$, and in $\sum_2$ the decreasing function $n^{-5/4}$, while the exponential factors (up to a constant) will be the same. This implies that the contributions both in $\sum_2$ and $\sum_3$ will be dominated by the contribution of $n \asymp T^{1/3}$. Thus essentially in the explicit formula (3.1) for $R(T)$ the "critical" values of n will be when $n \asymp T^{1/3}$. In the truncated formula $(1 \ll N \ll T)$

(3.3)

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}} x^{\frac14} \sum_{n\le N} (-1)^n d(n) n^{-\frac34}\cos\left(4\pi\sqrt{nx} - \frac14\pi\right) + O_\varepsilon\left(x^{\frac12+\varepsilon}N^{-\frac12}\right),$$

and in the integrated analogue for $E(T)$ (see (2.4)), if we want bounds of the type $\Delta^*(x) \ll_\varepsilon x^{s+\varepsilon}(E(T) \ll_\varepsilon T^{d+\varepsilon})$ with $s < 1/3$ (resp. $d < 1/3$) we have to take N in (3.3) with $N = x^{1-2s}$. This implies that $1-2s > 1/3$, hence the "critical" values for n in (3.3) will be larger that $x^{1/3}$, whereas in (3.2) they are of the order $T^{1/3}$. The "closeness" of $E(T)$ and $2\pi\Delta^*(T/(2\pi))$ is basically induced by this phenomenon.

To continue with the estimation of $R(T)$, we use the Taylor expansion of $f(T,n)$ (see (2.2)) with L terms, where L is chosen so large that the tails of the series will make a negligible contribution (i.e. $O(1)$). The sums in $\sum_2$ and $\sum_3$ are split into $O(\log T)$ subsums of the form (2.8), after the removal of the monotonic coefficients $n^{1/4}$ and $n^{-5/4}$ by partial summation. Applying Lemma 4 it follows that

$$T^{3/4}\sum\nolimits_1 \ll T^{1/4}T^{\kappa/2}\log^2 T\cdot T^{\frac13\left(\frac14+\frac12+\frac{\lambda}{2}\right)} = T^{\frac{1+\kappa}{2}+\frac{\lambda}{6}}\log^2 T,$$

and in a similar way one has the bound

$$T^{3/4}\sum\nolimits_2 \ll T^{\frac{1+\kappa}{2}+\frac{\lambda}{6}}\log^2 T.$$

Therefore we have the bound

(3.4) $$R(T) \ll T^{\frac{1+\kappa}{2}+\frac{\lambda}{6}}\log^2 T + T^{1/2}\log T \ll T^{\frac{1+\kappa}{2}+\frac{\lambda}{6}}\log^2 T,$$

since $0 \le \kappa \le \frac{1}{2} \le \lambda \le 1$. Already the trivial exponent pair (0,1) gives the bound $R(T) \ll T^{2/3}\log^2 T$, which improves the bound $R(T) \ll T^{3/4}$ that was mentioned in Section 1. The exponent in (3.4) does not exceed 2/3 if

(3.5) $$3\kappa + \lambda \le 1$$

holds, but it is not likely that the exponent $593/912 = 0.6502129\ldots$ in (1.11) of Theorem 1 can be attained in this fashion. To attain this exponent (more sophisticated present-day estimates can yield a slightly smaller exponent), one has to use estimates for two-dimensional exponential sums. In particular, for (1.11) one can use the bound of G. Kolesnik, worked out in Chapter 7 of [3]. This is ($c \ne 0,\ K \ll T^{1/2}$)

(3.6)
$$\sum_{K\le k\le K'\le 2K}(-1)^k d(k) e^{ic(kT)^{1/2}+idk^{3/2}T^{-1/2}} \ll_\varepsilon T^\varepsilon\left(T^{-\frac{1}{16}}K^{\frac{173}{152}} + T^{\frac{1}{16}}K^{\frac{119}{152}}\right),$$

as the terms $a_5k^{5/2}T^{-3/2} + \ldots$ in the (2.2) make a smaller contribution. The terms $n > T^{1/2}$ in (3.1) may be estimated by Lemma 4 with $(\kappa,\lambda) = = (2/18, 13/18) = ABA(1/6, 2/3)$ in the terminology of exponent pairs. The contribution is seen to be $O_\varepsilon(T^{11/18+\varepsilon}),\ 11/18 = 0.6111\ldots.$

