PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 3, March 2010, Pages 1093–1100
S 0002-9939(09)10115-6
Article electronically published on October 26, 2009

ON THE ROLE OF THE COLLECTION PRINCIPLE FOR Σ^0_2-FORMULAS IN SECOND-ORDER REVERSE MATHEMATICS

C. T. CHONG, STEFFEN LEMPP, AND YUE YANG

(Communicated by Julia Knight)

Abstract. We show that the principle PART from Hirschfeldt and Shore is equivalent to the Σ^0_2-Bounding principle $B\Sigma^0_2$ over RCA_0, answering one of their open questions.

Furthermore, we also fill a gap in a proof of Cholak, Jockusch and Slaman by showing that D^2_2 implies $B\Sigma^0_2$ and is thus indeed equivalent to Stable Ramsey's Theorem for Pairs (SRT^2_2). This also allows us to conclude that the combinatorial principles IPT^2_2, SPT^2_2 and SIPT^2_2 defined by Dzhafarov and Hirst all imply $B\Sigma^0_2$ and thus that SPT^2_2 and SIPT^2_2 are both equivalent to SRT^2_2 as well.

Our proof uses the notion of a bi-tame cut, the existence of which we show to be equivalent, over RCA_0, to the failure of $B\Sigma^0_2$.

1. Introduction and results

Let $\mathcal{M}$ be a model of RCA_0. In their paper on combinatorial principles implied by Ramsey's Theorem for pairs (RT^2_2), Hirschfeldt and Shore [7, section 4] introduced the following combinatorial principle:

Definition 1.1. The principle PART states: Let $\langle M, \prec \rangle$ be a recursive (i.e., Δ^0_1-definable in $\mathcal{M}$) linear ordering with least and greatest element. Assume that for any $x \in M$, exactly one of $\{y \in M : y \prec x\}$ and $\{y \in M : x \prec y\}$ is $\mathcal{M}$-finite. Then for any $\mathcal{M}$-finite partition $\{a_i : i \leq k\}$ of $\langle \mathcal{M}, \prec \rangle$, $\{y \in M : a_i \prec y \prec a_{i+1}\}$ is not $\mathcal{M}$-finite for exactly one $i < k$. (For simplicity, we will assume here that the endpoints of $\mathcal{M}$ under $\prec$ are a_0 and a_k. Note that we will always denote the universe of the model $\mathcal{M}$ by M.)

Received by the editors June 10, 2009, and, in revised form, June 23, 2009, and July 9, 2009.
2000 *Mathematics Subject Classification.* Primary 03F35; Secondary 03H15.
Key words and phrases. Reverse mathematics, Σ^0_2-bounding, linear order, tame cut, bi-tame cut, Ramsey's Theorem.

The main ideas were conceived during the workshop on "Computability, Reverse Mathematics and Combinatorics" at Banff International Research Station in December 2008. The authors wish to thank Banff International Research Station for its hospitality; Richard Shore for useful discussions and for pointing out an error in an earlier version; and Carl Jockusch, Richard Shore, Jeff Hirst and Damir Dzhafarov for mentioning related problems which could be solved by our main result. The research of the first author was supported in part by NUS grant WBS C146-000-025-001. The second author's research was partially supported by NSF grant DMS–0555381 as well as grant no. 13407 by the John Templeton Foundation entitled "Exploring the Infinite by Finitary Means". The third author was partially supported by NUS Academic Research grant R 146-000-114-112.

Note that the conclusion of PART clearly implies the hypothesis, which in turn implies that there is at most one $i < k$ for which $\{y \in M : a_i \prec y \prec a_{i+1}\}$ is not $\mathcal{M}$-finite. PART was introduced by Hirschfeldt and Shore [7] as one in a series of principles shown to be strictly weaker than Ramsey's Theorem for pairs. In fact, they studied the Chain Antichain Principle (CAC), the Ascending and Descending Sequence Principle (ADS), and their *stable* versions, denoted respectively as SCAC and SADS, and proved that CAC, and hence SCAC, is strictly weaker than RT^2_2 and that SCAC implies SADS, which strictly implies PART.

