01 Jan 1975

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Ronald J. Adrian
Department of Theoretical and Applied Mechanics
University of Illinois
Urbana, Illinois 61801

ABSTRACT

It is shown that conditional averages in the form of expected values of functions of the velocity at an arbitrary point given the velocities at a finite number of distinct points, appear naturally in certain types of turbulence theories and that the closure problems in such theories ultimately reduce to the approximation of these averages. Two exemplary theories are considered. The first is characteristic of turbulence models formulated in terms of probability density functions whereas the second is related to the derivation of optimal algorithms for the numerical integration of the turbulent Navier-Stokes equations at large Reynolds numbers. Some mathematical properties of conditional expected values, including relations between conditional and unconditional second order tensor moments and results for the special case of isotropic turbulence are also presented.

INTRODUCTION

If $g(\underline{u})$ is some function of the velocity field $\underline{u}(\underline{x}, t)$ and E is an event in the flow, then the conditional average of $g(\underline{u})$ is

$$< g(\underline{u}) \mid E > \; = \text{expected value of } g(\underline{u}) \text{ given that } E \text{ occurs.} \qquad (1)$$

To date virtually all applications of conditional averages in turbulence research have been in experimental studies of either intermittency or coherent turbulent structures, wherein the techniques of conditional averaging have been used to expose features of flow that are obscured, to varying degrees, by conventional unconditional averaging. (The term "coherent structures" is used generically here to encompass not only turbulent bursts in wall boundary layers, but more generally the phenomenon of recurrent turbulent flow patterns that appear to be quasi-deterministic in the sense that they occur repeatedly with patterns which, albeit random, can still be distinguished from more chaotic background motions.) While a variety of functions g and events E have been employed, it is primarily the definition of the event that distinguishes one type of conditional average from another. In the category of intermittency studies, which includes the investigations of Kovasznay, Kibens and Blackwelder (1970), Wygnanski and Fiedler (1970) and Thomas (1973), the event is the turbulent or non-turbulent state of the flow, and it is quantitatively defined in terms of approximations to the idealized criterion that the flow is turbulent if the vorticity fluctuations exceed a certain level. These approximations involve measurable quantities such as time derivatives of the velocity or high frequency components of the velocity and are different in each experiment. In coherent flow studies (Willmarth and Lu 1972, 1974, Grass 1971, Gupta, Laufer and Kaplan 1971, Brodkey, Wallace and Eckelmann 1974, Offen and Kline 1973, Zaric 1974, Adrian 1975) the event is the occurrence of a coherent flow pattern, but because of uncertainties about the best means of recognizing coherent flows a wide variety of mathematical definitions for E have been used in practice, and as noted by Brodkey, Wallace and Eckelmann (1974) this diversity of event definitions precludes a priori comparisons of the results of most investigations.

Even though differences are more the exception than the rule, one general property is common to almost all of the various definitions of E that have been employed: the events are defined in terms of inequalities between some combination of flow variables and some phenomenologically determined detection level. While theory suggests that events of this kind are appropriate for detecting turbulent versus non-turbulent flow, it offers little evidence for their propriety as a means of detecting coherent turbulence. This lack of theoretical guidance is not surprising since the classical statistical theory of turbulence is not naturally formulated in terms of conditional averages, and moreover it does not even suggest the existence of coherent flow patterns, which appear to be more amenable to deterministic analyses than statistical analyses. In fact, the philosophies underlying coherent flow analysis seem to be so different from those of the statistical approach that it is not unreasonable to question whether the two methodologies are compatible, and hence, by association, whether conditional averages can play a useful role in statistical theories.

The purpose of this paper is to demonstrate that conditional averages of a type that are different from those described above do appear naturally in certain statistical formulations of turbulence theory and, indeed, are of central importance in these formulations. The general form of these conditional averages is as follows. If $\underline{x}_a$ denotes the vector position of a point 'a', and $\underline{u}_a = \underline{u}(\underline{x}_a, t)$ is the total velocity at that point, then the (n + 1) - point conditional average is defined as

$$< g(\underline{u}) \mid \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n > = \text{expected value of } g(\underline{u}) \text{ given that } \underline{u}_a = \underline{v}_a \text{ for } a = 1, \dots n, \quad (2)$$

where $\underline{v}_1, \dots \underline{v}_n$ are n arbitrarily specified vectors. For brevity this conditional average will also be denoted by $< g(\underline{u}) \mid \underline{v}_1, \dots \underline{v}_n >$ or $< g(\underline{u}) \mid \underline{u}_1, \dots \underline{u}_n >$, depending on the context. It is not difficult to conceive of using the event defined in eqn. (2) as a coherent flow detector by selecting values of $\underline{x}_a$ and $\underline{v}_a$ to correspond to the coherent flow pattern under investigation. The primary distinction between this type of event and the events used in existing experimental studies is that the former involves equalities while the latter involve inequalities. The former is a more fundamental quantity in that conditional averages defined by inequality events could be derived from it by forming suitable combinations of $\underline{u}_1, \dots \underline{u}_n$ and averaging over sets of $\underline{v}$ values that satisfy the inequality.

While (n + 1) - point conditional averages may be useful in experimental studies of coherent turbulence, their role in statistical theories of turbulence is more general than pattern recognition, and the work to be presented will be directed mainly towards examining their theoretical function and significance. For this purpose two kinds of theoretical formulations will be considered in the first part of the paper. In the second part of the paper certain useful properties of conditional averages will be derived.

THEORY

Mathematical Definitions

Before proceeding it is necessary to present some definitions and relationships concerning conditional statistics and probability density functions, most of which are discussed in detail in the book by Papoulis (1965). Letting $\underline{u}_a = \underline{u}(\underline{x}_a, t)$ be the total velocity*, the unconditional n-point probability density function is defined such that

$$f_n(\underline{v}_1, \dots \underline{v}_n, \underline{x}_1, \dots \underline{x}_n, t)\, d^3\underline{v}_1 \dots d^3\underline{v}_n =$$
$$\text{Prob}\left\{\underline{v}_1 \le \underline{u}_1 < \underline{v}_1 + d\underline{v}_1, \text{ and} \dots, \text{ and } \underline{v}_n \le \underline{u}_n < \underline{v}_n + d\underline{v}_n\right\}, \quad (3)$$

where $\underline{v}_a$ are dummy variables in the p.d.f. . As usual f_n satisfies the reduction and normalization properties,

$$\int f_n(\underline{v}_1, \dots \underline{v}_n)\, d^3\underline{v}_n = f_{n-1}(\underline{v}_1, \dots \underline{v}_{n-1}) \quad (4)$$

$$\int f_n\, d^3\underline{v}_1, \dots d^3\underline{v}_n = 1, \quad (5)$$

and n-point unconditionally averaged moments are calculated from f_n according to the relation

$$< g(\underline{u}_1, \dots \underline{u}_n) > =$$
$$\int g(\underline{v}_1, \dots \underline{v}_n)\, f_n(\underline{v}_1, \dots \underline{v}_n)\, d^3\underline{v}_1 \dots d^3\underline{v}_n. \quad (6)$$