In the bound (3.6) it is the first term on the right-hand side that will make the larger contribution, which is found, similarly as in the derivation of (3.4), to be

$$\ll_\varepsilon T^{3/4+\varepsilon}\left\{\max_{K\le T^{1/3}} T^{-1/2-1/16}K^{1/4+173/152} + \max_{K\ge T^{1/3}} T^{-1/16}K^{-5/4+173/152}\right\}$$
$$\ll_\varepsilon T^{593/912+\varepsilon}.$$

4. Proof of Theorem 2

Combining Lemma 1 and Lemma 2 we obtain

$$R(T) = \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{T<n\le T^2} (-1)^{n+1} d(n) n^{-5/4} \sin\left(2\pi\sqrt{2nT} - \frac{1}{4}\pi\right) +$$

$$+\frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{n\le T} (-1)^n d(n) n^{-5/4} \times$$

$$\times \left\{ e_2(T,n)\sin f(T,n) - \sin\left(2\pi\sqrt{2nT} - \frac{1}{4}\pi\right)\right\} -$$

$$-2\sum_{n\le c_0T} d(n) n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2} \sin\left(T\log\left(\frac{T}{2\pi n}\right) - T + \frac{1}{4}\pi\right) + O(T^{1/4}).$$

We set, for $T \le t \le 2T$,

$$S_1(t) := \sum_{T<n\le T^2} (-1)^{n+1} d(n) n^{-5/4} \sin(2\pi\sqrt{2nT} - \frac{1}{4}\pi),$$

$$S_2(t) := 2\sum_{n\le c_0T} d(n) n^{-1/2}\left(\log\frac{t}{2\pi n}\right)^{-2} \sin\left(t\log\left(\frac{t}{2\pi n}\right) - t + \frac{1}{4}\pi\right).$$

Note that the mean square bound ($c \ne 0$)

(4.2)

$$\int_T^{2T}\left|\sum_{K<k\le 2K} (-1)^k d(k) e^{\sqrt{ckt}i}\right|^2 dt =$$

$$= T\sum_{K<k\le 2K} d^2(k) + \sum_{K<m\ne n\le 2K} (-1)^{m+n} d(m)d(n) \int_T^{2T} e^{\sqrt{ct}(\sqrt{m}-\sqrt{n})i} dt \ll$$

$$\ll TK\log^3 T + \sqrt{T} \sum_{K<m\ne n\le 2K} \frac{d(m)d(n)}{|\sqrt{m}-\sqrt{n}|} \ll_\varepsilon$$

$$\ll_\varepsilon TK\log^3 T + T^{1/2+\varepsilon} \sum_{K<m\ne n\le 2K} \frac{K^{1/2}}{|m-n|} \ll_\varepsilon$$

$$\ll_\varepsilon T^\varepsilon (TK + T^{1/2}K^{3/2})$$

holds for $1 \ll K \ll T^C \quad (C > 0)$, where we used the first derivative test (see Lemma 2.1 of [3]). The same bound also holds if in the exponential we have $f(t,k)$ (cf. (2.2)) instead of $\sqrt{ctk}$, as shown e.g. in the derivation of the mean square formula for $E(t)$ in Chapter 15 of [3]. Using (4.2) it follows that

$$\int_T^{2T} t^{3/2} S_1^2(t)dt \ll_\varepsilon T^{1+\varepsilon}, \tag{4.3}$$

and, similarly as in Chapter 15 of [3], one obtains

$$\int_T^{2T} S_2^2(t)dt \ll_\varepsilon T^{1+\varepsilon}. \tag{4.4}$$

By using (4.2) we also have the crude bound

$$\int_T^{2T} R^2(t)dt \ll_\varepsilon T^{2+\varepsilon}, \tag{4.5}$$

Therefore from (4.1) and (4.3)-(4.5) we infer, by using the Cauchy-Schwarz inequality for integrals, that (setting $A = \frac{1}{\pi\sqrt{2\pi}}$ for brevity)