A general problem that was discussed quite extensively in [7] is the strength of the first-order theory of these principles. Hirst [8] has shown that RT^2_2 implies the Σ^0_2-Bounding principle $B\Sigma^0_2$, and in [7, Proposition 4.1], $B\Sigma^0_2$ was proved to be strictly weaker than SCAC. On the other hand, while PART does not follow from Recursive Comprehension (RCA_0; see [7, Corollary 4.7]), it is a consequence of $B\Sigma^0_2$ over RCA_0 (see [7, Proposition 4.4]). Question 6.4 of [7] asks whether SADS or indeed PART implies, or is weaker than, $B\Sigma^0_2$. In the latter case, PART would have been the first "natural" principle of reverse mathematics strictly between RCA_0 and $B\Sigma^0_2$. However, we show in this paper that the former case holds, namely, that PART and $B\Sigma^0_2$ are indeed equivalent, adding further evidence to $B\Sigma^0_2$ being a very robust proof-theoretic principle:

Theorem 1.2. *The principle* PART *is equivalent to* $B\Sigma^0_2$ *over* RCA_0.

The key idea of the proof is to show that PART does not hold in any model of RCA_0 in which $B\Sigma^0_2$ fails. It turns out that the failure of $B\Sigma^0_2$ in a model of RCA_0 where, by definition, the Σ^0_1-induction scheme $I\Sigma^0_1$ holds is captured by the existence of cuts with a property we call "bi-tameness". The notion of a *tame* Σ^0_2-function was introduced by Lerman in α-recursion theory (Lerman [9], Chong [2]), and was later adapted to models of fragments of Peano arithmetic to study the complexity of infinite injury priority arguments in the context of reverse recursion theory (Chong and Yang [3]).

A *cut* is a nonempty bounded subset of M that is closed downward and under the successor function. A Σ^0_2-*cut* is a cut that is Σ^0_2-definable. The existence of a Σ^0_2-cut characterizes models of $B\Sigma^0_2$ in which the Σ^0_2 induction scheme $I\Sigma^0_2$ fails. Let I be a Σ^0_2-cut bounded by k. Then I is *bi-tame* if both it and $[0,k] \setminus I$ are tame Σ^0_2 (to be defined below). As we shall see, the existence of a bi-tame cut characterizes precisely the models of RCA_0 that do not satisfy $B\Sigma^0_2$. This fact will then be used to establish the failure of PART. Recall here that the existence of a Σ^0_2-cut characterizes the failure of $I\Sigma^0_2$ over the base theory $I\Sigma^0_1$. Thus bi-tameness separates the Σ^0_2-cuts I in models satisfying only $I\Sigma^0_1$ from those in models also satisfying $B\Sigma^0_2$.

We conclude this paper by proving Theorem 1.4, which implies some other consequences of Ramsey type combinatorial principles. To elaborate, our method allows one to fill a gap in Cholak, Jockusch and Slaman [1]. They define the following principle:

Definition 1.3. The principle D^2_2 states: For any Δ^0_2-definable subset A of M, there is an infinite subset B in $\mathcal{M}$ which is either contained in or disjoint from A.

It was claimed in [1, Lemma 7.10] that D^2_2 is equivalent to Stable Ramsey's Theorem for Pairs (SRT^2_2) over RCA_0. However, the argument that D^2_2 implies SRT^2_2 implicitly assumes $B\Sigma^0_2$ and thus contains a gap. We close this gap in the following

Theorem 1.4. *The principle D_2^2 implies $B\Sigma_2^0$ and is therefore equivalent to Stable Ramsey's Theorem (SRT_2^2) over RCA_0.*

Dzhafarov and Hirst [5] introduced the following "polarized" versions of Ramsey's Theorem, based on similar notions in Erdős and Rado [4, §9]:

Definition 1.5. Let $n, k \geq 1$ and $f : [M]^n \to k$.

(1) A *p-homogeneous* set for f is a sequence $\langle H_1, \dots, H_n \rangle$ of $\mathcal{M}$-infinite sets such that for some $c < k$, $f(\{x_1, \dots, x_n\}) = c$ for every $\langle x_1, \dots, x_n \rangle \in H_1 \times \cdots \times H_n$.