The (n + 1) - point conditional p.d.f. is defined by

$$f(\underline{v} \mid \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n)\, d^3\underline{v} = \quad (7)$$
$$\text{Prob}\left\{\underline{v} \le \underline{u}(\underline{x}, t) < \underline{v} + d\underline{v} \text{ given that } \underline{u}_1 = \underline{v}_1, \text{ and} \dots, \text{and } \underline{u}_n = \underline{v}_n\right\},$$

and it is related to the unconditional p.d.f. by

$$f_{n+1}(\underline{v}, \underline{v}_1, \dots \underline{v}_n) = f(\underline{v} \mid \underline{v}_1, \dots \underline{v}_n)\, f_n(\underline{v}_1, \dots \underline{v}_n), \quad (8)$$

where the abbreviated notation $f(\underline{v} \mid \underline{v}_1, \dots \underline{v}_n) = f(\underline{v} \mid \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n)$ has been used. In terms of the conditional p.d.f. the (n + 1) - point conditional average is

$$< g(\underline{u}) \mid \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n > = \int g(\underline{v})\, f(\underline{v} \mid \underline{v}_1, \dots \underline{v}_n)\, d^3\underline{v} \quad (9)$$

which, in view of eqn. (8) may be rewritten in the form

$$\int g(\underline{v})\, f_{n+1}(\underline{v}, \underline{v}_1, \dots \underline{v}_n)\, d^3\underline{v} =$$
$$< g(\underline{u}) \mid \underline{v}_1, \dots \underline{v}_n > f_n(\underline{v}_1, \dots \underline{v}_n). \quad (10)$$

A quantity that appears frequently is the unconditional average of the product of $g(\underline{u})$ with a series of Dirac delta functions, i.e., $< g(\underline{u})\, \delta(\underline{u}_1 - \underline{v}_1) \dots \delta(\underline{u}_n - \underline{v}_n) >$. Since

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*Greek subscripts are used to refer to points in space, and an underbar denotes a vector. The only exception to the subscript convention is subscript 'n' which denotes the last point in a set of n points. Where necessary, index notation with Latin indices for spatial components and the summation convention are also used.

the product has zero measure unless the event $E = \left\{\underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n\right\}$ occurs, it is clear that this unconditional average must be related to the conditional average of $g(\underline{u})$. In fact, direct computation using eqns. (6) and (10) plus the sifting property of the Dirac delta function shows that

$$< g(\underline{u})\, \delta(\underline{u}_1 - \underline{v}_1) \ldots \delta(\underline{u}_n - \underline{v}_n) > =$$

$$< g(\underline{u}) | \underline{v}_1, \ldots \underline{v}_n > f_n(\underline{v}_1, \ldots \underline{v}_n). \tag{11}$$

This equation provides a link between conditional and unconditional averages. Another important relation which performs a similar function is

$$< g(\underline{u}) > = << g(\underline{u}) | \underline{v}_1 \ldots \underline{v}_n >> \tag{12a}$$

$$= \int < g(\underline{u}) | \underline{v}_1, \ldots \underline{v}_n > f_n(\underline{v}_1, \ldots \underline{v}_n)\, d^3\underline{v}_1 \ldots d^3\underline{v}_n. \tag{12b}$$

That is, the average value of $g(\underline{u})$ is equal to the conditionally averaged value of $g(\underline{u})$ given that $\underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n$ averaged over all values of $\underline{v}_1, \ldots \underline{v}_n$.

One-Point Probability Density Function Equation

The equation governing the evolution of the probability density function for the total velocity at one point in a turbulent flow provides a relatively simple first example of a statistical formulation in which conditional averages figure prominently. Starting with the equations for an unbounded incompressible flow**,

$$\frac{\partial \underline{u}_1}{\partial t} + \underline{u}_1 \cdot \frac{\partial \underline{u}_1}{\partial \underline{x}_1} = -\frac{1}{4\pi} \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right) \frac{\partial}{\partial \underline{x}} \cdot \left(\underline{u} \cdot \frac{\partial}{\partial \underline{x}}\, \underline{u}\right) d^3\underline{x}$$

$$+ \nu \frac{\partial}{\partial \underline{x}_1} \cdot \frac{\partial}{\partial \underline{x}_1}\, \underline{u}_1 \tag{13}$$

and

$$\frac{\partial}{\partial \underline{x}_1} \cdot \underline{u}_1 = 0, \tag{14}$$

Lundgren (1967) has shown that the equation governing the one point p.d.f. is

$$\frac{\partial f_1}{\partial t} + \underline{v}_1 \cdot \frac{\partial f_1}{\partial \underline{x}_1} =$$

$$-\frac{\partial}{\partial \underline{v}_1} \cdot \left\{ -\frac{1}{4\pi} \int_{\underline{v}} \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right)\left(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}\right) : [\underline{v}\,\underline{v} f_2(\underline{v}, \underline{v}_1)]\, d^3\underline{x}\; d^3\underline{v} \right.$$

$$\left. + \lim_{\underline{x} \to \underline{x}_1} \nu \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} \int_{\underline{v}} \underline{v} f_2(\underline{v}, \underline{v}_1)\, d^3\underline{v} \right\}. \tag{15}$$

The terms on the right hand side of eqn. (15) arise respectively from the pressure gradient and viscous stress terms in the momentum equation. Both depend on the two-point p.d.f., and therefore they must either be directly approximated in terms of f_1 as in Lundgren (1969, 1971) or calculated from auxilliary equations. In the latter case the approaches taken by Lundgren (1972) and Fox (1971) were to derive auxilliary equations for f_2 which were closed by approximating f_3 in terms of f_2.

