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On the role of finite inertia and resistivity in axisymmetric pulsar magnetospheres

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Summary. Inertial and resistive effects in axisymmetric pulsar magnetospheres are examined as a function of the plasma density.

In the *quasineutral* limit the behaviour of the centrifugal-magnetic wind is consistent with torque-free formulations of the magnetic structure, the plasma inertia remaining small throughout the open field region. However, an anomalous resistivity, implying non-vanishing poloidal flow, must be invoked to prevent the two charge species from separating in the star's gravity field; unless the closed field region is charge separated, then, enhanced diffusion across field lines would be required in the vicinity of the equatorial plane.

In a steady, *charge-separated* magnetosphere, the effect of particle inertia and/or radiation reaction must be included, at least near the light cylinder, in the momentum equation and/or its divergence (i.e. the charge density), in order to satisfy charge conservation. If the near field is dipole-like and inertial effects are reasonably small near the star surface, the equatorial charge species will be restricted to toroidal motion, and either (a) the polar charge species are also so confined, or (b) the polar particles circulate back to the star. These possibilities are discussed in the light of various integrals of the motion.

1 Introduction

The only self-consistently computed model of a pulsar magnetosphere attempted so far has been that of Kuo-Petravic, Petravic & Roberts (1974). They numerically integrated equations representing momentum conservation and continuity for each charge species, together with Maxwell's equations. The star, initially at rest in vacuum and surrounded by an intrinsic magnetic dipole field aligned with the rotation axis, was gradually spun up to a final angular speed Ω; meanwhile, positively- or negatively-charged particles (but not both) were ejected from each point on the inner boundary, depending on the sign of $\mathbf{E}\cdot\mathbf{B}$ which became non-zero during the spin-up – thus conforming with the extraction mechanism originally discussed by Goldreich & Julian (1969). Their finding that the field lines remained roughly dipolar at distances greater than c/Ω from the rotation axis after the magnetosphere had filled, with the charged particles appearing to cross them freely, contradicted theoretical

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investigations based on the assumption $\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\simeq 0$ (*cf.* Goldreich & Julian 1969; or Michel 1973b), which required the field lines to open out beyond the light cylinder. Kuo-Petravic, Petravic & Roberts (1975) have attributed the complete breakdown of the frozen-in assumption which they observed to the charge-separated nature of their system. However, some doubt might be cast on their computations, because of the introduction of artificial diffusion terms (allowing coarser meshes) and increased particle masses (offsetting the effect of errors in the large electromagnetic terms), while it is not certain that a steady-state flow was actually achieved, particularly in view of the small size of their system.

The efforts of Kuo-Petravic *et al.* do suggest that it is essential to distinguish between charge-separated and quasineutral-plasma models of pulsar magnetospheres (*cf.* also Scharlemann 1974). In this investigation, which will be carried out within the axisymmetric framework, the roles of finite inertia and resistivity will be correlated with the degree of charge separation. By and large, models which purport to be self-consistent fail to take physical properties of the plasma into account (Kuo-Petravic *et al.*, for example, ignored the effect of radiation damping despite the enormous energies attained by their particles). In particular, the essential requirement of no net current outflow, that is $\nabla\cdot\mathbf{J}=0$ everywhere in a steady state, is often glossed over since inertial and dissipative effects would otherwise have to be incorporated. (On the other hand, Scharlemann (1974), in his treatment of inertial effects, seems to have overlooked the question of the overall charge conservation.) Rather than seeking a limited 'self-consistent' solution, we attempt to lay some groundwork on which a realistic pulsar-magnetosphere model might be based.

2 The generalized Ohm's law

The assumption

$$\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\simeq 0 \tag{1}$$

is justifiably the starting point for investigations of pulsar magnetospheres characterized by non-negligible plasma densities, as, in the intense magnetic field surrounding the compact object, charged particles should find it easier to move along rather than across the field lines, which thus become approximate equipotentials. $\boldsymbol{\beta}$ represents the velocity (normalized to c) of a frame of reference where the total momentum density of the plasma vanishes; the equality here is not shown as exact since it is generally necessary to allow for at least a small parallel component of electric field $|\mathbf{E}_{\|}|\ll|\boldsymbol{\beta}||\mathbf{B}|$ in order to take care of the balance of forces along $\mathbf{B}$ (for example, to provide support against gravity). Note also that approximation (1) does not necessarily imply a charge density given by $\rho_c\simeq -\nabla\cdot\boldsymbol{\beta}\times\mathbf{B}/4\pi$, since the divergence of the neglected terms might not itself be negligible (as illustrated in Section 4). In order to determine the extent to which the approximation is valid (and whether in fact it breaks down completely in some magnetospheric domains), it will be necessary to consider the dynamics of the individual charge species in one form or another. The problem is complicated by (primarily collisionless) interactions which on the one hand tend to prevent the charge species from separating under large-scale dynamical influences, but on the other result in small-scale electric fields which could have a cumulative effect – producing in particular a sizeable 'runaway' population (*cf.* Coppi & Ferrari 1970).

The plasma will be taken to consist of electrons of charge $-e$, mass m_e, and ions of charge Ze, mass m_i; correspondingly, the laboratory-frame number densities will be denoted by n_i, n_e. For each species, the particle velocities will be assumed isotropic in a (asterisked) frame where the momentum density vanishes (thus viscous effects are ignored), so that mean velocities $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_e$ (normalized to c) may be unambiguously defined, with the associated Lorentz factors $\gamma_i\equiv(1-|\boldsymbol{\beta}_i|^2)^{-1/2}$, $\gamma_e\equiv(1-|\boldsymbol{\beta}_e|^2)^{-1/2}$. Then, in a steady state, conservation of momentum for ions and electrons may be written

$$\nabla\cdot K_i = n_i Ze(\mathbf{E} + \boldsymbol{\beta}_i \times \mathbf{B}) + n_i m_i \nabla\left(\frac{GM}{r}\right) + \mathbf{R}_i - \mathbf{X}_{ei}, \tag{2a}$$

$$\nabla\cdot K_e = -n_e e(\mathbf{E} + \boldsymbol{\beta}_e \times \mathbf{B}) + n_e m_e \nabla\left(\frac{GM}{r}\right) + \mathbf{R}_e + \mathbf{X}_{ei}, \tag{2b}$$

where

$$\nabla\cdot K_i = \nabla\cdot[(\epsilon_i^* + P_i)\,\gamma_i\boldsymbol{\beta}_i\boldsymbol{\beta}_i] + \nabla P_i,$$

$$\nabla\cdot K_e = \nabla\cdot[(\epsilon_e^* + P_e)\,\gamma_e\boldsymbol{\beta}_e\boldsymbol{\beta}_e] + \nabla P_e.$$

Here K_i, K_e are the kinetic tensor densities (Rossi & Olbert 1970) corresponding to each charge species, ϵ_i^*, ϵ_e^* represent the respective comoving-frame energy densities, and P_i, P_e denote the individual scalar pressures. In addition to the Lorentz force, we have included in equations (2) gravity in the non-relativistic approximation, and (symbolically) radiation reaction $\mathbf{R}_i$, $\mathbf{R}_e$ as well as terms $\pm\mathbf{X}_{ei}$ representing the momentum exchange between the two charge species. If both ions and electrons are assumed 'cold' in the sense $\epsilon_i^* \simeq n_i^* m_i c^2$, $\epsilon_e^* \simeq n_e^* m_e c^2$, $P_i \simeq P_e \simeq 0$ (thereby avoiding the question of whether the particle velocities are indeed isotropic in the asterisked frame), then

$$\nabla\cdot K_i \simeq \nabla\cdot(n_i m_i c^2 \gamma_i \boldsymbol{\beta}_i\boldsymbol{\beta}_i), \tag{3a}$$

$$\nabla\cdot K_e \simeq \nabla\cdot(n_e m_e c^2 \gamma_e \boldsymbol{\beta}_e\boldsymbol{\beta}_e), \tag{3b}$$

while the steady-state continuity equations may be given as

$$\nabla\cdot(n_i\boldsymbol{\beta}_i) = 0, \tag{4a}$$

$$\nabla\cdot(n_e\boldsymbol{\beta}_e) = 0. \tag{4b}$$

Equations (2) provide a direct description of the magnetospheric dynamics which would be particularly appropriate if the two charge species manage to move more or less independently of each other except for coupling through the macroscopic electric field. This would be the case in a highly charge-separated magnetosphere. Then equations (2), (3) would reduce to

$$\mathbf{E} + \boldsymbol{\beta}_i \times \mathbf{B} \simeq 0, \tag{5a}$$

$$\mathbf{E} + \boldsymbol{\beta}_e \times \mathbf{B} \simeq 0 \tag{5b}$$

only if

$$\frac{c|\boldsymbol{\beta}_i|}{\omega_{Bi}} \ll \frac{\gamma_i|\boldsymbol{\beta}_i|^2}{|\boldsymbol{\beta}_i\cdot\nabla(\gamma_i\boldsymbol{\beta}_i)|}, \quad \frac{c|\boldsymbol{\beta}_e|}{\omega_{Be}} \ll \frac{\gamma_e|\boldsymbol{\beta}_e|^2}{|\boldsymbol{\beta}_e\cdot\nabla(\gamma_e\boldsymbol{\beta}_e)|}, \tag{6a, b}$$

where $\omega_{Bi} \equiv Ze|\mathbf{B}|/m_i c\gamma_i$, $\omega_{Be} \equiv e|\mathbf{B}|/m_e c\gamma_e$: the gyroradii must remain small compared with the scale length over which the particle acceleration takes place. In addition, of course, the conditions $|\mathbf{R}_i|$, $|\mathbf{X}_{ei}| \ll n_i Ze|\boldsymbol{\beta}_i||\mathbf{B}|$ and $|\mathbf{R}_e|$, $|\mathbf{X}_{ei}| \ll n_e e|\boldsymbol{\beta}_e||\mathbf{B}|$ must be satisfied.