(2) Such a sequence $\langle H_1, \dots, H_n \rangle$ of $\mathcal{M}$-infinite sets is called *increasing p-homogeneous* if (1) is required to hold only for increasing tuples $\langle x_1, \dots, x_n \rangle$.

(3) PT_2^2 (or IPT_2^2, respectively, for "(Increasing) Polarized Theorem") is the statement that every $f : [M]^2 \to 2$ has an (increasing) p-homogeneous set.

(4) SPT_2^2 (or SIPT_2^2, respectively, for "Stable (Increasing) Polarized Theorem") is the statement that every $f : [M]^2 \to 2$ has an (increasing) p-homogeneous set.

(There are natural extensions to k many colors and, for $(\mathsf{I})\mathsf{PT}_2^2$, to n-tuples.)

The following was proved in [5]:

Theorem 1.6 (Dzhafarov and Hirst [5])**.**

(1) *Over RCA_0, PT_2^2 implies $B\Sigma_2^0$.*
(2) *Over RCA_0, PT_2^2 implies IPT_2^2 and SPT_2^2.*
(3) *Over RCA_0, both IPT_2^2 and SPT_2^2 imply SIPT_2^2.*
(4) *Over RCA_0, SIPT_2^2 implies D_2^2.*
(5) *Over $\mathsf{RCA}_0 + B\Sigma_2^0$, IPT_2^2 implies SPT_2^2.*
(6) *Over $\mathsf{RCA}_0 + B\Sigma_2^0$, all of SRT_2^2, SPT_2^2 and SIPT_2^2 are equivalent.*

Since all these principles imply D_2^2, Theorem 1.4 considerably simplifies the picture (and answers Questions 5.1 (part 2) and 5.2 in [5]):

Theorem 1.7.

(1) *Over RCA_0, all of IPT_2^2, SPT_2^2 and SIPT_2^2 imply $B\Sigma_2^0$.*
(2) *SRT_2^2, SPT_2^2 and SIPT_2^2 are equivalent over RCA_0.*
(3) *Over RCA_0, IPT_2^2 implies SPT_2^2.*

The question of whether either or both of the middle two implications in

$$\mathsf{RT}_2^2 \Leftrightarrow \mathsf{PT}_2^2 \Rightarrow \mathsf{IPT}_2^2 \Rightarrow \mathsf{SPT}_2^2 \Leftrightarrow \mathsf{SRT}_2^2$$

are strict remains open.

The rest of this paper is devoted to the proofs of Theorems 1.2 and 1.4.

2. The proof of Theorem 1.2

We work in models of RCA_0. Since $I\Sigma_1^0$ is the most important consequence of RCA_0 that we use, we can work as if we were in first-order Peano arithmetic. (We refer the reader to Hájek and Pudlák [6] for background on first-order arithmetic and to Simpson [10] for background on second-order arithmetic and reverse mathematics.)

We will show the equivalence of PART and $B\Sigma_2^0$ in two steps after introducing the notion of bi-tame cuts.

Definition 2.1. Suppose $\mathcal{M}$ is a model of $I\Sigma_1^0$. We say a set I is a *bi-tame* cut in $\mathcal{M}$ iff

(1) I is a *cut*, i.e., closed under successor and closed downward.
(2) There are a point $k \notin I$ and a Σ_2^0-function $g : [0,k] \to M$ with recursive approximation $h(i,s) : [0,k] \times M \to M$ such that:
  (a) The domain of g is the whole interval $[0,k]$.
  (b) The range of g is unbounded in $\mathcal{M}$.
  (c) (Tame Σ_2^0 on I) For any $i \in I$, there is an s such that for all $j < i$, for all $t > s$, $h(j,t) = h(j,s)$; i.e., g settles down on all initial segments, and so, in particular, $g \restriction [0,i)$ is $\mathcal{M}$-finite.
  (d) (Tame Σ_2^0 on $\bar{I}$) For any $i < k$ not in I, there is an s such that for all j with $i < j \leq k$ and all $t > s$, $h(j,t) = h(j,s)$; i.e., g also settles down on all final segments, and so, in particular, $g \restriction (i,k]$ is $\mathcal{M}$-finite.