$$\int_T^{2T} R^2(t)dt = O_\varepsilon(T^{7/4+\varepsilon}) + \tag{4.6}$$

$$+A\int_T^{2T} t^{3/2}\left(\sum_{n\le T}(-1)^n d(n) n^{-\frac{5}{4}} \times\right.$$

$$\left.\times\left\{e_2(T,n)\sin f(T,n) - \sin\left(2\pi\sqrt{2nT} - \frac{1}{4}\pi\right)\right\}\right)^2 dt.$$

Further, for a given $\delta > 0$ we split

$$\sum_{n\le T} = \sum_{n\le\delta\sqrt{T}} + \sum_{\delta\sqrt{T}<n\le T} = S_3(t) + S_4(t),$$

say. It follows from (4.6) that

$$\int_T^{2T} R^2(t)dt = A\int_T^{2T} t^{3/2}\left(S_3^2(t)+S_4^2(t)+2S_3(t)S_4(t)\right)dt + O_\varepsilon(T^{7/4+\varepsilon}). \tag{4.7}$$

Again, by (4.2), it is seen that the integral with $S_4^2(t)$ is absorbed by the error term in (4.7).

Next, we consider the integral with $S_3(t)S_4(t)$ in (4.7), writing m for the integer variable in $S_3(t)$. If $m < T^{1/3}$, then we observe that

$$\sin f(t,m) - \sin(2\sqrt{2\pi mt}-\pi/4) = \sum_{k=1}^\infty \frac{(y-y_0)^k}{k!}\sin\left(y_0+\frac12 k\pi\right),$$

$y = f(t,m),\ y_0 = 2\sqrt{2\pi mt}-\pi/4,\ y-y_0 = d_3m^{3/2}t^{-1/2}+d_5m^{5/2}t^{-3/2}+\ldots.$ Therefore in

$$\int_T^{2T} t^{3/2}\sum_{m<T^{1/3}}\cdots\sum_{\delta\sqrt{T}<n\le T}\ldots dt$$

we shall encounter the exponential factor

$$\exp\left(\pm i\left\{f(t,m)-\sqrt{8\pi mt}\right\}\right)\exp\left(\pm i\left\{f(t,n)-\sqrt{8\pi nt}\right\}\right). \tag{4.9}$$

In the first exponential we use (4.8), and the dominant contribution comes from the term $k=1$. The first derivative test shows that the contribution is

$$\ll T^2\sum_{m\le T^{1/3}} d(m)m^{-5/4}\cdot m^{3/2}T^{-1/2}\sum_{\delta\sqrt{T}<n\le T} d(n)n^{-5/4}n^{-1/2} \ll$$
$$\ll T^{3/2}T^{\frac54\cdot\frac13}T^{-\frac12\cdot\frac34}\log^2 T = T^{37/24}\log^2 T.$$

In case when $T^{1/3} < m \le \delta\sqrt{T}$ in $S_3(t)$, we shall have exponentials of the form

$$\exp(\pm if(t,m)\pm i\sqrt{8\pi nt}),\quad \exp(\pm if(t,m)\pm if(t,n)),$$
$$\exp(\pm i\sqrt{8\pi mt}\pm i\sqrt{8\pi nt}),\quad \exp(\pm i\sqrt{8\pi mt}\pm if(t,n)),$$

with all possible combinations of signs. The most interesting case is that of

$$\exp(iF(t,m,n)),\quad F(t,m,n) := \sqrt{8\pi mt}-f(t,n),$$

when

$$\frac{d}{dt}F(t,m,n) = \sqrt{\frac{2\pi m}{t}} - 2\mathrm{arsinh}\sqrt{\frac{\pi n}{2t}} =$$
$$= \sqrt{\frac{2\pi}{t}}(\sqrt{m}-\sqrt{n}) + c_3n^{3/2}t^{-3/2} + c_5n^{5/2}t^{-5/2} + \ldots.$$