However, a *quasineutral* plasma would be characterized by rather strong coupling between electrons and ions through the momentum exchange terms $\pm\mathbf{X}_{ei}$ in equations (2); and it may then be more advantageous to employ a one-fluid description in which momentum balance for the plasma as a whole is supplemented by an equation describing the relative motion of the two charge species. It will be assumed that, in a (primed) frame where the total momentum density of the plasma vanishes, the kinetic energy density of each charge species is small compared with its rest-mass energy density, i.e. $\gamma_i' \simeq \gamma_e' \simeq 1$. Then, by transforming $\boldsymbol{K}_i, \boldsymbol{K}_e$ from the primed to the laboratory frame, it is readily shown that

$$\nabla \cdot \boldsymbol{K}_i \simeq \nabla \cdot (n_i m_i c^2 \gamma \boldsymbol{\beta}\boldsymbol{\beta}_i + n_i m_i c^2 \gamma \boldsymbol{\beta}_i \boldsymbol{\beta} - n_i m_i c^2 \gamma \boldsymbol{\beta}\boldsymbol{\beta}),$$

$$\nabla \cdot \boldsymbol{K}_e \simeq \nabla \cdot (n_e m_e c^2 \gamma \boldsymbol{\beta}\boldsymbol{\beta}_e + n_e m_e c^2 \gamma \boldsymbol{\beta}_e \boldsymbol{\beta} - n_e m_e c^2 \gamma \boldsymbol{\beta}\boldsymbol{\beta})$$

with $\boldsymbol{\beta} \simeq (n_i m_i \boldsymbol{\beta}_i + n_e m_e \boldsymbol{\beta}_e)/(n_i m_i + n_e m_e)$ (*cf.* Rossi & Olbert 1970; Ardavan 1976a). Substituting in these expressions, equations (2) may be added to yield momentum conservation for the plasma as a whole

$$\nabla \cdot (\boldsymbol{K}_i + \boldsymbol{K}_e) = \nabla \cdot (\rho c^2 \gamma \boldsymbol{\beta}\boldsymbol{\beta}) = \rho_c \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} + \rho \nabla \left(\frac{GM}{r}\right) + (\mathbf{R}_i + \mathbf{R}_e), \tag{7}$$

where $\rho = n_i m_i + n_e m_e$, $\rho_c = (n_i Z - n_e) e$, and $\mathbf{J} = (n_i Z \boldsymbol{\beta}_i - n_e \boldsymbol{\beta}_e) ec$ are respectively the total mass, charge and current densities in the laboratory frame, and ρ, $\boldsymbol{\beta}$ are related through the continuity equation

$$\nabla \cdot (\rho \boldsymbol{\beta}) = 0. \tag{8}$$

If the ion and electron momentum equations are first multiplied by the respective charge-to-mass ratios and then added, the result is a generalized Ohm's law relating $\mathbf{J}$ to $\mathbf{E}$:

$$\begin{aligned} \nabla \cdot \left(\frac{Ze}{m_i} \boldsymbol{K}_i - \frac{e}{m_e} \boldsymbol{K}_e\right) &= \nabla \cdot (c\gamma \boldsymbol{\beta} \mathbf{J} + c\gamma \mathbf{J} \boldsymbol{\beta} - c^2 \rho_c \gamma \boldsymbol{\beta}\boldsymbol{\beta}) \\ &= \frac{Ze^2 \rho}{m_i m_e} (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \left(\frac{m_i - Z m_e}{m_i m_e}\right) e \left(\rho_c \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}\right) + \rho_c \nabla \left(\frac{GM}{r}\right) \\ &\quad + \left(\frac{Ze}{m_i} \mathbf{R}_i - \frac{e}{m_e} \mathbf{R}_e\right) - \left(\frac{Ze}{m_i} + \frac{e}{m_e}\right) \mathbf{X}_{ei}, \end{aligned}$$

or, making use of equation (7),

$$\begin{aligned} \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B} = &\frac{m_i}{Ze\rho} \nabla \cdot \left\{\gamma \left[c^2 \xi \rho \boldsymbol{\beta}\boldsymbol{\beta} + \frac{m_e c}{e} (\boldsymbol{\beta} \mathbf{J}' + \mathbf{J}' \boldsymbol{\beta})\right]\right\} - \frac{m_i \xi}{Ze} \nabla \left(\frac{GM}{r}\right) \\ &- \frac{m_i}{Ze\rho} \left(\mathbf{R}_i - \frac{Z m_e}{m_i} \mathbf{R}_e\right) + \frac{m_i}{Ze\rho} \left(1 + \frac{Z m_e}{m_i}\right) \mathbf{X}_{ei}, \end{aligned} \tag{9}$$

where

$$\xi \equiv 1 - Z m_e/m_i + m_e \rho_c / e\rho, \quad \mathbf{J}' \equiv \mathbf{J} - \rho_c c \boldsymbol{\beta}.$$

The factor of γ accompanying the quadratic terms (*cf.* Ardavan 1976a) is the only formal difference between equations (7), (9) and the non-relativistic versions given by, e.g. Rossi & Olbert (1970). It is again emphasized that this one-fluid description is only applicable to a 'near-equilibrium' plasma and precludes the runaway of one charge species with respect to the other. Although Coulomb collisions will be rare in the low-density and/or relativistic circumpulsar plasma, the restriction becomes less severe if the 'anomalous' resistivity (Coppi & Ferrari 1970), associated with microinstabilities induced by the relative streaming of electrons and ions, is taken into account. Accordingly, it will be appropriate to identify

$$\frac{m_i}{Ze\rho} \left(1 + \frac{Z m_e}{m_i}\right) \mathbf{X}_{ei} \to \eta_{an} \mathbf{J}'$$

in equation (9). As a rough upper limit, $\eta_{\rm an} \lesssim 4\pi/\omega_{\rm p}$, where $\omega_{\rm p} \equiv (4\pi e^2\rho/m_{\rm i}m_{\rm e}\gamma)^{1/2}$ will be taken as a characteristic plasma frequency.* Of course, radiation reaction will also serve to impede the runaway of electrons.

In the limit of total charge separation, equations (7) and (9) reduce properly to equation (2a) or (2b) (with kinetic tensor given by equation (3a) or (3b), and $\mathbf{X}_{\rm ei} = 0$), and obviously there is little point in referring to a generalized Ohm's law if the degree of charge separation is high.† The relation to the degree of charge separation can be illustrated by rewriting equation (9) in the form

$$\rho_{\rm c}\mathbf{E}' = \frac{m_{\rm i}\rho_{\rm c}}{Ze\rho}\mathbf{F}_{\rm eff} \tag{10}$$

where

$$\mathbf{E}' \equiv \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B},$$

$$\mathbf{F}_{\rm eff} \equiv \boldsymbol{\nabla}\cdot\left\{\gamma\left[c^2\xi\rho\boldsymbol{\beta}\boldsymbol{\beta} + \frac{m_{\rm e}c}{e}(\boldsymbol{\beta}\mathbf{J}' + \mathbf{J}'\boldsymbol{\beta})\right]\right\} - \xi\rho\boldsymbol{\nabla}\left(\frac{GM}{r}\right) - \left(\mathbf{R}_{\rm i} - \frac{Zm_{\rm e}}{m_{\rm i}}\mathbf{R}_{\rm e}\right) + \frac{Ze\rho\eta_{\rm an}}{m_{\rm i}}\mathbf{J}'.$$

Equation (10) indicates that, if resistive effects are small, setting $\mathbf{E}' = 0$ in momentum conservation (7) results in an error $\sigma(|m_{\rm i}\rho_{\rm c}/Ze\rho|)$ relative to the non-electromagnetic terms in the latter equation. The point is just that assumption (1) *can* be perfectly compatible with equation (7) *if* the plasma is quasineutral, in the sense $|m_{\rm i}\rho_{\rm c}/Ze\rho| \ll 1$. The terms in $\mathbf{F}_{\rm eff}$ involving $\mathbf{J}'$, which have no counterpart in the momentum equation, will only be significant where

$$\frac{|\mathbf{B}|}{|\boldsymbol{\nabla}\times\mathbf{B}|} \sim \frac{1}{\gamma|\boldsymbol{\beta}|}\frac{c}{\omega_{\rm p}}, \tag{11}$$

that is within current layers (such as might occur about a symmetry plane, see Section 3.3) where the magnetic field varies on a scale comparable with the plasma skin depth.

For two relevant cases the generalized Ohm's law may be simplified substantially. If $Zm_{\rm e}/m_{\rm i} \ll 1$, $|m_{\rm e}\rho_{\rm c}/e\rho| \ll 1$ (a mild restriction on the concentration of electrons relative to ions), and $m_{\rm e}|\mathbf{J}'|/e\rho c|\boldsymbol{\beta}| \ll 1$ (otherwise electrons would be in runaway or a current layer would be implied), then equation (9) or (10) reduces to

$$\begin{aligned}\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} &\simeq \frac{m_{\rm i}c^2}{Ze}\,\boldsymbol{\beta}\cdot\boldsymbol{\nabla}(\gamma\boldsymbol{\beta}) - \frac{m_{\rm i}}{Ze}\,\boldsymbol{\nabla}\left(\frac{GM}{r}\right) + \eta_{\rm an}\mathbf{J}' \\ &\simeq \left[\left(\eta_{\rm an}\mathbf{J}' + \frac{m_{\rm i}}{Ze\rho}\frac{1}{c}\mathbf{J}'\times\mathbf{B}\right)\Big/\left(1 - \frac{m_{\rm i}\rho_{\rm c}}{Ze\rho}\right)\right],\end{aligned} \tag{12}$$

noting that $\xi \simeq 1$, neglecting radiation reaction since $|\mathbf{R}_{\rm i}| \ll |\mathbf{R}_{\rm e}|$ while the contribution due to electrons is reduced by a factor $Zm_{\rm e}/m_{\rm i}$, and making use of momentum conservation (7) and continuity (8). A necessary condition for assumption (1) to be valid is then that

$$\frac{c|\boldsymbol{\beta}|}{\omega_{\rm B}} \ll \frac{\gamma|\boldsymbol{\beta}|^2}{|\boldsymbol{\beta}\cdot\boldsymbol{\nabla}(\gamma\boldsymbol{\beta})|}, \quad \omega_{\rm B} \equiv \frac{Ze|\mathbf{B}|}{m_{\rm i}c\gamma} \tag{13}$$

* Kaplan & Tsytovich (1973), however, argue that the resistivity may be further enhanced through the generation of Alfvén-wave turbulence in a strong magnetic field.