Remark. Throughout this paper, we use boldface definability, so, e.g., Σ_2^0 is really $\Sigma_2^0(\mathcal{M})$, i.e., with parameters from M. Observe that any bi-tame cut I is Δ_2^0, as both I and $[0,k] \setminus I$ are Σ_2^0.

First we show that the failure of PART is equivalent to the existence of bi-tame cuts.

The failure of PART can be stated as: There is a recursive linear ordering $\mathcal{M} = (M, \prec)$ together with an $\mathcal{M}$-finite partition $\{a_i : i \leq k\}$ (which we refer to as *landmarks*) such that for any $x \in M$, exactly one of $\{y : y \prec x\}$ and $\{y : x \prec y\}$ is $\mathcal{M}$-finite, but that for all $i < k$, the interval $\{y : a_i \prec y \prec a_{i+1}\}$ is $\mathcal{M}$-finite.

Lemma 2.2. *The existence of a linear ordering witnessing the failure of* PART *is equivalent to the existence of a bi-tame cut.*

A pictorial version of the proof proceeds as follows: Imagine the graph of the function g which witnesses the bi-tameness of I as consisting of k many vertical columns, the i-th one of which is of height $g(i)$. Now "push the columns from both ends" as in Domino to produce a linear ordering. The resulting horizontal picture is more or less the linear order.

For the converse, just "un-Domino" the horizontal picture. We get the bi-tameness from the condition that either the initial segment or the final segment is $\mathcal{M}$-finite.

Proof. ($\Leftarrow$) Let I be a bi-tame cut with witness Σ_2^0-function $g : [0,k] \to M$ and recursive approximation $h(x,s)$ of $g(x)$ as in Definition 2.1. We recursively enumerate the linear order $\prec$ as follows.

Stage 0 (laying out the landmarks): Set $a_i = i$ for $i \leq k$ and $a_i \prec a_j$ iff $i < j$.

Stage $s+1$: Suppose we have specified the order up to $m \in M$ at the end of the stage s. Then, for each $i < k$ in increasing order, if $h(i,s+1) > h(i,s)$, then set $k = h(i,s+1) - h(i,s)$ and insert the next k many elements of $\mathcal{M}$ between the landmarks a_i and a_{i+1} to the right of all the elements previously inserted between these landmarks.

We check that the linear order $\prec$ works: It is a recursive linear ordering because a linear order $\prec$ is recursive iff it has an r.e. copy. Once the approximation $h(i,s)$ of $g(i)$ has settled down on the initial segment $[0,i+1]$ or the final segment $[i,k]$, depending on whether $i \in I$ or not, no elements will enter the interval $\{y : a_i \prec y \prec a_{i+1}\}$; hence this interval is $\mathcal{M}$-finite. Finally, a one-point partition leaves either an initial segment or a final segment $\mathcal{M}$-finite because of the bi-tameness, and it cannot leave both $\mathcal{M}$-finite as g is unbounded.

($\Rightarrow$) Suppose $\prec$ is such a recursive linear order with landmarks $\{a_i : i \leq k\}$. Fix a recursive enumeration of $\mathcal{M}$. Define a recursive approximation $h(i,s) = x$ of $g : [0,k] \to M$ by taking x to be the maximal element enumerated into the interval $\{y : a_i \prec y \prec a_{i+1}\}$ up to stage s. g is unbounded since every element of M appears in the enumeration. Define the Σ_2^0-cut I by $i \in I$ iff $i < k$ and the initial segment of the one-point partition by a_i is $\mathcal{M}$-finite. We check that I is bi-tame. We only need to show that if $i \in I$, then g settles down on the initial segment $[0,i]$, as the other case for the final segment is symmetric. Since the interval $[0,a_i] = \{y : y \prec a_i\}$ is $\mathcal{M}$-finite, apply $B\Sigma_1^0$ to the formula $\forall y \in [0,a_i]\,\exists s\,[y \prec a_i \text{ at stage } s]$ to obtain a uniform upper bound t such that no element enters the interval $[0,a_i]$ after stage t. Thus $h(i,s) = h(i,t)$ for all $s > t$. □

The second part of the proof is to link the existence of bi-tame cuts to $B\Sigma_2^0$. The essential idea is based on the proof of the equivalence of $B\Sigma_2^0$ and $I\Delta_2^0$ by Slaman [11].