Here we have $|\sqrt{m}-\sqrt{n}| \gg \sqrt{n}$ for $|n-m| \gg n$, namely for $m \ll n$. In that case the contribution is clearly, by the first derivative test, $\ll_\varepsilon T^{7/4+\varepsilon}$. If $m \gg n$, this means that

$$\delta\sqrt{T} < n \ll \delta\sqrt{T}. \tag{4.10}$$

Then we have

$$\left|\sqrt{\frac{2\pi}{t}}\,(\sqrt{m}-\sqrt{n})\right| \gg \left(\frac{n}{t}\right)^{3/2}$$

for $|m-n| \gg n^2/T$, which certainly holds in view of (4.10) if $\delta > 0$ is sufficiently small, since $|m-n| \ge 1$ when $m \ne n$. The total contribution of such pairs m,n is

$$\ll T^2 \sum_{m\le\sqrt{T}} d(m)m^{-5/4} \sum_{n\ne m, n\asymp\delta\sqrt{T}} \frac{d(n)}{|m-n|}n^{-5/4}n^{1/2} \ll T^{7/4}.$$

In a similar fashion it is seen that all other cases make the same total contribution which is $\ll T^{7/4}$. Thus we have

$$\int_T^{2T} R^2(t)dt = A\int_T^{2T} t^{3/2}S_3^2(t)dt + O_\varepsilon(T^{7/4+\varepsilon}).$$

In $S_3(t)$ we replace $e_2(t,n)$ by 1 (see (2.2)), making an error which is absorbed in the error term above. Thus it is shown that

$$\int_T^{2T} R^2(t)dt = O_\varepsilon(T^{7/4+\varepsilon}) + \tag{4.11}$$

$$+A\int\limits_T^{2T} t^{3/2}\left(\sum_{n\le\delta\sqrt{T}}(-1)^n d(n)n^{-5/4}\left\{\sin f(t,n)-\sin\left(\sqrt{8\pi nt}-\frac14\pi\right)\right\}\right)^2 dt =$$

$$= A\int\limits_T^{2T} t^{3/2}\sum_{n\le\delta\sqrt{T}} d^2(n)n^{-5/2}\left\{\sin f(t,n)-\sin\left(\sqrt{8\pi nt}-\frac14\pi\right)\right\}^2 dt+$$

$$+O_\varepsilon(T^{7/4+\varepsilon}).$$

Namely, when we square out the first sum above, then we encounter diagonal terms ($m = n$) which account for the main contribution. There are also the non-diagonal terms ($m \ne n$), which are estimated similarly as in the preceding case, and which make a total contribution of $O_\varepsilon(T^{7/4+\varepsilon})$.

At this point we invoke the elementary formula

$$(\sin\alpha-\sin\beta)^2=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta =$$

$$=1-\frac12(\cos 2\alpha+\cos 2\beta)+\cos(\alpha+\beta)-\cos(\alpha-\beta)$$

with

$$\alpha = f(t,n),\quad \beta=\sqrt{8\pi nt}-\frac14\pi,$$

and insert it in (4.11). To deal with the contribution of

$$-\frac12(\cos 2\alpha+\cos 2\beta)+\cos(\alpha+\beta)$$

we split the sum over n in (4.11) at $n = T^\rho,\ 0<\rho<\frac12$. Using $|\sin\alpha-\sin\beta|\le$ $\le|\alpha-\beta|$ for $n<T^\rho$ and the first derivative test for the remaining n we obtain a contribution which is

(4.12)
$$\ll T^{5/2}\sum_{n<T^\rho} d^2(n)n^{1/2}T^{-1}+T^2\sum_{n\ge T^\rho} d^2(n)n^{-2}\ll$$
$$\ll T^{\frac32+\frac32\rho}\log^3 T+T^{2-\rho}\log^3 T\ll T^{9/5}\log^3 T$$

with the choice $\rho = 1/5$. Using $1-\cos(\alpha-\beta) = 2\sin^2(\frac12(\alpha-\beta))$ and invoking the asymptotic expansion (2.2) for $f(T,n)$, we have altogether