It is, of course, through the pressure gradient and viscous stress terms of eqn. (15) that two-point conditional averages enter the theory, and it is not difficult to show that they are the only unknown two-point quantities in the equation. This is most apparent for the viscous term since it follows immediately from eqn. (10) that

$$\lim_{\underline{x} \to \underline{x}_1} \nu \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} \int_{\underline{v}} \underline{v} f_2(\underline{v}, \underline{v}_1)\, d^3\underline{v}$$

$$= \lim_{\underline{x} \to \underline{x}_1} \nu \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} [< \underline{u} | \underline{v}_1 > f_1(\underline{v}_1)] \tag{16a}$$

$$= \nu f_1(\underline{v}_1) \lim_{\underline{x} \to \underline{x}_1} [\frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} < \underline{u} | \underline{v}_1 >]\,. \tag{16b}$$

Similarly, for the pressure term

$$\int_{\underline{v}} \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right)(\underline{v}\,\underline{v}) : \left(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}\right) f_2(\underline{v}, \underline{v}_1)\, d^3\underline{v}\; d^3\underline{x}$$

$$= \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right)\left(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}\right) : \left[\int_{\underline{v}} \underline{v}\,\underline{v} f_2(\underline{v}, \underline{v}_1)\, d^3\underline{v}\right] d^3\underline{x} \tag{17a}$$

$$= \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right)\left(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}\right) : [< \underline{u}\,\underline{u} | \underline{v}_1 > f_1(\underline{v}_1)]\, d^3\underline{x}, \tag{17b}$$

where eqn. (17b) follows from eqn. (17a) by application of eqn. (10). Combining eqns. (15), (16b) and (17b) gives the following equation for f_1 in terms of conditional averages:

$$\frac{\partial f_1}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f_1 =$$

$$-\frac{\partial}{\partial \underline{v}_1} \cdot \left\{ f_1(\underline{v}_1)\left[-\frac{1}{4\pi} \int_{\underline{x}} \left(\frac{\partial}{\partial \underline{x}_1} \frac{1}{|\underline{x}_1 - \underline{x}|}\right)\left(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}\right) : < \underline{u}\,\underline{u} | \underline{v}_1 > d^3\underline{x} \right.\right.$$

$$\left.\left. + \lim_{\underline{x} \to \underline{x}_1} \nu \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} < \underline{u} | \underline{v}_1 > \right]\right\}. \tag{18}$$

**In equation (13) the first term on the right hand side is the pressure gradient $(-\rho^{-1}\partial p_1/\partial x_1)$ which has been expressed in terms of the velocity field in the usual way by taking the divergence of the Navier-Stokes equations, applying eqn. (14), and solving the resulting Poisson's equation for the pressure.

Thus the closure problem at the one-point level reduces from one of approximating the entire two-point p.d.f. to one of approximating the two-point conditional averages $<\underline{u}|\underline{v}_1>$ and $<\underline{u}\,\underline{u}|\underline{v}_1>$ which are only integral moments of the two-point p.d.f. Although this is a considerable simplification, the difficulties of approximating the conditional averages are still formidable. For example, it will be shown later (c.f. eqns. (39) and (44)) that the two-point spatial correlation may be derived from $<\underline{u}|\underline{v}_1>$ and $f_1(\underline{v}_1)$, and this implies that $<\underline{u}|\underline{v}_1>$ contains all of the two-point statistical information that is available in the correlation tensor, including such fundamental quantities as the Kolmogorov microscale and the integral length scale. Since the prediction of even these quantities is extremely demanding, it must be expected that satisfactory closure approximations for the conditional averages would be at least as difficult to develop.

The two-point conditional averages appear in the pressure and viscous terms for distinctly different reasons. In the former, the instantaneous pressure at a point $\underline{x}_1$ is determined non-locally by the non-linear instantaneous accelerations experienced by the fluid at every point in the flow, and the quantity $<\underline{u}\,\underline{u}|\underline{v}_1>$ in the integrand represents the average of these contributions when $\underline{u}_1 = \underline{v}_1$. An examination of the pressure integral indicates that its value is determined primarily by the value of $<\underline{u}\,\underline{u}|\underline{v}_1>$ for separations whose magnitudes $|\underline{x}-\underline{x}_1|$ are roughly on the order of the integral length scale, i.e., by the large scale structure of the flow. In contrast, the value of the viscous term is, as is obvious from the limit in eqn. (16b), determined entirely by the small scale structure of $<\underline{u}|\underline{v}_1>$ near the point $\underline{x}_1$. In both terms $\underline{v}_1$ and $\underline{x}$ must be allowed to assume arbitrary values.

Numerical Solution of the Navier-Stokes Equations for Large Reynolds Numbers

The development of a numerical algorithm for the solution of the three-dimensional Navier-Stokes equations provides a second example of a theoretical model of turbulence in which conditional averages occur. The two principal approaches to this problem are to perform the computations either in physical space (Deardorff 1970a, 1970b) or in spectral space (Orszag 1971a, 1971b). In either method computer storage and computation time limitations restrict the number of grid points, making it impossible to resolve turbulent motions on scales as small as the Kolmogorov microscale when the Reynolds number is large. Consequently closure approximations are required. In physical space calculations the method used by Deardorff (1970a) is to select the grid scale spacing such that maximum wave number of the resolvable motions falls within the inertial subrange. The governing equations are averaged over a grid volume, resulting in the case of the momentum equation in an equation for the grid volume averaged velocity that contains a Reynolds stress tensor representing the effects of subgrid scale motions. Closure is accomplished by postulating a relation between the subgrid scale Reynolds stresses and the local deformation of the grid volume averaged velocity field.

In order to investigate the role of conditional averages in numerical models of turbulence, it is convenient to formulate the problem in a manner which, though somewhat different from Deardorff's formulation of the subgrid scale model, is sufficiently similar to permit comparison. In particular, it is supposed that at time 't' n random, unaveraged, total velocities are given at grid points $\underline{x}_a$ by equations $\underline{u}_a(t) = \underline{v}_a(t)$, $a = 1, \ldots n$, and that it is desired to predict the velocities at a later time by direct forward integration. Two sources of error in the numerical prediction would be the finite grid spacing and the finite size of the integration time step Δt, but by assuming that Δt is sufficiently small the time step error can be eliminated from consideration. Then, if the grid spacing were also vanishingly small (i.e., $n \to \infty$), a simple, accurate prediction formula would be $\underline{u}_a(t+\Delta t) = \underline{v}_a(t) + (\partial \underline{u}_a/\partial t)_t \, \Delta t$, where $(\partial \underline{u}_a/\partial t)_t$ could be evaluated from eqn. (13) and the initial data $\underline{u}_a(t) = \underline{v}_a(t)$. However, when the grid scale is finite and large with respect to the Kolmogorov microscale the velocity field between the grid points is unknown and cannot be accurately estimated by conventional numerical interpolation formulae because the existence of subgrid scale motions renders smooth variations of the inter-grid field unlikely. Therefore $\partial \underline{u}_a/\partial t$ cannot be calculated exactly. Suppose that $\dot{\underline{v}}_a(t)$ is an estimate of $\partial \underline{u}_a/\partial t$ that is based on the grid data $\underline{u}_a(t) = \underline{v}_a(t)$. Then the predicted velocity $\check{\underline{u}}_a(t+\Delta t)$, given by

$$\check{\underline{u}}_a(t+\Delta t) = \underline{v}_a(t) + \dot{\underline{v}}_a(t)\,\Delta t, \tag{19}$$

is not in general equal to the true velocity $\underline{u}_a(t+\Delta t)$, and since Δt has been assumed to be very small, the error $[\underline{u}_a(t+\Delta t) - \check{\underline{u}}_a(t+\Delta t)]$ is due entirely to the error in the estimation of $(\partial \underline{u}_a/\partial t)_t$. Thus, a reasonable formulation of the problem is to attempt to find an estimator $\dot{\underline{v}}_a$ that minimizes the error $[\partial \underline{u}_a/\partial t - \dot{\underline{v}}_a]$ in some average sense.