Lemma 2.3. *Suppose $\mathcal{M} \models I\Sigma_1^0$. Then $\mathcal{M} \not\models B\Sigma_2^0$ iff there exists a bi-tame cut in $\mathcal{M}$.*

Proof. Clearly, if $\mathcal{M} \models B\Sigma_2^0$, then there is no Δ_2^0-cut and thus a fortiori no bi-tame cut.

On the other hand, suppose that $\mathcal{M} \not\models B\Sigma_2^0$. We need to construct a bi-tame cut. We start by proving two claims:

Claim 2.4. Suppose that $\mathcal{M} \models I\Sigma_1^0$ and $\mathcal{M} \not\models B\Sigma_2^0$. Then there are an element $k \in M$ and a function $f : [0,k) \to M$ such that

(a) f is injective;
(b) the domain of f is $[0,k)$ and the range of f is unbounded; and
(c) the graph of f is Π_1^0.

Proof. Let $\forall t\,\psi(x,y,t)$ be a Π_1^0-formula which witnesses the failure of $B\Pi_1^0$ (which is equivalent to $B\Sigma_2^0$) on some interval $[0,k)$. We define a Π_1^0-function $f : [0,k) \to M$ by setting $f(x) = \langle x, \langle y,s\rangle\rangle$ iff

$$\forall t\,\psi(x,y,t) \wedge \forall z < y\,\exists t < s\,\neg\psi(x,z,t) \wedge \forall t < s\,\exists z < y\,\forall v < t\,\psi(x,z,v).$$

Intuitively, the first coordinate x of $f(x)$ just makes f injective. So, f essentially maps x to the least y such that $\forall t\,\psi(x,y,t)$, but this alone would give us only a Δ_2^0-graph. To make the graph Π_1^0, we observe furthermore that for any $z < y$, $\exists t_z\,\neg\psi(x,z,t_z)$. The number s in $f(x)$ is the least upper bound on all such t_z, which exists by $B\Sigma_1^0$. It is now easy to check that f works, concluding the proof of Claim 2.4. □

We may think of $f(x)$ as the stage at which x is enumerated into $[0,k)$. We now construct a "tame" Σ_2^0-function g which enumerates the interval $[0,k)$. Here, "tameness" means that g settles down on all initial segments. This tameness constitutes the essential difference between g and f. More precisely, we have the following:

Claim 2.5. Let k and f be as in Claim 2.4. Then there are a Σ_2^0-cut I and a Σ_2^0-function $g : I \to [0,k)$, together with a recursive approximation h to g, such that

(a) g is 1-1 from I *onto* $[0,k)$;
(b) g is "tame"; i.e., for all $i \in I$, there is a stage s such that for all $j < i$ and all $t > s$, $h(j,t) = h(j,s) = g(j)$, so, in particular, $g \restriction i$ is $\mathcal{M}$-finite; and
(c) g is not "coded" on $I \times [0,k)$; i.e., $g \neq X \cap (I \times [0,k))$ for any $\mathcal{M}$-finite set X. (Informally, there is no $\mathcal{M}$-finite "end-extension" of (the graph of) g.)

Proof. We start with the definition of a function F. Let $\theta(x,y,u)$ be a Σ^0_0-formula such that $(x,y) \in f$ iff $\forall u\, \theta(x,y,u)$. For each $s \in M$, we will define $F(s)$ as the approximation to f at stage s; since f is a $1-1$ function, $F(s)$ can be made a $1-1$ function as well (possibly with a smaller domain). $F(s)$ is defined as follows. Set $(x,y) \in F(s)$ iff

$$x < k \wedge y \leq s \wedge \forall u < s\, \theta(x,y,u) \wedge \forall y' < y\, \exists u' < s\, \neg\theta(x,y',u')$$
$$\wedge \neg\exists x' < x\, [\forall u < s\, \theta(x',y,u) \wedge \forall y' < y\, \exists u' < s\, \neg\theta(x',y,u')].$$