† The claim by Ardavan (1976a) that assumption (1) is valid in the relativistic regime, independently of the degree of charge separation, seems to have resulted from making order-of-magnitude comparisons between various groupings of terms in his equation (15), but overlooking cancellations between individual terms.

(*cf.* equations (6)), or alternatively, if $|m_i\rho_c/Ze\rho| \ll 1$, the 'Hall' current arising from the inertial drifts must satisfy $|\mathbf{J}'| \ll Ze\rho c|\boldsymbol{\beta}|/m_i$. The resistive term has been retained in equation (12) as the principal contribution to $\mathbf{E}_\parallel$. In the case of an electron–positron plasma, $Z = 1$, $m_i = m_e$, and $\xi = m_e\rho_c/e\rho$; then in equation (10) $\rho_c\mathbf{E}'$ is $\sigma(|m_e\rho_c/e\rho|^2)$ compared to the non-electromagnetic forces acting on the plasma, and the generalized Ohm's law may be approximated as $\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B} \simeq \eta_{an}\mathbf{J}'$.

The suggested relationship of inertial and dissipative effects to the degree of charge separation will be taken up further. Because of the obvious advantages of applying a one-fluid description to a quasineutral plasma, this limit will be considered separately.

3 Flux-freezing in a quasineutral-plasma magnetosphere

In this section the existence of a time-independent, quasineutral-plasma magnetosphere is postulated, with the star acting as an appropriate source. It will be apparent that quasineutrality can only be maintained if relative streaming between the two charge species is allowed for, and reconciled with the requirement of a steady current flow through the magnetosphere. Thus we are not claiming to develop a self-consistent model, but simply explore some consequences of imposing quasineutrality on the system.

Two sets of coordinates will be employed, cylindrical polars (R, ϕ, z) centred about the rotation (z) axis, and spherical polars (r, θ, ϕ) centred about the star (appropriate for $R \ll c/\Omega$ or $R \gg c/\Omega$). As $\nabla \cdot \mathbf{B} = 0$ and the system is assumed axisymmetric, the poloidal magnetic field may be expressed as

$$\mathbf{B}_p = -\frac{1}{2}B_0\frac{a^2}{R}e_\phi \times \nabla\Psi, \tag{14}$$

Ψ representing a magnetic surface $(\mathbf{B} \cdot \nabla\Psi = 0)$. Typical values of $|\mathbf{B}_p|$, $|B_\phi|$ and ρ/m_i at the star surface $r = a$ will be denoted by B_0, B_{T0} and n_0, respectively. An immediate consequence of assumption (1), time-independence and axisymmetry ($\mathbf{E} = -\nabla\chi$, $E_\phi = -R^{-1}\partial\chi/\partial\phi = 0$), is that the plasma flow must obey

$$\boldsymbol{\beta} \simeq \left(\frac{\boldsymbol{\beta}_p \cdot \mathbf{B}_p}{|\mathbf{B}_p|^2}\right)\mathbf{B} + \frac{\Omega(\Psi)R}{c}e_\phi \tag{15}$$

(Mestel 1961), where $\Omega(\Psi)$, representing the local angular speed of rotation of the star, is constant along a given magnetic surface; because of the rigidity of the neutron star crust, Ω is assumed constant everywhere. With $\boldsymbol{\beta}$ of the form (15), equation (1) becomes

$$\mathbf{E} + \frac{\Omega R}{c}e_\phi \times \mathbf{B} \simeq 0; \tag{16}$$

if the divergence of the neglected inertial and resistive terms is also negligible (but *cf.* Section 4), Poisson's equation then yields

$$\rho_c \simeq -\frac{\Omega B_z}{2\pi c} + \frac{\Omega R J_\phi}{c^2}. \tag{17}$$

Moreover, continuity (8) may be expressed as

$$\frac{\rho|\boldsymbol{\beta}_p|}{|\mathbf{B}_p|} \simeq \left(\frac{n_0 m_i \Omega a}{cB_0}\right)\mathcal{N}(\Psi), \tag{18}$$

which simply states that constant mass flux is enclosed between given magnetic surfaces.

3.1 THE CONSERVATION OF ANGULAR MOMENTUM AND ENERGY

Equations (15), (16) and (18) can be consistent with the constraint $\beta_\phi < 1$ only if $|\boldsymbol{\beta}_\mathrm{p}| \neq 0$ along all magnetic surfaces that cross the light cylinder*; and since $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\Psi \simeq 0$, a steady outflow of plasma could occur only along open ($|\mathbf{B}_\mathrm{p}| \to |B_\mathrm{r}|$) field lines. According to the discussion of the preceding section, however, the flux-freezing assumption is only valid if (a) the plasma inertia remains reasonably small (condition (13)), and (b) outside of current layers (11) where resistive effects must be taken into account. As regards the magnitude of the inertial drifts, the conditions under which a plasma flow of the form (15) is consistent with angular momentum and energy conservation can be established. Even if $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\Psi \to 0$, in general $\mathbf{J}\cdot\boldsymbol{\nabla}\Psi \neq 0$ and angular momentum is exchanged between field and plasma. This is described by the toroidal component of momentum conservation (7), which, making use of equations (15) and (18), setting $\mathbf{J}_\mathrm{p} = (c/4\pi)\,\boldsymbol{\nabla}\times(B_\phi e_\phi)$, and ignoring radiation reaction, may be cast into

$$B_\phi \simeq -B_{T0}\frac{a}{R}\Phi(\Psi) + \mathrm{sgn}\,\{\boldsymbol{\beta}_\mathrm{p}\cdot\mathbf{B}_\mathrm{p}\}\left(\frac{4\pi n_0 m_\mathrm{i} c\Omega a}{B_0}\right)\mathcal{N}(\Psi)\,\gamma\beta_\phi, \tag{19}$$

where $\mathrm{sgn}\{\Phi\} = \mathrm{sgn}\{\boldsymbol{\beta}_\mathrm{p}\cdot\mathbf{B}_\mathrm{p}\}$ for swept-back field lines. The two terms in this expression for the toroidal magnetic field represent respectively the contribution of field-aligned current, $\mathbf{J}_{\mathrm{p}\parallel} = \alpha\mathbf{B}_\mathrm{p}$, less a correction due to the Coriolis force experienced by the plasma, which drives a drift current $\mathbf{J}_{\mathrm{p}\perp} = [-\rho c^3\boldsymbol{\beta}\cdot\boldsymbol{\nabla}(R\gamma\beta_\phi)/R\,|\mathbf{B}_\mathrm{p}|^2]\,e_\phi\times\mathbf{B}_\mathrm{p}$. From equation (12),

$$\mathbf{J}_{\mathrm{p}\perp} \simeq \frac{Ze\rho c}{m_\mathrm{i}}\left(\boldsymbol{\beta}_{\mathrm{p}\perp} + \eta_\mathrm{an}J'_\phi\frac{e_\phi\times\mathbf{B}_\mathrm{p}}{|\mathbf{B}_\mathrm{p}|^2}\right), \tag{20}$$

which again makes it apparent that the approximations of magnetic flux-freezing ($|\boldsymbol{\beta}_{\mathrm{p}\perp}| \ll |\boldsymbol{\beta}_\mathrm{p}|$) and a 'torque-free' magnetic field ($|\mathbf{J}_{\mathrm{p}\perp}| \ll |\rho_\mathrm{c}|c$) are quite distinct for the quasineutral case ($|\rho_\mathrm{c}| \ll Ze\rho/m_\mathrm{i}$).

The balance of forces along $\mathbf{B}$ bears critically on the whole magnetospheric structure. Assuming that $|m_\mathrm{i}\rho_\mathrm{c}/Ze\rho| \ll 1$, equation (12) indicates that

$$\mathbf{E}_\parallel \simeq \eta_\mathrm{an}\mathbf{J}'_{\mathrm{p}\parallel}. \tag{21}$$

The scalar product of the momentum equation (7) with $\boldsymbol{\beta}$ (again ignoring radiation reaction) then yields energy conservation in the form

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left\{\gamma\left[1-\left(\frac{\Omega R}{c}\right)\beta_\phi\right] - \frac{GM}{c^2 r}\right\} \simeq \frac{\rho_\mathrm{c}\eta_\mathrm{an}}{\rho c^2}\mathbf{J}'_\mathrm{p}\cdot\mathbf{B}_\mathrm{p}, \tag{22}$$

after making use of equations (15), (16) and (18). Assuming

$$\eta_\mathrm{an} \lesssim 4\pi/\omega_\mathrm{p} \equiv (4\pi m_\mathrm{i} m_\mathrm{e}\gamma/e^2\rho)^{1/2}$$

as an upper limit for the parallel resistivity, equations (21) and (22) suggest that acceleration due to an electrostatic field maintained through plasma turbulence might be substantial for relatively low plasma densities or within current layers. Otherwise centrifugal acceleration, represented by the term in equation (22) involving β_ϕ, would dominate beyond the point where $(\Omega R/c)\,\gamma\beta_\phi \sim GM/c^2 r$. Normalizing to the Crab pulsar's light cylinder, an upper limit for the critical density separating the two regimes is

$$\frac{\rho}{m_\mathrm{i}} \sim \frac{3\times10^7\,\mathrm{cm}^{-3}}{\gamma}\left(\frac{c}{\Omega R\beta_\phi}\right)^{2/3}\left(\frac{\Omega}{190\,\mathrm{s}^{-1}}\right)^{2/3}\left(\frac{|\mathbf{B}|}{10^6\,\mathrm{gauss}}\right)^{4/3}\left(\frac{R\,|\boldsymbol{\nabla}\times\mathbf{B}|}{|\mathbf{B}|}\right)^{2/3}; \tag{23}$$

* Pilipp (1974) has shown that, if equation (16) holds, the body of plasma cannot be confined within the light cylinder (although his analysis does not include the possibility of 'gaps' within the space-charge region).

by way of comparison, the number density corresponding to a charge-separated magnetosphere would be of order $n_c \equiv \Omega|\mathbf{B}|/4\pi ce$ or $\rho/m_i \sim 1\times10^6\,\mathrm{cm^{-3}}(\Omega/190\,\mathrm{s^{-1}})(|\mathbf{B}|/10^6\,\mathrm{gauss})$. In Section 4 it is pointed out that if the magnetosphere is highly charge separated near the star, charge mixing accompanied by streaming instabilities might occur beyond the centrifugal–gravity balance surface.