An obvious candidate for $\dot{\underline{v}}_a$ is

$$\dot{\underline{v}}_a = \left< \frac{\partial \underline{u}_a}{\partial t} \,\Big|\, \underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n \right>, \tag{20}$$

because the conditionally averaged time derivative is the best non-linear estimate of $\partial \underline{u}_a/\partial t$ in the mean square sense. That is, the mean square estimation error $<(\partial \underline{u}_a/\partial t - \dot{\underline{v}}_a)\cdot(\partial \underline{u}_a/\partial t - \dot{\underline{v}}_a)>$ is a minimum if $\dot{\underline{v}}_a$ is given by eqn. (20) (Papoulis p. 217), and therefore the mean square error $<|\underline{u}_a(t+\Delta t) - \check{\underline{u}}_a(t+\Delta t)|^2>$ is also minimized. (It should be observed that although this formulation minimizes the error incurred at each time step, it has not been shown that it minimizes the accumulated error over many time steps, although this seems likely intuitively.)

An equation for $<\partial \underline{u}_a/\partial t | \underline{v}_1, \dots \underline{v}_n>$ can be derived from the momentum equation (with the pressure term in its more common form, $\rho^{-1}\,\partial p_a/\partial \underline{x}_a$) by application of the operator $<(\quad)|\underline{v}_1, \dots \underline{v}_n>$. This yields

$$< \frac{\partial \underline{u}_a}{\partial t} | \underline{v}_1, \dots \underline{v}_n > = - < (\frac{\partial}{\partial \underline{x}_a} \cdot [\underline{u}_a \underline{u}_a]) | \underline{v}_1, \dots \underline{v}_n >$$

$$- < (\rho^{-1} \frac{\partial p_a}{\partial \underline{x}_a}) | \underline{v}_1, \dots \underline{v}_n > \tag{21}$$

$$+ < (\nu \frac{\partial}{\partial \underline{x}_a} \cdot \frac{\partial}{\partial \underline{x}_a} \underline{u}_a) | \underline{v}_1, \dots \underline{v}_n > .$$

An equivalent, but more informative version of eqn. (21) can be derived by observing that

$$< \frac{\partial \underline{u}_a}{\partial t} \delta(\underline{u}_1 - \underline{v}_1) \dots \delta(\underline{u}_n - \underline{v}_n) > =$$

$$< (\lim_{\Delta t \to 0} \frac{\underline{u}_a(t+\Delta t) - \underline{u}_a(t)}{\Delta t}) \delta(\underline{u}_1 - \underline{v}_1) \dots \delta(\underline{u}_n - \underline{v}_n) > = \tag{22a}$$

$$< (\lim_{\Delta t \to 0} \frac{\underline{u}_a(t+\Delta t) - \underline{u}_a(t)}{\Delta t}) | \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n > f_n(\underline{v}_1, \dots \underline{v}_n) = \tag{22b}$$

$$< \frac{\partial \underline{u}_a}{\partial t} | \underline{u}_1 = \underline{v}_1, \dots \underline{u}_n = \underline{v}_n > f_n(\underline{v}_1, \dots \underline{v}_n), \tag{22c}$$

and hence that

$$< \frac{\partial \underline{u}_a}{\partial t} | \underline{v}_1, \dots \underline{v}_n > = f_n^{-1} < \frac{\partial \underline{u}_a}{\partial t} \delta(\underline{u}_1 - \underline{v}_1) \dots \delta(\underline{u}_n - \underline{v}_n) > . \tag{23}$$

Substituting $\partial \underline{u}_a/\partial t$ from eqn. (13) into the right hand side of eqn. (23) and performing a series of manipulations similar to those used in deriving the f_1 equation (c.f. Lundgren 1967) gives

$$< \frac{\partial \underline{u}_a}{\partial t} | \underline{v}_1, \dots \underline{v}_n > = \lim_{\underline{x} \to \underline{x}_a} \left\{ - \frac{\partial}{\partial \underline{x}} \cdot < \underline{u}\,\underline{u} | \underline{v}_1, \dots \underline{v}_n > + \nu \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} < \underline{u} | \underline{v}_1, \dots \underline{v}_n > \right\}$$

$$- \frac{1}{4\pi} \int_{\underline{x}} (\frac{\partial}{\partial \underline{x}_a} \frac{1}{|\underline{x}_a - \underline{x}|})(\frac{\partial}{\partial \underline{x}} \frac{\partial}{\partial \underline{x}}) : < \underline{u}\,\underline{u} | \underline{v}_1, \dots \underline{v}_n > d^3 \underline{x}. \tag{24}$$

It is immediately apparent from this equation that the determination of the estimator $\dot{\underline{v}}_a$ for an n-point grid reduces to a problem of finding two (n + 1) - point conditional averages, $<\underline{u}|\underline{v}_1, \dots \underline{v}_n>$ and $<\underline{u}\,\underline{u}|\underline{v}_1, \dots \underline{v}_n>$, in which $\underline{u}$ is the velocity at a point $\underline{x}$ that can lie anywhere in the flow volume. The conditional averages in eqn. (24) may be interpreted as stochastic interpolation formulae whose functions are to predict the average behavior of the unknown continuous velocity field in the regions between the grid points in terms of the known velocities at the grid points. In general, if the set of velocities at the grid points is the only information that is available, these conditional averages are the best non-linear estimates of the inter-grid velocity field in the mean square sense, and they are sufficient to minimize the m.s. error associated with the estimator $\dot{\underline{v}}_a$. This situation will be termed 'optimal'. However, since closure approximations are required for the (n + 1) point conditional averages, the integration errors will not be absolutely minimized, and therefore any closed theory derived from eqn. (24) will be 'sub-optimal'. The extent to which the errors in a sub-optimal theory approach the absolute minimum will depend entirely on the accuracy of the approximations for $<\underline{u}|\underline{v}_1, \dots \underline{v}_n>$ and $<\underline{u}\,\underline{u}|\underline{v}_1, \dots \underline{v}_n>$.