Since $F(s)$ is an $\mathcal{M}$-finite set of pairs, we can list all its elements $(x_0,y_0),\dots,(x_e,y_e)$ (for some $e = e_s < k$, say) ordered by their second coordinates, i.e., such that $y_i < y_j$ iff $i < j$. We define $h(i,s) = x_i$ for all $i \leq e$. Formally, we define $h(i,s) = x$ iff there is $c < 2^{\langle k,s\rangle}$ which is a code of an $\mathcal{M}$-finite sequence $\langle c_0,\dots,c_i\rangle$ of length $i+1$, say, such that

(1) for each $j \leq i$, c_j is a pair $\langle x_j, y_j\rangle$;
(2) $x = x_i$;
(3) $\forall j \leq i\, ((x_j,y_j) \in F(s))$;
(4) $\forall j < k \leq i\, (y_j < y_k)$; and
(5) $\forall j < i\, \forall z < y_{j+1}\, [y_j < z \to \forall x < k\, ((x,z) \notin F(s))]$.

Let $I = \{i : \exists s\, \forall j \leq i\, \forall t > s\, [h(j,s) = h(j,t)]\}$ and, for each $i \in I$, let $g(i) = \lim_s h(i,s)$. We first note that since F and h are Δ^0_0, both I and the graph of g are Σ^0_2.

We now check that I, g and h satisfy statements (a)-(c) from the claim.

(a) We first show that g is $1-1$. Observe that if $g(i) = x$, then there is s such that $\forall t > s\, (h(i,t) = x)$; so, in particular, there is some y such that for all $t > s$, $(x,y) \in F(t)$, i.e., $f(x) = y$. Suppose that $i_1 < i_2$ are two elements in I, and that $g(i_1) = x_1$ and $g(i_2) = x_2$. By definition of I, there is a stage s such that g settles down at both i_1 and i_2. Thus there are y_1 and y_2 such that $(x_1,y_1),(x_2,y_2) \in F(t)$ for all $t > s$. By the choice of $F(t)$, $y_1 \neq y_2$ and thus $x_1 \neq x_2$.

Next we show that g is onto $[0,k)$. For any $m < k$, $f(m)$ is defined, thus $f \restriction [0,k) \times [0,f(m))$ is a bounded Π^0_1-set and hence coded. Therefore, its complement $[0,k) \times [0,f(m)) \setminus f$ is an $\mathcal{M}$-finite Σ^0_1-set. Now, by $B\Sigma^0_1$, there is a uniform bound s such that for each (x,y) in this complement, $\exists u < s\, \neg\theta(x,y,u)$. Hence for all $t > s$, $(x,y) \notin F(t)$. Hence, if $(m,f(m))$ is the e-th pair in $F(s)$, $g(e) = m$. This establishes (a).

(b) follows from the definition of g and h.

(c) Observe that I is indeed a cut. Suppose that $i \in I$. Let s be the (least) stage by which g settles down on $[0,i]$. At stage s, we will see the $\mathcal{M}$-finite set $F(s)$, say, $\{(x_0,y_0),\dots,(x_i,y_i)\}$, listed with increasing y-coordinates. Since f is unbounded, let z be the (least) number in the range of f such that $z > y_i$. Then $g(i+1) = z$. Finally, we show that g is not coded on $I \times [0,k)$. Suppose otherwise, say, $X \cap (I \times [0,k)) = g$ for some $\mathcal{M}$-finite set X. Then

$$g = \{(i,m) \in X : m < k \text{ and } i \text{ is the least } j \text{ such that } (j,m) \in X\},$$

which is $\mathcal{M}$-finite. Hence its domain I would be $\mathcal{M}$-finite, a contradiction. This concludes the proof of Claim 2.5. □

Finally, we use g and h to obtain a bi-tame cut J with its approximation $l(j,s)$.

We start with the interval $[0,k^k]$ and initially place two markers l and r at 0 and k^k, respectively. At each stage s, the construction is performed in e many steps, where e is the least number not in the domain of $h(\cdot,s)$. At the end of each step, we shrink the gap between l and r by a factor of k.