If, however, a quasineutral-plasma magnetosphere is postulated, an anomalous resistivity *must* be invoked inside the centrifugal–gravity surface. The parallel electric field required to counteract gravity, $|\mathbf{E}_\parallel| \sim GM\rho|B_r|/r^2|\rho_c||\mathbf{B}|$, will greatly exceed the 'runaway' value above which Coulomb collisions occur too infrequently to prevent electrons from separating *en masse* from ions, viz. $|\mathbf{E}_c| \sim m_e v_{Te}\nu_{ei}/e \sim 4\pi e\omega_{pe}^2/v_{Te}^2$, where $v_{Te} = (kT_e/m_e)^{1/2}$ is the electron thermal speed, $\omega_{pe} = (4\pi e^2 n_e/m_e)^{1/2}$ the (non-relativistic) electron plasma frequency, and $\nu_{ei} \lesssim \omega_{pe}^4/n_e v_{Te}^3$ the electron–ion collision frequency: thus, near the star surface*

$$\frac{|\mathbf{E}_\parallel|}{|\mathbf{E}_c|} \sim 2\times10^4\left(\frac{T_e}{10^7\,\mathrm{K}}\right)\left(\frac{M}{10^{33}\,\mathrm{g}}\right)\left(\frac{10^6\,\mathrm{cm}}{r}\right)^2\left(\frac{190\,\mathrm{s^{-1}}}{\Omega}\right)\left(\frac{10^{12}\,\mathrm{gauss}}{|\mathbf{B}|}\right).$$

In fact, runaway is unlikely to occur, for plasma turbulence will be excited and the effective collision frequency increased by a factor not orders of magnitude less than

$$\omega_{pe}/\nu_{ei} \sim 1\times10^7(n_c/n_e)^{1/2}(T_e/10^7\,\mathrm{K})^{3/2}(\Omega/190\,\mathrm{s^{-1}})^{-1/2}(|\mathbf{B}|/10^{12}\,\mathrm{gauss})^{-1/2};$$

the current density needed to maintain the coupling is reduced by the same factor, from the improbable value $|\mathbf{J}_{p\parallel}| \sim (|\mathbf{E}_\parallel|/|\mathbf{E}_c|)\,en_e v_{Te}$. In principle a plasma as dense as $\sim10^5 n_c$ (near the star surface

$$n_c \sim \Omega B_0/4\pi ce \sim 1\times10^{12}\,\mathrm{cm^{-3}}\,[\Omega/190\,\mathrm{s^{-1}}][B_0/10^{12}\,\mathrm{gauss}])$$

could be supported against gravity; in the absence of the anomalous resistivity, on the other hand, the two charge species would be cleanly segregated into different regions under the action of the macroscopic $\mathbf{E}_\parallel$. Of course, the relative streaming between the charge species which gives rise to the anomalous resistivity must be reconciled with the overall current distribution: the actual degree of charge separation will depend largely on the requirement of steady current flow through the magnetosphere. For example, quasineutrality could not be maintained along magnetic surfaces which close well within the light cylinder, unless both charge species circulate along them back to the star (such a flow would probably be unsteady, leading to a rapid dissipation of energy).

3.2 THE CENTRIFUGAL-MAGNETIC WIND

The gravitational and resistive contributions to energy conservation (22) will now be ignored, and only the term representing centrifugal driving retained. This amounts to considering the plasma flow well outside the centrifugal–gravity balance surface, invoking plasma densities in excess of the critical value (23), and supposing current layers to be absent. Then equation (22) simplifies to†

$$\gamma \simeq [\epsilon(\Psi)]\Big/\left[1-\left(\frac{\Omega R}{c}\right)\beta_\phi\right] \simeq 1\Big/\left[1-\left(\frac{\Omega R}{c}\right)\beta_\phi\right], \tag{24}$$

* $|\mathbf{E}_c|$ has in fact been overestimated since a magnetic field sufficiently strong that $kT_e \ll \hbar\omega_{Be}$ has the effect of reducing the Coulomb collision rate by constraining electrons to move one-dimensionally (*cf.* Basko & Sunyaev 1975).

† The integrated equations (15), (18), (19) and (24) were employed by Michel (1969) in his treatment of relativistic stellar winds, in the spherically symmetric case, and are completely analogous to the non-relativistic versions given by Mestel (1968).

where, to make the second step, the plasma flow is assumed non-relativistic for $R \ll c/\Omega$. As $\gamma \equiv (1-\beta_\phi^2 - |\boldsymbol{\beta}_\mathrm{p}|^2)^{-1/2}$, the poloidal and toroidal bulk velocities are related through

$$|\boldsymbol{\beta}_\mathrm{p}| \simeq \left\{2\left(\frac{\Omega R}{c}\right)\beta_\phi - \left[1+\left(\frac{\Omega R}{c}\right)^2\right]\beta_\phi^2\right\}^{1/2} \tag{25}$$

The existence of a 'centrifugal-magnetic wind' is thus implied by equations (15), (18), (19) and (24); moreover, since $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\Psi \simeq 0$, an open field structure is required (if the possibility of plasma inflow to the star is excluded). The question of how much energy the centrifugally-driven plasma acquires is considered in this subsection.

Eliminating $|\boldsymbol{\beta}_\mathrm{p}|$ and B_ϕ from equation (15) by means of equations (19), (24) and (25), yields the algebraic equation

$$\beta_\phi \simeq x - x[2x\beta_\phi - (1+x^2)\beta_\phi^2]^{1/2}\left(T - N\frac{x\beta_\phi}{1-x\beta_\phi}\right), \tag{26}$$

or alternatively, eliminating β_ϕ in favour of γ,

$$\gamma \simeq \frac{1}{1-x^2} - \frac{x^2}{1-x^2}\left[\gamma^2 - \left(\frac{\gamma-1}{x}\right)^2 - 1\right]^{1/2}[T - N(\gamma-1)] \tag{26$'$}$$

(the latter form being more convenient for the relativistic regime). Here $x \equiv \Omega R/c$,

$$N(x,z) \equiv \frac{4\pi\rho c^2|\boldsymbol{\beta}_\mathrm{p}|}{x^2|\mathbf{B}_\mathrm{p}|^2} \simeq \frac{4\pi n_0 m_\mathrm{i} c\Omega a}{B_0}\,\frac{\mathcal{N}(\Psi)}{x^2|\mathbf{B}_\mathrm{p}|}, \tag{27a}$$

$$T(x,z) \equiv \frac{|B_{\phi 0}|}{x|\mathbf{B}_\mathrm{p}|} \equiv \frac{B_{T0}\Omega a}{c}\,\frac{|\Phi(\Psi)|}{x^2|\mathbf{B}_\mathrm{p}|} \tag{27b}$$

where $|B_{\phi 0}| \equiv B_{T0}(\Omega a/c)|\Phi(\Psi)|/x \geqslant |B_\phi|$ represents the contribution of field-aligned current to the toroidal magnetic field – *cf.* equation (19). If $|\mathbf{B}_\mathrm{p}| \propto x^{-k}$ with $k<2$, N and T would vanish in the limit $x\to\infty$ and γ could not remain finite and non-zero, whereas with $k>2$ the field lines would close: therefore $\Psi\to\Psi(\theta)$ and $x^2|\mathbf{B}_\mathrm{p}|$ must eventually become constant along a given magnetic surface. Typically, then, $N \ll 1$. Moreover, according to equation (15), $|B_\phi| \to x|\mathbf{B}_\mathrm{p}|/|\boldsymbol{\beta}_\mathrm{p}|$ in the limit $x\to\infty$, so that $T \gtrsim 1$ so long as $|\boldsymbol{\beta}_\mathrm{p}| \lesssim 1$.

Equations (26) may be converted into quartics, the roots of which can be expressed analytically (*cf.* Abramowitz & Stegun 1965). However, the salient features of the plasma flow can be gleaned directly from the above forms. Thus, well inside the light cylinder,

$$\beta_\phi \simeq x - x(2x\beta_\phi - \beta_\phi^2)^{1/2}\,T$$

or

$$\beta_\phi \simeq x\left[1 - \frac{Tx}{(1+T^2x^2)^{1/2}}\right] \qquad x \ll 1. \tag{28}$$

At the light cylinder equation (26′) reduces to $1 \simeq 2^{1/2}(\gamma-1)^{1/2}[T - N(\gamma-1)]$, yielding the two solutions

$$\gamma - 1 \simeq \frac{1}{2T^2},\quad \left[1 - \left(\frac{N}{2T}\right)^{1/2}\frac{1}{T}\right]\frac{T}{N} \qquad x = 1. \tag{29}$$

Finally, well outside the light cylinder equation (26′) takes the form

$$\gamma \simeq (\gamma^2 - 1)^{1/2}[T - N(\gamma - 1)];$$

unless $T \simeq 1$, there will again be low- and high-energy roots, respectively

$$\gamma \simeq 1 \Big/ \left(1 - \frac{1}{T^2}\right)^{1/2}, \quad \left(1 - \frac{1}{T}\right)\frac{T}{N} \qquad x \gg 1. \tag{30}$$

However, as may be verified by numerically tracing the roots of the corresponding quartics, only the low-γ branch extends continuously from the vicinity of the star to the far region, with β_ϕ increasing as $\sim x$ inside the light cylinder, reaching a value of $1/(1 + 2T^2)$ at $x = 1$, and falling off outside as $\sim x^{-1}$, so that $\gamma \to 1/(1 - 1/T^2)^{1/2}$ in the limit $x \to \infty$. Apparently, by minimizing T with respect to γ using the asymptotic form of equation (26′), an upper limit can be set on γ, for given N. Thus

$$\left(\frac{\partial T}{\partial \gamma}\right)_N = 0 \simeq \frac{\partial}{\partial \gamma}\left[\frac{\gamma}{(\gamma^2 - 1)^{1/2}} + N(\gamma - 1)\right],$$

whence

$$\gamma \lesssim \left(1 + \frac{1}{N^{2/3}}\right)^{1/2} \simeq \frac{1}{N^{1/3}} \qquad x \to \infty \tag{31}$$

and

$$T \gtrsim (1 + N^{2/3})^{3/2} - N \simeq 1 + \tfrac{3}{2}N^{2/3} \qquad x \to \infty. \tag{32}$$