(4.13)
$$\int\limits_T^{2T} R^2(t)dt =$$

$$= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \sum_{T^{1/5}\le n\le \delta\sqrt{T}} \frac{d^2(n)}{n^{5/2}} \int_T^{2T} t^{3/2} \sin^2\left(\frac{1}{2}a_1 n^{3/2} t^{-1/2}\right) dt + O(T^{\frac{15}{8}}\log^3 T).$$

In the integral above we make the change of variable

$$\frac{1}{2}a_1 n^{3/2} t^{-1/2} = y, \quad dt = -\frac{1}{2}a_1^2 n^3 y^{-3} dy.$$

We set $z = (4(y/a_1)^2)^{1/3}$ so that, after changing the order of integration and summation, the main term on the right-hand side of (4.13) becomes

$$\frac{1}{\pi\sqrt{2\pi}} \sum_{T^{1/5}\le n\le \delta\sqrt{T}} d^2(n) n^{-5/2} \int_{\frac{1}{2}a_1 n^{3/2}(2T)^{-1/2}}^{\frac{1}{2}a_1 n^{3/2}T^{-1/2}} \left(\frac{1}{2}a_1\right)^3 n^{9/2} y^{-3} \sin^2 y \cdot$$

$$\cdot a_1^2 n^3 y^{-3} dy = \tag{4.14}$$

$$= \frac{a_1^5}{8\pi\sqrt{2\pi}} \int_{2^{-3/2}a_1 T^{-1/5}}^{\frac{1}{2}a_1\delta^{3/2}T^{1/4}} \sum_{\max(T^{1/5},z)\le n\le \min(\delta\sqrt{T},2^{1/3}z)} d^2(n) n^5 \cdot \frac{\sin^2 y}{y^6} dy.$$

The range of summation for n is the interval $[z, 2^{1/3}z]$ if

$$y \in J, \qquad J := \left[\frac{1}{2}a_1 T^{-1/8},\ \frac{\delta^{3/2}}{2\sqrt{2}} a_1 T^{1/4}\right].$$

If we replace the interval of integration in the second integral in (4.14) by J, then by using

$$|\sin y| \le \min(1, |y|) \tag{4.15}$$

it follows that the error that is made is $\ll_\varepsilon T^{9/5+\varepsilon}$.

Now we use Lemma 3 ((2.9) with $a = 5$) to obtain that the integral over J equals, with suitable constants d_j, e_j,

$$\int_{2^{-3/2}a_1 T^{-1/5}}^{\frac{1}{2}a_1\delta^{3/2}T^{1/4}} \left(T^2 y^4 \sum_{j=0}^{3} d_j \log^j (y^{2/3}T^{1/3}) + O_\varepsilon(T^{11/6+\varepsilon} y^{11/3})\right) \frac{\sin^2 y}{y^6} dy =$$

$$= T^2 \int\limits_{2^{-3/2}a_1T^{-1/5}}^{\frac12 a_1\delta^{3/2}T^{1/4}} \left(\sum_{j=0}^{3} e_j \log^j(y^2T)\right) \frac{\sin^2 y}{y^2} dy + O_\varepsilon\left(T^{11/6+\varepsilon}\right),$$

since by using (4.15) we have

$$\int\limits_{2^{-3/2}a_1T^{-1/5}}^{\frac12 a_1\delta^{3/2}T^{1/4}} T^{\frac{11}{6}+\varepsilon}\cdot\frac{\sin^2 y}{y^{7/3}} dy =$$

$$= T^{\frac{11}{6}+\varepsilon}\left(\int\limits_{2^{-3/2}a_1T^{-1/5}}^{1} \frac{\sin^2 y}{y^{7/3}} dy + \int\limits_{1}^{\frac12 a_1\delta^{3/2}T^{1/4}} \frac{\sin^2 y}{y^{7/3}} dy\right) \ll$$