It is worthwhile to note that the closure problem for eqn. (24) is identical to the problem that would be encountered if one attempted to solve the n-point p.d.f. equation on the set of points $\underline{x}_a$, and so, in this sense, the solution of f_n equation and the numerical solution of the Navier-Stokes equation on an n-point grid are equivalent. This conclusion is more obvious when one observes that the information available from the n-point numerical solution would be just sufficient to determine the n-point p.d.f. for the velocities at the grid points. The closure of eqn. (24) is also related to the subgrid scale closure problem. For example $<\underline{u}|\underline{v}_1, \dots \underline{v}_n>$ in the viscous term contains information which specifies the average curvature (or more precisely the average Laplacian) of the velocity field at $\underline{x}_a$. Since the local curvature is intimately related to the turbulent viscous dissipation and this, as is well known, is determined predominantly by small scale motions, it is evident that closure approximations for $<\underline{u}|\underline{v}_1, \dots \underline{v}_n>$ must attempt to model the small scale structure. There are, in addition, large scale effects because the velocity at any point $\underline{x}_a$ has a long range (on the order of the integral scale) influence on the conditionally averaged stress tensor $<\underline{u}\,\underline{u}|\underline{v}_1, \dots \underline{v}_1>$ that can appreciably affect the value of the pressure integral. The long range influence is also present in $<\underline{u}|\underline{v}_1, \dots \underline{v}_n>$, but its effect in the viscous term is less important than in the pressure integral. Both large and small scale structure appear to be important in the first term on the right hand side of eqn. (24).

In order to illustrate the type of numerical algorithm that eqn. (24) produces, and to illuminate the interpolative nature of the conditional averages more clearly, stochastic estimation theory will be used to develop a partial closure approximation in which the conditional averages are approximated by relatively simple forms. In what follows, $< \underline{u} | \underline{u}_1, \ldots \underline{u}_n >$ is used to denote the expected value of $\underline{u}$ given the velocities $\underline{u}_1, \ldots \underline{u}_n$ at $\underline{x}_1, \ldots \underline{x}_n$. Since $\underline{u}_1, \ldots \underline{u}_n$ are the actual, random velocities, $< \underline{u} | \underline{u}_1, \ldots \underline{u}_n >$ is also a random variable that must be conceptually distinguished from $< \underline{u} | \underline{v}_1, \ldots \underline{v}_n >$, which is simply a number that depends on the non-random parameters $\underline{v}_1, \ldots \underline{v}_n$. Of course, when the random velocities are such that $\underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n$, the two types of conditional averages are equal.

A linear estimate for the ith component of $< \underline{u} | \underline{u}_1, \ldots \underline{u}_n >$ is obtained by assuming that it can be represented by a linear combination of all component velocities at the grid points $\underline{x}_\beta$:

$$< u_i | \underline{u}_1, \ldots \underline{u}_n > = a_{\beta ik} u_{\beta k}, \ \beta = 1, \ldots n, \quad i, k = 1, 2, 3 \tag{25}$$

and choosing the coefficients $a_{\beta ik}$ to minimize the mean square error given by

$$\text{m.s. error} = < \left| < u_i | \underline{u}_1, \ldots \underline{u}_n > - a_{\beta ik} u_{\beta k} \right|^2 >. \tag{26}$$

(Index notation with the summation convention is used to avoid confusion). If the m.s. error is to be a minimum, it is necessary that

$$\frac{\partial}{\partial a_{\delta \ell m}} < \left| < u_i | \underline{u}_1, \ldots \underline{u}_n > - a_{\beta ik} u_{\beta k} \right|^2 > = 0 \tag{27}$$

for $\delta = 1, \ldots n$, $\ell = 1,2,3, m = 1,2,3$. Hence, for fixed i eqn. (27) yields the following set of 3n linear algebraic equations for the 3n coefficients $a_{\beta ik}$:

$$< u_{\beta k} u_{\gamma \ell} > a_{\beta ik} = < u_i \, u_{\gamma \ell} >. \tag{28}$$

The coefficients $a_{\beta ik}$ can be expressed solely in terms of the two-point spatial correlations of the total velocity, $< u_{\beta k} u_{\gamma \ell} > = < u_k (\underline{x}_\beta, t) \, u_\ell (\underline{x}_\gamma, t) >$ and $< u_i \, u_{\gamma \ell} > = < u_i (\underline{x}, t) \, u_\ell (\underline{x}_\gamma, t) >$, and therefore they are two-point functions of the continuously variable position $\underline{x}$ and the parameters $\underline{x}_\beta$ which are determined by the configuration of the grid points. Thus, the linear estimate in eqn. (25) has reduced an (n + 1) point closure problem to a two-point closure problem.

Regardless of the value of 'n' any conditionally averaged velocity must satisfy the continuity equation

$$\frac{\partial}{\partial x_i} < u_i | \underline{u}_1, \ldots \underline{u}_n > = 0, \tag{29}$$

and therefore it is essential for any estimate of the conditional average to satisfy eqn. (29) also. Substituting the linear estimate postulated in eqn. (25) into eqn. (29) shows that $u_{\beta k} \, \partial a_{\beta ik} / \partial x_i = 0$, and since the $u_{\beta k}$ are arbitrary, it follows that the coefficients must satisfy

$$\frac{\partial a_{\beta ik}}{\partial x_i} = 0 \tag{30}$$

for $\beta = 1, \ldots n$ and $k = 1, 2, 3$. Taking the divergence of eqn. (28) yields a set of 3n linear equations for the variables in eqn. (30)

$$< u_{\beta k} u_{\gamma \ell} > \frac{\partial a_{\beta ik}}{\partial x_i} = < \frac{\partial u_i}{\partial x_i} u_{\gamma \ell} > = 0, \tag{31}$$

in which the last equality follows from the continuity equation. But if eqn. (28) has unique solutions $a_{\beta ik}$ for $i = 1, 2, 3$ then the rank of the coefficient matrix $< u_{\beta k} u_{\gamma \ell} >$ (after some re-labeling to put the equations in standard form) must be exactly 3n, and hence the determinant of the coefficient matrix in eqn. (31) is not equal to zero. Therefore, the only solutions of eqn. (31) are the trivial ones $\partial a_{\beta ik} / \partial x_i = 0$, and it is concluded that the continuity equation is automatically satisfied when $a_{\beta ik}$ is (uniquely) determined from eqn. (28). (The point here is that less general linear estimates such as $< \underline{u}_i | \underline{u}_1, \ldots \underline{u}_n > = a'_{\beta k} u_k$ may not, in general, satisfy continuity.)