For $T < 1 + 3N^{2/3}/2$, γ becomes complex (the low- and high-energy branches merge asymptotically with $T \simeq 1 + 3N^{2/3}/2$). (These results agree with Michel's (1969), who argued, in the spherically symmetric case, that current outflow should occur in such a way that minimum torque is exerted on the star, for a given mass flux.) Evidently condition (13) will be satisfied out to a point

$$x_{\rm lim} \sim \left(\frac{\Omega a}{c}\right)^3 \frac{ZeB_0}{m_i c \gamma \Omega} \ggg 1,$$

beyond which the plasma decouples from the magnetic field and the field lines would be allowed to close. For $x \ll x_{\rm lim}$, the flux-freezing assumption has been applied consistently, at least as regards the neglect of plasma inertia in the generalized Ohm's law. Equations (31) and (32) can be interpreted quite simply. The plasma continues to be accelerated so long as the centrifugal force $\rho c^2 \gamma \beta_\phi^2/R$ which it experiences does not overcome the restraining magnetic tension

$$|B_z'|^2/4\pi R \simeq |B_z|^2/4\pi\gamma^2 R,$$

in a frame instantaneously comoving with the plasma where $\mathbf{E}' \simeq \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \simeq 0$. Applying this condition at the light cylinder yields

$$\gamma \lesssim \left(\frac{|\mathbf{B}_\mathrm{p}|^2}{4\pi\rho c^2}\right)^{1/3}\Bigg|_{R \simeq c/\Omega} \equiv \gamma_\mathrm{c},$$

which in turn implies a lower bound on $|B_\phi|$ through equations (15), (24).★

3.3 THE TORQUE-FREE APPROXIMATION

From equations (19) and (24), the current drift driven by the Coriolis force can be neglected if

$$\gamma - 1 \ll \frac{B_{T0}B_0}{4\pi n_0 m_i c^2}, \tag{33}$$

and in fact equations (31) and (32) indicate that $\gamma \lesssim (B_{T0}B_0/4\pi n_0 m_i c^2)^{1/3}$ so that $\mathbf{J}_p \simeq \alpha \mathbf{B}_p$ ($B_{T0}B_0/4\pi n_0 m_i c^2 \gg 1$ by assumption). Moreover, by means of the R component of momentum conservation (7) (neglecting gravity and radiation reaction), the toroidal current density may be expressed as

$$J_\phi \simeq -\left\{2 - \frac{\alpha B_\phi}{(\Omega R/c)^2(cB_z/4\pi R)} - \frac{4\pi\rho c^2 e_R\cdot[R\boldsymbol{\beta}\cdot\boldsymbol{\nabla}(\gamma\boldsymbol{\beta})]}{(\Omega R/c)^2\,|B_z|^2}\right\}\frac{(\Omega R/c)^2}{1-(\Omega R/c)^2}\frac{cB_z}{4\pi R}.$$

The centrifugal drift term is at most of order γ_c^{-2} relative to the other terms, and can be neglected. Thus the total current density has the form

$$\mathbf{J} \simeq \alpha\mathbf{B} + \rho_c \Omega R e_\phi, \tag{34}$$

where $B_\phi \simeq B_{\phi 0} \equiv -B_{T0}\,a\Phi(\Psi)/R$, $\alpha \simeq -(cB_{T0}/2\pi a B_0)\,d\Phi/d\Psi$ and ρ_c is given by equation (17).

The conclusion that the plasma inertia remains small is subject to some obvious restrictions. Thus, in order for the asymptotic behaviour $|\mathbf{B}_p| \to |B_r|$, $|B_\phi| \to (\Omega R/c)|B_r|/\beta_r$ to be realized, the appropriate currents must be generated within the plasma. Using equations (15) and (34) and requiring that $B_\theta \to 0$, Maxwell's equations become in the limit $R \to \infty$

$$\frac{c}{4\pi r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi) \simeq J_r \simeq \rho_c c\beta_r, \tag{35a}$$

$$-\frac{c}{4\pi r}\frac{\partial B_r}{\partial\theta} \simeq J_\phi \simeq \rho_c c\left(\frac{c}{\Omega r\sin\theta}\right) + \frac{cB_r\cos\theta}{2\pi r\sin\theta}, \tag{35b}$$

where

$$\rho_c \simeq -\frac{\Omega}{4\pi c\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta B_r). \tag{35c}$$

It will be noticed that (i) a current density $\mathbf{J} \simeq \rho_c\Omega R e_\phi$ ($\alpha \simeq 0$) would not be consistent with equations (35b) and (35c), which imply $\rho_c \propto B_r$, $J_\phi \propto B_r/r$; (ii) the poloidal current density

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* On the basis of the integrated equations (15), (18), (19) and (24), Ardavan (1976b) asserts that the plasma flow necessarily becomes discontinuous just inside the light cylinder. At the point in question, $x = 1/(1+N/T)^{1/2}$, equation (26′) may be factorized as

$$(1 - T\gamma|\boldsymbol{\beta}_p|)[T - N(\gamma - 1)] = 0,$$

which yields the three roots

$$\gamma = 1 + \frac{T}{N},\quad 1 + \frac{T}{N} \pm \frac{T}{N}\left(1 - \frac{N}{T^3}\right)^{1/2}.$$

The root $\gamma = 1 + T/N$ entails $\beta_\phi = x$ and $B_\phi = 0$, and becomes complex for $x < 1/(1+N/T)^{1/2}$; while $\gamma > 1 + T/N$ implies $\beta_\phi > x$, that is forward-bent field lines. On the other hand, the root $\gamma = 1 + T/N - (T/N)(1 - N/T^3)^{1/2} \simeq 1 + 1/(2T^2)$ belongs to the low-energy branch which passes continuously through $x = 1/(1+N/T)^{1/2}$ and satisfies $\beta_\phi < x$ everywhere. Ardavan's conclusion therefore appears to be incorrect.

should be asymptotically purely convective, equation (35a), in contrast to its conductive behaviour ($\boldsymbol{\beta}_{ip} \neq \boldsymbol{\beta}_{ep}$) well inside the light cylinder, where the charge density changes sign along the surface $B_z \simeq 0$ (see equation (17)) which cuts across the magnetic surfaces along which current flow takes place; (iii) the toroidal current density cannot be asymptotically purely convective since equation (35b) indicates that J_ϕ does not change sign with ρ_c (whereas $J_\phi \simeq \rho_c \Omega R$ for $R \ll c/\Omega$). Conclusions (ii) and (iii) are entirely consistent with the dynamics of the plasma flow with finite inertia taken into account (Section 3.2): asymptotically $\beta_r \simeq 1$, implying (within the framework of the one-fluid description) that the poloidal current density must be primarily convective; whereas the behaviour $\beta_\phi \propto r^{-1}$ allows the toroidal velocities of ions and electrons to differ substantially, although the difference can still be small if the plasma density is sufficiently high. By definition, of course, the required conduction currents could not be generated in a charge-separated magnetosphere. This suggests that, in a low-density plasma, the magnetic field would not be distorted into an open, wound structure, as required if the flux-freezing assumption is to hold beyond the light cylinder.

A portion of the magnetosphere will of course remain closed. According to equation (25), plasma would be transported towards the equatorial plane from both sides, where cross-field diffusion must take place if the plasma flow is to be steady. In order that $\boldsymbol{\beta}_p \rightarrow \boldsymbol{\beta}_{p\perp}$, the value of $\eta_{an}|J'_\phi|$ must be greatly enhanced (see equation (20)); this requires B_R to vanish abruptly in the plane of symmetry. Likewise, plasma would have to accumulate in the equatorial plane until $\mathbf{J}_p \rightarrow \mathbf{J}_{p\perp} \simeq [-\rho c^3 \boldsymbol{\beta} \cdot \boldsymbol{\nabla}(R\gamma\beta_\phi)/RB_z]\, e_R$. Put the other way round, if finite resistivity and inertia were neglected, the z-component of momentum conservation would reduce to

$$\frac{1}{8\pi}\frac{\partial}{\partial z}\left\{\left[1-\left(\frac{\Omega R}{c}\right)^2\right]B_R^2 + B_\phi^2\right\} = 0 \qquad z = 0. \tag{36}$$

For $R \lesssim c/\Omega$, both $\partial B_R/\partial z$ and $\partial B_\phi/\partial z$ would have to remain finite at $z = 0$, additional constraints which will not generally be satisfied. The terms involving $\mathbf{J}'$ should therefore be retained in the generalized Ohm's law (9), as well as the plasma inertia term in the momentum equation (7). (In the open field region likewise, a current sheet would be implied by any reasonable choice of the boundary condition $\Psi(r = a, \theta)$, as several authors have pointed out. The dynamical significance of this equatorial layer remains unclear, since equation (36) is nominally satisfied in the limit $R \rightarrow \infty$ if $|B_\phi| \rightarrow (\Omega R/c)|B_r|$.) It is questionable whether a steady flow through the closed portion of the magnetosphere can be established. However, by postulating a high degree of charge separation along these magnetic surfaces, field-aligned streaming could be electrostatically suppressed.

4 The electrodynamics of a charge-separated magnetosphere

4.1 CONSERVATION OF CHARGE

It was emphasized in Section 3 that, in order for the two charge species to coexist in the presence of the gravity field near the star, the anomalous resistivity must be invoked. This apparently requires both charge species to be supplied continuously along a given field line. However, in view of the fairly substantial ionic work function expected in the strong surface magnetic field of a neutron star (*cf.* Flowers *et al.* 1977), it is not obvious that, even with collective effects taken into account, a sufficient degree of coupling between the charge species will exist at the star surface for the star to act as a plasma source. Alternatively, inflow and outflow of the two charge species along the same field line (implying some form of circulation unless external particle sources are present) might be contemplated, but as a result of the two-stream instability one charge species would almost certainly end up dragging the other along with it. (This is not to rule out the possibility that some form of counterstreaming, accompanied by energy dissipation, takes place locally.) Quasineutrality might also be maintained under the special circumstances required for the creation of electron–positron pairs (*cf.* Cheng & Ruderman 1977).