$$\ll T^{\frac{11}{6}+\varepsilon}\left(\int\limits_0^1 y^{-1/3}dy + \int\limits_1^\infty y^{-7/3}dy\right) \ll T^{\frac{11}{6}+\varepsilon},$$

which accounts for the error term in Theorem 2. Replacing the segment of integration in the integral on the right-hand side of (4.16) by $(0,\infty)$, we make an error which is $\ll_\varepsilon T^{9/5+\varepsilon}$. Namely, for $0<\alpha<1<\beta$, $j=0,1,\dots$ fixed, we have

$$\int\limits_\alpha^\beta \frac{\sin^2 y}{y^2}\log^j dy = \int\limits_0^\infty \frac{\sin^2 y}{y^2}\log^j dy + O(\alpha) + O\left(\beta^{-1}\log^j\beta\right), \tag{4.17}$$

where we used again (4.15). Hence the expression in (4.16) becomes, on using (4.17) with $\alpha = 2^{-3/2}a_1T^{-1/5}$, $\beta = 2a_1\delta^{3/2}T^{1/4}$,

$$T^2\sum_{j=0}^{3} b_j \log^j T + O\left(T^{11/6+\varepsilon}\right)$$

with some constants b_j ($b_3>0$) which may be explicitly evaluated, and Theorem 2 follows.

5. Proof of Theorem 3

The proof of the fourth moment estimate in (1.14) follows by employing the method of [5] used in the proof of (1.5) and (1.6), and therefore we shall be brief. The chief ingredient of the proof is Lemma 6 with $k = 2$, since raising the sum

$$\left|\sum_{K<k\le 2K}(-1)^k d(k)e^{\sqrt{ckt}i}\right|$$

to the fourth power leads to expressions of the form $\sqrt{n_1}+\sqrt{n_2}-\sqrt{n_3}-\sqrt{n_4}$ ($n_j \in \mathbb{N}$) in the exponential. Care has also to be taken when one takes the first two terms in the asymptotic expansion (2.2) of $f(t,k)$, namely of the term $k^{3/2}t^{-1/2}$. This is achieved by using the approach of M. Jutila [7, part II], as embodied in e.g. Lemma [5, part I]. As already explained, the major contribution will come from the terms $n \asymp T^{1/3}$ in (3.1). The contribution of the terms $n \le T^{1/3}$, corresponding to Σ_1 in the proof of Theorem 1, will be

$$\ll T\log T \max_{K\ll T^{1/3}} \int_T^{2T}\left|\sum_{K<n\le K'\le 2K}(-1)^n d(n)n^{1/4}e^{\sqrt{ckt}i}\right|^4 dt, \tag{5.1}$$

and the integral is, up to a small error term,

$$\ll \max_{K\ll T^{1/3}} K \int_{T/2}^{5T/2}\Bigg|\sum\nolimits^*_{K<m,n,k,l\le K'\le 2K} \times \times(-1)^{m+n+k+l}d(m)d(n)d(k)d(l)\exp(i\Delta\sqrt{t})\Bigg|\,dt, \tag{5.2}$$

where $\sum^*$ means that $|\Delta| \le T^{\varepsilon-1/2}$ holds, and

$$\Delta := \sqrt{8\pi}(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{l}).$$

Now we use Lemma 6 ((2.10) with $k = 2, \delta \asymp K^{-1/2}|\Delta|$), estimating the integral on the right-hand side of (5.2) trivially. It follows that the total contribution will be

$$T^{1+\varepsilon} \ll_\varepsilon \max_{K\ll T^{1/3}} T(K^{9/2}T^{-1/2}+K^3) \ll_\varepsilon$$

$$\ll_\varepsilon T^{3/2+3/2+\varepsilon} + T^{3+\varepsilon} \ll_\varepsilon T^{3+\varepsilon},$$

and the same final bound follows for the contribution of the terms n in (5.1) satisfying $n > T^{1/3}$. The proof of Theorem 3 is complete.

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