An estimate is also needed for the conditionally averaged Reynolds stress tensor, and an appropriate quadratic form is

$$< u_i \, u_j | \underline{u}_1, \ldots \underline{u}_n > = b_{\beta \gamma ijk\ell} \, u_{\beta k} \, u_{\gamma \ell}, \tag{32}$$

where the coefficients could be found as before by minimizing the mean square error. For present purposes such complexity is not necessary and the sub-optimal estimate $< u_i \, u_j | \underline{u}_1, \ldots \underline{u}_n > = < u_i | \underline{u}_1, \ldots \underline{u}_n > \ < u_j | \underline{u}_1, \ldots \underline{u}_n >$ will be satisfactory. This implies that

$$b_{\beta \gamma ijk\ell} = a_{\beta ik} \, a_{\gamma j\ell}. \tag{33}$$

The final equation for the predictor $\dot{\underline{v}}_\alpha$, obtained by combining eqns. (20), (24), (25), (32), and (33), and setting $\underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n$, is

$$\dot{v}_{\alpha i} = - (\Phi_{\alpha \beta \gamma ik\ell} + \Psi_{\alpha \beta \gamma ik\ell}) \, v_{\beta k} \, v_{\gamma \ell} + \nu \, \Delta_{\alpha \beta ik} \, v_{\beta k} \tag{34}$$

where

$$\Phi_{\alpha\beta\gamma ik\ell} = \lim_{\underline{x}\to\underline{x}_\alpha} \left\{ \frac{\partial}{\partial x_p} (a_{\beta ik}\, a_{\gamma p\ell}) \right\} \tag{35}$$

$$\Psi_{\alpha\beta\gamma ik\ell} = \frac{1}{4\pi} \int_{\underline{x}} \left(\frac{\partial}{\partial x_{\alpha i}} \frac{1}{|\underline{x}_\alpha - \underline{x}|}\right)\left(\frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} a_{\beta pk}\, a_{\gamma q\ell}\right) d^3\underline{x} \tag{36}$$

$$\Delta_{\alpha\beta ik} = \lim_{\underline{x}\to\underline{x}_\alpha} \left\{ \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} a_{\beta ik} \right\} . \tag{37}$$

Because of the approximations for $< \underline{u} | \underline{v}_1, \ldots \underline{v}_n >$ and $< \underline{u}\,\underline{u} | \underline{v}_1, \ldots \underline{v}_n >$ in eqns. (25) and (32) eqn. (34) is a suboptimal estimate for $\dot{\underline{v}}_\alpha$. However, if the velocity field were Gaussian, i.e., if the velocities $\underline{u}, \underline{u}_1, \ldots \underline{u}_n$ were joint normally distributed, then eqns. (25) and (32) would be exact (Papoulis, p. 256), and eqn. (34) would be an optimal numerical algorithm. The right hand side of eqn. (34) has the same form as any finite difference formulation of the Navier-Stokes equations in that it consists of linear and quadratic velocity terms that are multiplied by weight coefficients derived from the governing equations. The essential difference is that, unlike a conventional finite difference equation, the coefficients are not derived explicitly from finite difference approximations to spatial derivatives, but are derived instead from spatial derivatives of continuous approximations to the intergrid velocity field that are based on the finite data available on the grid. In this sense the present method resembles the Galerkin method used by Orszag (1971b), except that in the latter the effects of motions on length scales less than the grid scale are not accounted for.

The coefficients in eqn. (34) are ultimately functions of two-point spatial correlations of the total velocity, and since the spatial correlations are not known a priori, they must be approximated in terms of the known quantities. Superficially, it does not appear that this should be difficult because the wealth of information about the large scale turbulence structure that is contained in the velocity data at the grid points should be sufficient to determine the values of the spatial correlation functions for large values of the separations $(\underline{x}_\beta - \underline{x}_\gamma)$ or $(\underline{x} - \underline{x}_\beta)$, and universal similarity laws for the inertial subrange immediately suggest themselves as a means of modeling the small scale structure of the correlations, i.e., their behavior for separations $(\underline{x} - \underline{x}_\alpha)$ less than the grid scale. However, any approach of this type immediately encounters the fundamental problem of attempting to estimate quantities (the spatial correlations) that are averages over a large ensemble of individual realizations of the velocity field in terms of quantities that refer to a single realization (the instantaneous grid velocity data). The same type of problem arises in the subgrid scale closure used by Deardorff (1970a) where Smagorinsky's (1963) closure approximation is used to estimate the subgrid scale eddy coefficient in terms of the deformation of the local, instantaneous grid volume averaged velocity field. A related formula (eqn. (3.5) in Deardorff 1970a) gives the local turbulent dissipation in terms of the instantaneous grid volume averaged velocity field, and it is possible that this could be extended to the present formulation wherein a knowledge of the dissipation would probably be enough to model the small scales adequately. Further, the integral length scale could be estimated from the dissipation and the turbulence intensity so that the long range behavior of the spatial correlations could also be modeled. Several potential difficulties can be anticipated in such a procedure, and therefore it is not being proposed as a viable closure scheme, but merely as an example of an 'instantaneous/unconditional average' type of closure. An alternative to closures of this type is the 'unconditional average/unconditional average' type of closure. For example, in the present numerical algorithm one approach would be to assume an initial form for the spatial correlation and integrate forward in time until enough grid data were available to calculate new estimates of the large scale spatial correlations between the grid points. Then, as before, estimates of the correlations for large scale separations $(\underline{x} - \underline{x}_\beta)$ intermediate to the grid points could be obtained by interpolation while estimation for small scale separations $(\underline{x} - \underline{x}_\beta)$ less than the grid spacing could be achieved by recourse to universal laws for the inertial subrange. This type of implicit approach is clearly cumbersome, and moreover, there is no guarantee that it would converge.

Regardless of the type of closure scheme, it must be born in mind that spatial correlations appear in the theory only as the result of certain approximations which were, in fact, designed to reduce the conditional averages in eqn. (24) to functions of the more familiar and much more extensively researched spatial correlations. These approximations would be correct only if the velocity field were Gaussian. The important conclusions are that the fundamental closure problem in the present numerical model of turbulence is one of modeling the intergrid velocity field, and that conditional averages are intrinsically appropriate to the description of this field.