Apart from the emissivity of the star surface, that is the assumed relation between the particle outflow and the applied electric field, an essential constraint on the magnetospheric structure and dynamics is that there should be no net current outflow. If the magnetosphere is charge-separated, it is difficult to reconcile this constraint with the space-charge distribution (17) corresponding to the complete neglect of particle inertia (a dilemma which has been recognized since the work of Goldreich & Julian (1969)). For example, in a magnetic dipole field

$$\Psi(r,\theta) = \pm \frac{a \sin^2\theta}{r},$$

equation (17) indicates that the two charge species are separated along the conical surface $\sin\theta \simeq (2/3)^{1/2}$; on the other hand, from equations (5) and continuity (4) it follows that

$$\boldsymbol{\beta}_s \simeq \left(\frac{\boldsymbol{\beta}_{sp}\cdot\mathbf{B}_p}{|\mathbf{B}_p|^2}\right)\mathbf{B} + \frac{\Omega R}{c}\, e_\phi, \tag{37a}$$

$$\frac{n_s|\boldsymbol{\beta}_{sp}|}{|\mathbf{B}_p|} \simeq \left(\frac{\Omega^2 a}{2\pi c^2 e}\right)\mathcal{N}_s(\Psi) \tag{37b}$$

(*cf.* equations (15), (18)), where subscript 's' will hereinafter stand for either ions or electrons ($s \equiv i$ or e). Equations (37) imply an outflow of the 'polar' charge species at colatitudes $\sin\theta \lesssim (2/3)^{1/2}(\Omega a/c)^{1/2}$ (see footnote), which must then be balanced by an outflow of charges of the opposite sign at colatitudes $\sin\theta \gtrsim (2/3)^{1/2}$ (unless external particle sources are invoked). These 'equatorial' particles must then manage to cross magnetic surfaces well inside the light cylinder, contrary to equation (37a).*

Evidently, then, the inertialess, dissipationless assumption is untenable if the charge species are completely segregated near the star. In order to allow for finite inertia, it will be necessary to return to the two-fluid momentum equations (2), (3), which, using continuity (4) and ignoring radiation reaction and direct interactions between the two charge species at this point in the development, reduce to

$$m_s c^2 \boldsymbol{\beta}_s\cdot\nabla(\gamma_s\boldsymbol{\beta}_s) = q(\mathbf{E} + \boldsymbol{\beta}_s\times\mathbf{B}) + m_s\nabla\left(\frac{GM}{r}\right), \tag{38}$$

where q stands for the charge of either species ($q \equiv Ze$ or $-e$). Introducing the electrostatic potential $\mathbf{E} = -\nabla\chi$, this equation may be rewritten

$$\mathbf{E}_s + \boldsymbol{\beta}_s\times\mathbf{B}_s = 0 \tag{38$'$}$$

(*cf.* equation (1)), where $\mathbf{E}_s \equiv -\nabla\chi_s$,

$$\chi_s \equiv \chi + \frac{m_s c^2}{q}\gamma_s - \frac{GMm_s}{qr} \tag{39}$$

* This difficulty is not, of course, peculiar to the assumption of aligned magnetic and rotation axes: if the dipole axis were inclined at any angle less than $\sim\pi/2 - (\Omega a/c)^{1/2}$ the polar cap would still be 'confined' by charges of only one sign.

and

$$\mathbf{B}_s \equiv \mathbf{B} + \frac{m_s c^2}{q} \nabla \times (\gamma_s \boldsymbol{\beta}_s). \tag{40}$$

As $\nabla \cdot \mathbf{B}_s = 0$ and the system is axisymmetric, it is possible to define an effective magnetic surface function Ψ_s by

$$\mathbf{B}_{sp} = -\frac{1}{2} B_0 \frac{a^2}{R} e_\phi \times \nabla \Psi_s,$$

with $\mathbf{B}_s \cdot \nabla \Psi_s = 0$ (*cf.* definition (14)); from equation (40) and the definitions of Ψ and Ψ_s, it is evident that

$$\Psi_s = \Psi + \frac{2 m_s c^2}{q B_0 a^2} R \gamma_s \beta_{s\phi}. \tag{40'}$$

Since $\partial \chi_s / \partial \phi = 0$, the toroidal component of equation (38′) implies $\boldsymbol{\beta}_{sp} = \kappa_s \mathbf{B}_{sp}$, where κ_s is a scalar, so that Ψ_s in fact represents the particle stream function (unless $\kappa_s = 0$); the poloidal component of this equation becomes

$$\nabla \chi_s = \frac{1}{2} B_0 a^2 \left(\frac{\beta_{s\phi} - \kappa_s B_{s\phi}}{R} \right) \nabla \Psi_s, \tag{41}$$

whence $\beta_{s\phi} - \kappa_s B_{s\phi} = (2R/B_0 a^2)\, d\chi_s / d\Psi_s \equiv \Omega_s(\Psi_s) R/c$. Thus

$$\boldsymbol{\beta}_s = \left(\frac{\boldsymbol{\beta}_{sp} \cdot \mathbf{B}_{sp}}{|\mathbf{B}_{sp}|^2} \right) \mathbf{B}_s + \frac{\Omega_s(\Psi_s) R}{c} e_\phi \tag{42}$$

(*cf.* equation (37a)), where $\Omega_s(\Psi_s) \simeq \Omega$ along all streamlines that intersect the star (assuming inertial drifts to be small near its surface). The derivation of equation (42), first given by Mestel (unpublished), is exactly analogous to that of equation (15), but with the replacements $\chi \to \chi_s$, $\mathbf{B} \to \mathbf{B}_s$. The incorporation of finite inertia has the effect that the charged particles are tied not to the magnetic field lines, but to 'quasified lines' pointing in the direction $\mathbf{B}_s$ (*cf.*, for example, Rossi & Olbert 1970). Continuity (4) may be written

$$\frac{n_s |\boldsymbol{\beta}_{sp}|}{|\mathbf{B}_{sp}|} = \left(\frac{\Omega^2 a}{2\pi c^2 e} \right) \mathcal{N}_s(\Psi_s) \tag{43}$$

(*cf.* equation (37b)). Combining equations (38′), (39), (40) and (42) (with $\Omega_s(\Psi_s) = \Omega$), the electric field in the corotating frame may be expressed as

$$\mathbf{E} + \frac{\Omega R}{c} e_\phi \times \mathbf{B} = \frac{m_s c^2}{q} \nabla \left\{ \gamma_s \left[1 - \left(\frac{\Omega R}{c} \right) \beta_{s\phi} \right] - \frac{GM}{c^2 r} \right\} \tag{44}$$

(*cf.* equation (16)) so that, to within an additive constant,

$$\chi = \frac{1}{2} B_0 a^2 \frac{\Omega}{c} \Psi - \frac{m_s c^2}{q} \gamma_s \left[1 - \left(\frac{\Omega R}{c} \right) \beta_{s\phi} \right] + \frac{GMm_s}{qr} \equiv \frac{3q_0}{2a} (\Psi + \eta), \tag{45}$$

where $q_0 \equiv \Omega B_0 a^3 / 3c$ and η represents the deviation from magnetic flux-freezing due to the particle mass. Poisson's equation then yields for the charge density

$$\rho_c = -\frac{\Omega B_z}{2\pi c} + \frac{\Omega R J_\phi}{c^2} + \frac{m_s c^2}{4\pi q} \nabla^2 \left\{ \gamma_s \left[1 - \left(\frac{\Omega R}{c} \right) \beta_{s\phi} \right] \right\} \tag{46}$$

(*cf.* equation (17)).

Equation (46) gives an explicit relationship between the charge distribution and particle inertia. In principle, for example, both charge species could be emitted at the polar cap if the acceleration of the 'foreign' particles (having sign opposite to that of $-\Omega B_z/2\pi c$) were such that $|\nabla^2\gamma_s| \gtrsim |2q\Omega B_z/m_s c^3|$ outside the star; and for a hypothetical 'quasivacuum' magnetosphere $\nabla^2\gamma_s \simeq 2q\Omega B_z/m_s c^3$. Such possibilities cannot be ruled out *a priori*; again the actual particle densities must be consistent with the steady-state condition $\nabla\cdot\mathbf{J} = 0$ as well as the emissivity of the star surface. Steep gradients in the particle Lorentz factor would likewise result in deviations from magnetic flux-freezing, the poloidal drift due to Coriolis force being given by

$$\boldsymbol{\beta}_{\mathrm{sp}\perp} = -\frac{m_s c^2 \boldsymbol{\beta}_s\cdot\nabla(R\gamma_s\beta_{s\phi})}{qR\,|\mathbf{B}_p|}\, e_\phi \times \frac{\mathbf{B}_p}{|\mathbf{B}_p|}. \tag{47}$$

However, the cumulative effect of this drift will remain small so long as

$$\gamma_s \ll \left(\frac{\Omega a}{c}\right)^2 \left(\frac{|q|B_0}{m_s c\Omega}\right) |\Psi_s| \Big/ \left(\frac{\Omega R}{c}\right)\beta_{s\phi}$$

(equation (40′)). Even if the net stellar charge (that is the sum of the charge carried in the star's highly conducting interior and the residual surface charge) were totally unshielded, its contribution to the electrostatic potential χ would only be of magnitude q_0/r, and $|\eta|$ in equation (45) would be at most of order unity. This it is reasonable to expect that, as an upper limit, $\gamma_s \lesssim (\Omega a/c)^2(|q|B_0/m_s c\Omega)$; then equation (40′) indicates that $|\Psi_s - \Psi| \lesssim (\Omega R/c)\,\beta_{s\phi}$, so that particle streamlines practically coincide with magnetic field lines in the neighbourhood of the star.* Actually, for extremely relativistic electrons, radiation reaction might well become dynamically more significant than inertia itself, leading to larger poloidal drifts than indicated by (47) (*cf.* Section 4.3). However, if the energy source is ultimately the star's rotation, the ratio of the net loss of energy to that of angular momentum must equal Ω (*cf.* Cohen & Treves 1972), so that the dissipation of too much energy in the region $R \ll c/\Omega$ could not be matched by the appropriate outflow of angular momentum.