PROPERTIES OF CONDITIONAL AVERAGES

In the first part of this section the conditionally averaged counterparts of a few common unconditional statistical quantities are defined, and in the second part some special relations for isotropic turbulence are derived. Throughout, rather than dealing with the statistics of total velocities, it is convenient to use Reynolds decomposition to divide the total velocity $\underline{u}$ and the dummy variable $\underline{v}$

into mean and fluctuating parts in the usual fashion:

$$\underline{u} = \underline{U} + \underline{u}', \quad \underline{v} = \underline{U} + \underline{c}, \quad \underline{U} = \langle \underline{u} \rangle . \tag{38}$$

Then the conditional average of $\underline{u}$ is related to that of $\underline{u}'$ by

$$\langle \underline{u} | \underline{u}_1 = \underline{v}_1, \ldots \underline{u}_n = \underline{v}_n \rangle =$$

$$\underline{U} + \langle \underline{u}' | \underline{u}'_1 = \underline{c}_1, \ldots \underline{u}'_n = \underline{c}_n \rangle . \tag{39}$$

Second Order Conditional Moments

(n + 2) - Point Conditional Covariance Tensor. This quantity is defined as the covariance of the velocities at two points $\underline{x}_\alpha$, $\underline{x}_\beta$ given the velocities at n other points, none of which coincide with $\underline{x}_\alpha$ or $\underline{x}_\beta$. That is,

$$R_{ij}(\underline{x}_\alpha, \underline{x}_\beta | \underline{c}_1, \ldots \underline{c}_n) \equiv \langle u'_{\alpha i} u'_{\beta j} | \underline{u}'_1 = \underline{c}_1, \ldots \underline{u}'_n = \underline{c}_n \rangle . \tag{40}$$

As with the unconditional two-point covariance tensor, the continuity equation implies that

$$\frac{\partial R_{ij}}{\partial x_{\alpha i}} (\underline{x}_\alpha, \underline{x}_\beta | \underline{c}_1, \ldots \underline{c}_n) = \frac{\partial R_{ij}}{\partial x_{\beta j}} (\underline{x}_\alpha, \underline{x}_\beta | \underline{c}_1, \ldots \underline{c}_n) = 0 . \tag{41}$$

(n + 1) - Point Conditional Covariance Tensor. A special case of the (n + 2) - point conditional covariance occurs when one of the conditional points, say $\underline{x}_1$, coincides with $\underline{x}_\beta$. Then, after setting $\underline{u}'_\beta = \underline{u}'_1 = \underline{c}_1$, and dropping the subscript α for convenience, eqn. (40) reads

$$R_{ij}(\underline{x}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) = \langle u'_i u'_{1j} | \underline{u}'_1 = \underline{c}_1, \ldots \underline{u}'_n = \underline{c}_n \rangle \tag{42a}$$

$$= c_{1j} \langle u'_i | \underline{c}_1, \ldots \underline{c}_n \rangle , \tag{42b}$$

from which it is clear that the (n + 1) - point conditional covariance depends simply on the (n + 1) - point conditionally averaged velocity. As in eqn. (41) the (n + 1) - point conditional covariance has zero divergence at the point $\underline{x}$. According to eqns. (12b) and (42) the unconditional covariance tensor $R_{ij}(\underline{x}, \underline{x}_1) = \langle u'_i u'_{1j} \rangle$ can be calculated from the conditionally averaged velocity according to the formula

$$R_{ij}(\underline{x}, \underline{x}_1) = \langle R_{ij}(\underline{x}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) \rangle \tag{43a}$$

$$= \int c_{1j} \langle u'_i | \underline{c}_1, \ldots \underline{c}_n \rangle f_n(\underline{c}_1, \ldots \underline{c}_n) d^3\underline{c}_1 \ldots d^3\underline{c}_n . \tag{43b}$$

The last equation indicates that much of the structural information in the (n + 1) - point conditionally averaged velocity is not available in the unconditional covariance. Indeed, even the two-point conditionally averaged velocity should provide greater detail than the unconditional covariance because the relation

$$R_{ij}(\underline{x}, \underline{x}_1) = \int c_{1j} \langle u'_i | \underline{u}'_1 = \underline{c}_1 \rangle f_1(\underline{c}_1) d^3\underline{c}_1 \tag{44}$$

suggests that the fine structure in $\langle u'_i | \underline{u}_1 = \underline{c}_1 \rangle$ is smoothed out by the integration over $\underline{c}_1$.

Three-Dimensional (n + 1) - Point Conditional Spectral Density. The three-dimensional Fourier transform of the (n + 1) - point conditional covariance with $\underline{x} = \underline{x}_1 + \underline{r}$ yields an (n + 1) - point conditional spectral density

$$S_{ij}(\underline{k}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) \equiv \frac{1}{8\pi^3} \int e^{-j\underline{k}\cdot\underline{r}} R_{ij}(\underline{x}_1 + \underline{r}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) d^3\underline{r} \tag{45}$$

that is related to the unconditional three-dimensional spectral density tensor

$$S_{ij}(\underline{k}, \underline{x}_1) = \frac{1}{8\pi^3} \int e^{-j\underline{k}\cdot\underline{r}} R_{ij}(\underline{x}_1 + \underline{r}, \underline{x}_1) d^3\underline{r} \tag{46a}$$

by

$$S_{ij}(\underline{k}, \underline{x}_1) = \int S_{ij}(\underline{k}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) f_n(\underline{c}_1, \ldots \underline{c}_n) d^3\underline{c}_1 \ldots d^3\underline{c}_n \tag{46b}$$

Equations (42b) and (45) suggest that it is natural to define

$$S_i(\underline{k}, x_1 | \underline{c}_1, \ldots \underline{c}_n) \equiv \frac{1}{8\pi^3} \int e^{-j\underline{k}\cdot\underline{r}} \langle u'_i(\underline{x}_1 + \underline{r}) | \underline{c}_1, \ldots \underline{c}_n \rangle d^3\underline{r} \tag{47}$$

such that

$$S_{ij}(\underline{k}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n) = c_{1j} S_i(\underline{k}, \underline{x}_1 | \underline{c}_1, \ldots \underline{c}_n). \tag{48}$$

The conditional vector spectrum, being a Fourier transform of the velocity field, is more amenable to physical interpretation than the spectral density tensor, and as a corollary to the comments concerning the information content of the (n + 1) - point conditional covariance tensor, it also contains more information. The loss of information is illustrated by considering the relation between the unconditional spectral density and the two-point conditional vector spectrum in homogeneous flows where $S_{ij}(\underline{k}, \underline{x}_1) = S_{ij}(\underline{k})$ and $S_i(\underline{k}, \underline{x}_1 | \underline{c}_1) = S_i(\underline{k} | \underline{c}_1)$:

$$S_{ij}(\underline{k}) = \int c_{1j} S_i(\underline{k} | \underline{c}_1) f_1(\underline{c}_1) d^3\underline{c}_1 . \tag{49}$$

Isotropic Turbulence

Because of its simplicity isotropic turbulence is an obvious starting place for investigations, both theoretical and experimental, of conditional averages. Moreover, the importance of the small scale, and therefore presumably locally isotropic, behavior of conditionally averaged velocity field has already been demonstrated. In this section the relations governing conditional averages in isotropic turbulence are presented in an order that essentially parallels Batchelor's (1960) development for isotropic correlation tensors.

Isotropic Representation. In general, $\langle \underline{u}' | \underline{c}_1, \ldots \underline{c}_n \rangle$ is a vector function of (n + 1) position vectors

$\underline{x}, \underline{x}_1, \ldots \underline{x}_n$ and n velocity vectors $\underline{c}_1, \ldots \underline{c}_n$. Since isotropic turbulence is also homogeneous the conditional average must be invariant with respect to translations, and therefore

$$< \underline{u}' | \underline{c}_1, \ldots \underline{c}_n > = \text{function of } (\underline{r}_1, \ldots \underline{r}_n, \underline{c}_1, \ldots \underline{c}_n), \quad (50)$$

where $\underline{r}_\alpha = \underline{x} - \underline{x}_\alpha$. If such a vector function is also isotropic, then representation theory (Smith 1971) shows that it must have the form

$$< u_i' | \underline{c}_1, \ldots \underline{c}_n > = G_\alpha c_{\alpha i} + H_\alpha r_{\alpha i}, \quad (51)$$

where repeated indices are summed, and G_α and H_α are isotropic scalar functions of the isotropic invariants $\underline{r}_\beta \cdot \underline{r}_\gamma$, $\underline{c}_\beta \cdot \underline{c}_\gamma$ and $\underline{r}_\beta \cdot \underline{c}_\gamma$, $\beta = 1, \ldots n$, $\gamma = 1, \ldots n$.

Continuity Equation for $< \underline{u}' | \underline{c}_1 >$. For the special case $n = 1$ eqn. (51) reduces to

$$< u_i' | \underline{c}_1 > = G_1(r_1^2, c_1^2, \underline{r}_1 \cdot \underline{c}_1) c_{1i} + H_1(r_1^2, c_1^2, \underline{r}_1 \cdot \underline{c}_1) r_{1i}. \quad (52)$$

Since $< u_i | \underline{c}_1 >$ must satisfy the continuity equation (c.f. eqn. (29)), G_1 and H_1 are not independent, and a straightforward calculation shows that they are related by

$$\frac{\partial G_1}{\partial r_{1i}} c_{1i} + \frac{\partial H_1}{\partial r_{1i}} r_{1i} + 3H_1 = 0. \quad (53)$$

This equation has a familiar analog in the theory of isotropic covariance tensors (c.f. Batchelor, eqn. (3.4.2)).

Determination of G_1 and H_1. In isotropic turbulence the nine elements of the covariance tensor can be expressed in terms of two scalar functions, and this greatly simplifies their experimental determination. A similar result holds for the isotropic two-point conditionally averaged velocity vector, but in this case the simplification is not so dramatic. Let $\hat{r}_1$ be a unit vector in the direction of $\underline{r}_1 = \underline{x} - \underline{x}_1$, $\hat{n}_1$ be any unit vector that is normal to $\hat{r}_1$, and denote the conditional averages of the velocity components $\underline{u}' \cdot \hat{r}_1$ and $\underline{u}' \cdot \hat{n}_1$ by

$$g(\underline{r}_1, \underline{c}_1) = < \underline{u}' \cdot \hat{n}_1 | \underline{c}_1 > \quad (54a)$$

and

$$h(\underline{r}_1, \underline{c}_1) = < \underline{u}' \cdot \hat{r}_1 | \underline{c}_1 >. \quad (55)$$

Then from eqn. (52)

$$g(\underline{r}_1, \underline{c}_1) = \hat{n}_1 \cdot < \underline{u}' | \underline{c}_1 > = G_1 \underline{c}_1 \cdot \hat{n}_1, \quad (56)$$

and

$$h(\underline{r}_1, \underline{c}_1) = \hat{r}_1 \cdot < \underline{u}' | \underline{c}_1 > = G_1 \underline{c}_1 \cdot \hat{r}_1 + H_1 r_1. \quad (57)$$

These equations may be used to eliminate G_1 and H_1 from eqn. (52), thereby resulting in the equation

$$< \underline{u}' | \underline{u}_1' = \underline{c}_1 > = \frac{(\underline{c}_1 - (\underline{c}_1 \cdot \hat{r}_1)\hat{r}_1)}{\underline{c}_1 \cdot \hat{n}_1} g + h\hat{r}_1. \quad (58)$$

Thus, it is necessary to measure only two components of $\underline{u}'$, but unfortunately all three components of $\underline{c}_1$ must be measured, and g and h are functions of $\underline{r}_1 \cdot \underline{c}_1$ as well as r_1^2 and c_1^2.

SUMMARY AND CONCLUSIONS

It has been shown that the closure problems in at least two theoretical formulations of turbulence ultimately reduce to the approximation of conditional averages in terms of lower order statistics. In the equation for f_1 the two-point conditional averages specify the average velocity field surrounding the point of interest, and are needed to determine the conditional pressure and viscous stresses at that point. In the numerical model conditional averages may be interpreted as interpolators or, more precisely, as estimators of the average inter-grid velocity field that contain enough information to determine the stresses at the grid points (including the sub-grid scale Reynolds stresses) in such a way as to minimize the errors due to the coarseness of the grid. It has also been shown that even the lowest order conditional averages, i.e., $< \underline{u} | \underline{u}_1 = \underline{v}_1 >$ or $< \underline{u}' | \underline{u}'_1 = \underline{c}_1 >$ contain more of the turbulence structure than conventional spatial correlations. Therefore, although this question was not directly addressed, it is probable that simple conditional averages of the type discussed could be usefully employed in studies of coherent flow structures. One advantage in using this type of conditional average would be that the event $E = \{\underline{u}_1 = \underline{v}_1\}$ is relatively simple and would have significance in any type of flow.

Because so little is known about conditional averages (empirical data is especially lacking), it is difficult to speculate about the ultimate importance of their role in turbulence research or their utility in turbulence modeling. Nonetheless, it is apparent that they do have theoretical significance and that they can provide useful insight into the structure of turbulence. Consequently, it appears that further studies of conditional averages are warranted, and it is suggested that experimental measurements of $< \underline{u} | \underline{u}_1 = \underline{v}_1 >$ or $< \underline{u}' | \underline{u}_1' = \underline{c}_1 >$ would be especially valuable contributions.

ACKNOWLEDGEMENT

This study was supported by the National Science Foundation under Grant NSF DES 74-14962.

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