It will be tentatively assumed that the two charge species are completely separated near the star surface, that no exterior particle sources are present, and that $|\nabla^2\gamma_s| \ll |2q\Omega B_z/m_s c^3|$ at $r \simeq a$ so that the charge distribution there is at least crudely given by (17), with $\mathbf{B}$ dipole-like. Then, requiring that no net current leave the star but expecting charged particles to remain rigidly tied to field lines in the near region, it follows that the overall space-charge distribution must adjust so as either (a) to restrict both charge species to predominantly toroidal motion, or (b) to allow the polar particles to circulate back to the star (*cf.* Jackson 1976).

Whatever the behaviour of the polar charge species, the motion of the equatorial particles in a dipole-like magnetic field must be almost purely toroidal since a steady outflow would imply trans-$\mathbf{B}$ drifts comparable with the parallel component of velocity in the immediate vicinity of the star. Moreover, as polar particles 'intervene' along magnetic surfaces connecting most of the equatorial charge region with the star (Fig. 1), it would be difficult to resupply equatorial particles that stream away. Accordingly, setting $\boldsymbol{\beta}_{\mathrm{sp}} = 0$ for this charge species, equation (42) reduces to

$$\boldsymbol{\beta}_s = \frac{\Omega_s(\Psi_s)\,R}{c}\, e_\phi. \tag{48}$$

* In the computed model of Kuo-Petravic *et al.* (1974), the free crossing of magnetic surfaces by 'equatorial particles' well inside the light cylinder is probably due to their increased particle masses and/or their use of artificial diffusion terms.

This does not mean that the equatorial charge region must corotate with the star, as again they are not in direct contact except along the magnetic surfaces $2/3 \lesssim \Psi \leqslant 1$. In particular a potential gap might develop between the two charge regions (*cf.* Holloway 1973): the physical motivation for such a gap inside the centrifugal–gravity balance surface would be to stop the diffusion of equatorial charges into the polar region, whence they would be driven into the star by the electric field that supports the polar charges against gravity; in the outer electric field that confines the equatorial charges against centrifugal forces (see Section 4.3). Note that Ψ_s in equation (48) is not to be interpreted as a particle stream surface, but simply as a surface defined by (40′).

4.2 A 'QUASISTATIC' MAGNETOSPHERE

Setting $\kappa_s = 0$ in equation (41), it follows that

$$\frac{\beta_{s\phi}}{R} = \frac{2}{B_0 a^2} \frac{d\chi_s}{d\Psi_s}(\Psi_s), \tag{49}$$

where χ_s, Ψ_s are defined by (39), (40′). Since $\beta_{s\phi} \cong \Omega R/c$ at $r = a$, it appears that rigid corotation must occur along all surfaces of constant Ψ_s that intersect the star, that is

$$\beta_{s\phi} \simeq \frac{\Omega R}{c}$$

wherever

$$\Psi + \frac{2m_s c}{qB_0 a^2} \frac{\Omega R^2}{[1-(\Omega R/c)^2]^{1/2}} = \Psi_s \; (r = a, \theta). \tag{50}$$

The *polar* charge species, if likewise assumed to be confined to toroidal motion, would then be forced to corotate out to a point just within the light cylinder, being 'bound' to the star along surfaces of constant Ψ_s.* Equation (50) sets a definite limit on the extent of the corotating region; with $\Psi(r = a, \theta) \simeq \pm \sin^2\theta$, corotation certainly cannot continue past the point $R_c \cong [1-(2m_s c^3/q\Omega B_0 a^2)^2/2]\, c/\Omega$. Thus there is no need *a priori* to demand that the polar particles be confined within the light cylinder. In any case the supposition that these particles can be prevented from streaming outward must be tested. (Certainly if electrons are involved, radiation damping would lead to a breakdown of corotation well inside $R \simeq R_c$ as well as to substantial poloidal drifts.)

In the absence of any streaming, the two charge species could not coexist (Section 3) and the magnetosphere would necessarily be completely charge separated. Then Poisson's and Ampere's equations may be written

$$\nabla^2\chi = -4\pi q n_s, \tag{51a}$$

$$\nabla^2\Psi - \frac{2}{R}\frac{\partial\Psi}{\partial R} = -\frac{8\pi q n_s \beta_{s\phi} R}{B_0 a^2}. \tag{51b}$$

By elimination between equations (51) and the divergence of momentum conservation (38) (setting $\boldsymbol{\beta}_s = \beta_{s\phi} e_\phi$), the particle density may be expressed as

$$n_s = \left[-qB_0 a^2 \frac{\beta_{s\phi}}{R^2}\frac{\partial\Psi}{\partial R} - \frac{1}{2} qB_0 a^2 \nabla\left(\frac{\beta_{s\phi}}{R}\right)\cdot\nabla\Psi - m_s c^2 \frac{1}{R}\frac{\partial}{\partial R}(\gamma_s \beta_{s\phi}^2)\right]\Big/[4\pi q^2(1-\beta_{s\phi}^2)], \tag{52a}$$

* One class of corotating solutions has been examined recently by Endean (1976).

while $\Psi(R, z)$ is obtained by solving

$$(1-\beta_{s\phi}^2)\nabla^2\Psi - \frac{2}{R}\frac{\partial\Psi}{\partial R} = R\beta_{s\phi}\nabla\left(\frac{\beta_{s\phi}}{R}\right)\cdot\nabla\Psi + \frac{2m_s c^2}{qB_0 a^2}\beta_{s\phi}\frac{\partial}{\partial R}(\gamma_s\beta_{s\phi}^2). \tag{52b}$$

(Note that the charge distribution is independent of gravity.)

In the corotating region, these equations reduce to

$$n_s = \left[-2q\frac{\Omega}{c}B_z - m_s\Omega^2\gamma_s^3(2-x^2)\right] \Big/ [4\pi q^2(1-x^2)], \tag{53a}$$

$$(1-x^2)\frac{\partial^2\Psi}{\partial x^2} - (1+x^2)\frac{1}{x}\frac{\partial\Psi}{\partial x} + (1-x^2)\frac{\partial^2\Psi}{\partial\hat{z}^2} = \frac{2m_s c^3}{q\Omega B_0 a^2}\gamma_s^3 x^2(2-x^2), \tag{53b}$$

where $x \equiv \Omega R/c$, $\hat{z} \equiv \Omega z/c$, $\gamma_s = (1-x^2)^{-1/2}$. According to equation (53a), $n_s \to 0$ as $\gamma_s \to (-2qB_z/m_s c\Omega)^{1/3}$ within the corotating region. Here, apparently, while particle inertia is still insignificant in the momentum balance itself $(\mathbf{E} + xe_\phi \times \mathbf{B} \simeq 0)$, the *divergence* of this term cannot be neglected near the light cylinder but sets severe constraints on the magnetospheric structure. If the particle density is to remain positive definite, the field lines would have to bend upward towards the z axis on approaching the light cylinder, with a drastic increase in the magnetic field strength. A particular solution to equation (53b) in the corotating region is

$$\Psi_p \simeq \frac{2m_s c^3}{q\Omega B_0 a^2}\frac{1}{(1-x^2)^{1/2}} \qquad x \lesssim 1,$$

while the homogeneous solution near the light cylinder, found by separation of variables, has the form

$$\Psi_h \simeq f(z) + g(z)\ln(1-x^2) \qquad x \lesssim 1.$$

If the corotating region extended past the light cylinder, as in Michel's (1973a) massless solution, then the R-dependent component of Ψ_h would have to be discarded as singular, so that $B_z \to 0$ as $R \to c/\Omega$. Clearly the inclusion of finite inertia alters this conclusion: corotation breaks down inside the light cylinder and in general $g(z) \neq 0$. However, even though it may be possible to counterbalance the growth of the inertial term in equation (53a) by making $g(z)$ (considered as a boundary condition on the magnetic field) sufficiently large, a huge magnetic pressure would then be transmitted across the light cylinder, which could not be balanced without making unphysical assumptions about the magnetospheric structure beyond. It is also evident from equation (53a) that unless B_z becomes very small near the light cylinder, there would be a considerable 'heaping up' of charge just inside the cut-off point (a possibility which should not be excluded *a priori*).

The problem cannot be resolved simply by allowing the charge density to change sign, since the inertial contribution to n_s (equation (53a)) is negative for both charge species. However, it might be possible to relax the corotation constraint (50) by introducing a gap between the polar charge region, extending out to the point $x = [1-(-m_s c\Omega/2qB_z)^{2/3}]^{1/2}$, and another particle region lying somewhere beyond. Difficulties in realizing such a model, without requiring detailed field lines, are discussed elsewhere (Wang & Pryce, in preparation). Otherwise, poloidal motion must be introduced (or the implicit assumption of a steady state abandoned).

4.3 IS STEADY POLAR CIRCULATION POSSIBLE?

With the assumptions of Section 4.1, any outward flow of polar particles must eventually be reversed and return to the star, in a steady state. But the conservation of angular momentum states that, if dissipative processes can be ignored, the function

$$\Psi_s = \Psi + (2m_s c^2/qB_0 a^2)\, R\gamma_s\beta_{s\phi}$$

is constant along a particle streamline; if a polar particle is then to return to the star along a different field line from that along which it was emitted, it would have to gain a large amount of angular momentum and energy, which would be reabsorbed by the star. In reality much of this excess could be dissipated near or beyond the light cylinder by radiation (and perhaps viscous) processes.* Denoting the radiation reaction force per charged particle by $\mathbf{f}_s^{\rm rad}$, the rate at which it drifts across surfaces of constant Ψ_s is given by

$$\boldsymbol{\beta}_s \cdot \nabla\Psi_s = \frac{2Rf_{s\phi}^{\rm rad}}{qB_0 a^2}. \tag{54}$$

Note that the guiding-centre drifts due to Coriolis force and radiation reaction are both directed toward lower-latitude field lines (for polar particles), so long as $R\gamma_s\beta_{s\phi}$ increases along the streamline and $f_{s\phi}^{\rm rad} < 0$. Thus a polar particle initially drifts outward (towards increasing R), but the direction of the drift would reverse where B_z changes sign.

Once poloidal motion is introduced, the possibility of some charge mixing must be considered. According to equation (46), charge mixing takes place if the surface

$$-\frac{\Omega B_z}{2\pi c} + \frac{m_s c^2}{4\pi q}\nabla^2\left\{\gamma_s\left[1 - \left(\frac{\Omega R}{c}\right)\beta_{s\phi}\right]\right\} = 0 \tag{55}$$

(*unless* it coincides with $(\Omega R/c)\,\beta_{s\phi} = 1$) intersects a polar-particle stream surface. The subsequent behaviour of the polar particles is then determined by the electromagnetic

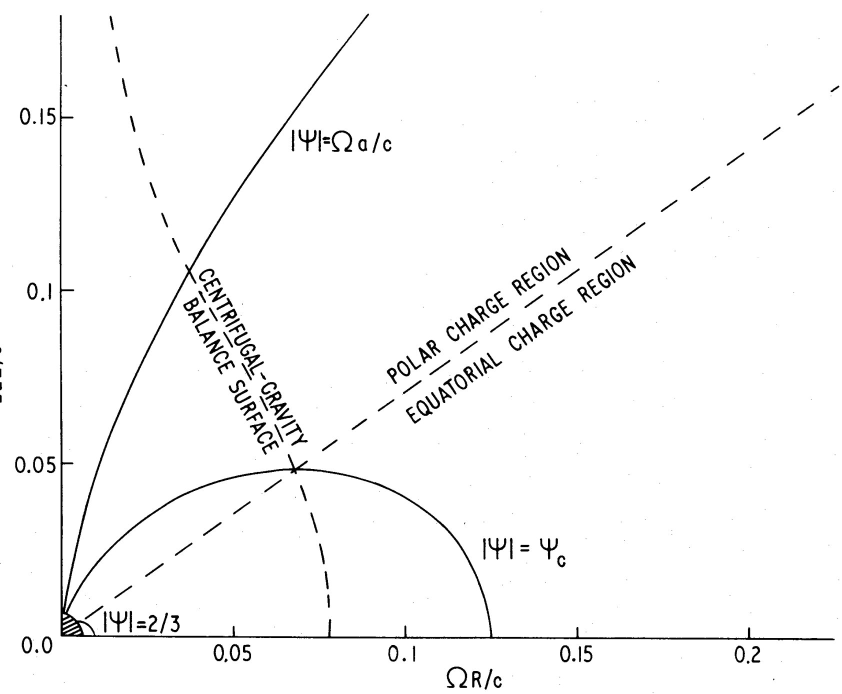

Figure 1. A charge-separated magnetosphere (inner region) with dipole magnetic field,

$\Psi(R, z) = \pm\, aR^2/(R^2 + z^2)^{3/2}$.

The significance of the magnetic surfaces shown is discussed in the text.

* In constructing models involving dissipation, it is important to keep in mind that the ratio of the net loss of energy to that of angular momentum must equal Ω, no matter what processes are involved, so long as the energy source is the star's rotation.

fields which support and confine the 'equatorial' charge species against inertial (gravity and centrifugal) forces. The magnetosphere can then be divided into 'dead' and potentially active zones (Fig. 1), assuming a dipole-like magnetic field near the star. The magnetic surface $|\Psi| \simeq 2/3$ encloses a small region consisting solely of equatorial particles. Both charge species are to be found along poleward magnetic surfaces $0 < |\Psi| \lesssim 2/3$, although the equatorial charge region is there 'detached' from the star surface. Consider the magnetic surface $|\Psi| \simeq (2/3)^{7/6}(\Omega^2 a^3/GM)^{1/3} \equiv \Psi_c$, which intersects the charge interface $\sin\theta \simeq (2/3)^{1/2}$ at $r \simeq (GM/\Omega^2 \sin\theta)^{1/3}$, where centrifugal and gravity forces just balance. Any polar particles which stray into the equatorial charge region within the range $\Psi_c \lesssim |\Psi| \lesssim 2/3$ would be decelerated by an $\mathbf{E}_\parallel$ which supports the equatorial particles against gravity; thus the entire range $\Psi_c \lesssim |\Psi| \leqslant 1$ effectively constitutes a dead zone. On the other hand, equatorial particles along the magnetic surfaces $0 < |\Psi| \lesssim \Psi_c$ experience a centrifugal force which must be balanced by an $\mathbf{E}_\parallel$ directed so as to accelerate incoming polar charges toward the light cylinder. This suggests that a significant amount of energy might be dissipated through microinstabilities, with electrons streaming through a confined background of protons or heavy ions so that

$$m_e c^2 \mathbf{B} \cdot [\boldsymbol{\beta}_e \cdot \nabla(\gamma_e\ \boldsymbol{\beta}_e)] \simeq \frac{m_i c^2 \beta_{i\phi}^2}{ZR} \frac{B_R}{(1-\beta_{i\phi}^2)^{1/2}}, \tag{55}$$

where $\beta_{i\phi}$ has the form (48). As mentioned previously, however, such charge mixing could in principle be prevented by interposing a potential gap between the two charge regions.

Near the star the charge interface occurs where $B_z \simeq 0$. But toward the light cylinder the last two terms on the right-hand side of equation (46) could become significant and B_z could change sign within the polar charge region; this might happen, for example, where $(\Omega R/c)\,\beta_{s\phi} > 1$. If the polar particles continued to acquire angular momentum, their guiding-centre drifts would then be directed back toward the star. The likelihood that on the one hand the region beyond the light cylinder will be depleted of the equatorial charge species (due to obstruction of the lines of communication with the star), while on the other hand polar charges that have dissipated much of their energy might accumulate there, is consistent with such a picture.

From another point of view, the conservation of energy in the absence of dissipative processes (obtained from the scalar product of momentum conservation (38′) with $\boldsymbol{\beta}_s$) states that

$$\chi_s \equiv \chi + m_s c^2 \gamma_s/q - GMm_s/qr$$

is constant along a particle streamline $\Psi_s = \Psi + 2m_s c^2 R\gamma_s\beta_{s\phi}/qB_0a^2$. Using the definition $\chi \equiv (\Omega B_0 a^2/c)(\Psi + \eta)$ and neglecting the initial particle energy, it follows that

$$\gamma_s \simeq \left[1 - \frac{1}{2}\left(\frac{\Omega a}{c}\right)^2 \left(\frac{qB_0}{m_s c\Omega}\right)(\eta - \eta_0)\right] \Big/ \left[1 - \left(\frac{\Omega R}{c}\right)\beta_{s\phi}\right] \tag{56}$$

along a particle streamline – *cf.* equation (24). Initially $(\Omega R/c)\,\beta_{s\phi}$ increases along a streamline – this is the centrifugal-magnetic driving discussion in relation to quasineutral winds. If this quantity approaches or becomes greater than unity (somewhere beyond the light cylinder), as may happen if the poloidal currents are limited so that $\beta_{s\phi}$ (*cf.* equation (37a) or (42)) and the slingshot effect are maximized, $\mathbf{E}_\parallel$ must then point in a direction to oppose outward flow, according to equation (56).

Finally, it should be emphasized that, in order for a steady circulation to be set up, the particle inertia or radiation-reaction force must be large enough for free crossing of magnetic surfaces to take place near and beyond the light cylinder. Otherwise the 'centrifugal barrier' would halt any attempt by polar particles to return to the star: since the field lines are swept backward, $\beta_{s\phi} > 1$ for a hypothetical particle returning through the light cylinder but obeying the inertialess approximation (37a).

5 Conclusions

In a quasineutral plasma large currents can be generated through small relative motions of ions and electrons – thus the magnetic field can be strongly distorted while the plasma energy density remains small compared with that of the magnetic field (in the laboratory frame), except possibly in localized regions. Such conduction currents are needed to maintain an open field zone; then the properties of the 'imbedded' centrifugal-magnetic wind can be described readily by means of the various field-line integrals, and a modest upper limit can be set on the plasma Lorentz factors. In order to provide such a relatively dense plasma, an anomalous resistivity must be invoked near the star to prevent the charge species from separating under gravity. However, in a steady state, any streaming along closed field lines would require enhanced diffusion to take place about the equatorial plane; whereas if the closed-field region were charge separated, the particle motion could be purely toroidal (although the equatorial charge species need not exactly corotate with the star) and this potential difficulty would be removed. If the star can act as a plasma source, then, the picture is that of an open, 'torque-free' field region beyond the light cylinder through which escapes a dense plasma, and a zone of field lines closing within the light cylinder, where the plasma is less mixed.

In a magnetosphere characterized by total charge separation near the star surface, the inertialess approximation cannot be used without encountering some version of the 'Goldreich–Julian paradox'. Inertial and (or) radiation-reaction terms must be included in either the momentum equation, its divergence (yielding the charge density), or both. It is more appropriate to divide the magnetosphere into 'polar' and 'equatorial' charge zones (rather than 'open' and 'closed') since there is no need to assume the existence of an open field region if the charged particles are no longer tied to field lines. If the charge distribution near the star surface is not too different from the inertialess form (17), the equatorial charge species must be restricted to almost purely toroidal motion in the dipole-like field within the light cylinder. Then either (a) the polar charge species are also confined to toroidal motion, or (b) the polar particles circulate back to the star. Alternative (a) implies corotation out to a point which is always within the light cylinder, but a difficulty is encountered in the rapid growth of the inertial contribution to the charge density, which can be resolved only if it is possible to introduce a gap near the light cylinder between the polar charge region and a non-corotating particle region lying somewhere beyond. Alternative (b) requires acceleration of the polar particles to very high energies and a complete breakdown of the frozen-in assumption, so that both particle inertia and radiation reaction must be included in momentum conservation.

Based on these conclusions, further investigation along the following lines would appear to be worthwhile:

(i) The conditions under which both charge species can be extracted simultaneously from a given point on the star surface should be established. In this regard, the role of the anomalous resistivity in coupling together the two charge species must be taken into account. There is also a possibility that sufficient electron–positron pairs might be produced near the star – through processes discussed by Cheng & Ruderman (1977) – (or possibly closer to the light cylinder for rapid rotators) to generate the type of quasineutral-plasma, torque-free wind discussed in Section 3.

(ii) If it is concluded that the star or magnetosphere can act as a source of quasineutral plasma, then the behaviour of the plasma and current flow about the equatorial plane, particularly in the vicinity of the light cylinder, should be examined, since most of the energy dissipation may be associated with this region.

(iii) If only one charge species can be supplied from a given point on the star surface, and no exterior particle sources are present, the problem of how the star conserves charge must be resolved. This involves a consideration of alternatives (a) and (b) above.

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