Step 0. Calculate $h(0,s)$. Set $l(0,s)=h(0,s)k^{k-1}$ and $r(0,s)=l(0,s)+k^{k-1}$.

Step i. Suppose $l(i-1,s)$ and $r(i-1,s)$ are the current positions of the markers and $r(i-1,s)-l(i-1)=k^{k-i}$. Calculate $h(i,s)$, and let $l(i,s)=l(i-1,s)+h(i,s)k^{k-i-1}$ and $r(i,s)=l(i,s)+k^{k-i-1}$.

Now let $J=\{x : \exists s\,\exists i\,\forall t>s\,\forall j<i\,[l(j,s)=l(j,t)\wedge x<l(i,s)]\}$ and $\bar{J}=\{x : \exists s\,\exists i\,\forall t>s\,\forall j<i\,[r(j,s)=r(j,t)\wedge x>r(i,s)]\}$. Then J and $\bar{J}$ are both Σ_2^0, and when h settles down on the initial segment $[0,i]$, then both $l(i,s)$ and $r(i,s)$ settle down as well. Clearly, J and $\bar{J}$ are disjoint, so it remains to show that $J\cup\bar{J}=[0,k^k]$, i.e., that there is no "gap" left. Suppose m belongs to the gap. Then write m as a k-ary number. We can then read out $g(i)$ from m for all $i\in I$, contradicting the fact that g is not coded on $I\times[0,k)$.

This concludes the proof of Lemma 2.3. □

Lemmas 2.2 and 2.3 now immediately establish Theorem 1.2 as desired.

3. The proof of Theorem 1.4

Using Theorem 1.2, it suffices to prove PART from D_2^2. So suppose that $(M,\prec)$ is a linear order in $\mathcal{M}$ such that for any $x\in M$, exactly one of $\{y\in M : y\prec x\}$ and $\{y\in M : x\prec y\}$ is $\mathcal{M}$-finite. Let A be the set of all $x\in M$ such that $\{y\in M : y\prec x\}$ is $\mathcal{M}$-finite or, equivalently, such that $\{y\in M : x\prec y\}$ is $\mathcal{M}$-infinite. Thus A is a Δ_2^0-definable subset of M. Applying D_2^2 (and by symmetry), let B be an infinite subset of A which exists in the second-order model $\mathcal{M}$. Then the $\prec$-downward closure C of B is a Σ_1^0-definable subset of M, and by our assumption on $(M,\prec)$, $C=A$. Now fix any $\mathcal{M}$-finite partition $\{a_i : i\le k\}$ of $\langle\mathcal{M},\prec\rangle$ (where a_0 and a_k are the least and greatest element), and assume that for each $i<k$, the interval $[a_i,a_{i+1}]$ is $\mathcal{M}$-finite. By Σ_1^0-induction, we then have that for each $i\le k$, the set $\{a_0,a_1,\dots,a_i\}$ is a subset of C, and thus $[a_0,a_i]$ is $\mathcal{M}$-finite. But clearly $a_k\notin C$, giving the desired contradiction.

As a final remark, we note that Jockusch later observed a shorter but less direct proof, using Hirschfeldt and Shore's result [7, Proposition 4.6] that SADS implies $B\Sigma_2^0$ and thus requiring only a proof of SADS from D_2^2 as in the first half of the previous paragraph. Once the infinite set B is obtained, one can argue immediately that it has order-type $\mathcal{M}$ or $\mathcal{M}^*$ and has thus established SADS.

References

[1] Cholak, Peter A., Carl G. Jockusch Jr. and Theodore A. Slaman, *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic **66** (2001), 1–55. MR1825173 (2002c:03094)

[2] Chong, C. T., *An α-finite injury method of the unbounded type*, J. Symbolic Logic **41** (1976), 1–17. MR0476456 (57:16019)

[3] Chong, C. T. and Yue Yang, *Σ_2-induction and infinite injury priority arguments. II. Tame Σ_2-coding and the jump operator*, Ann. Pure Appl. Logic **87** (1997), 103–116. MR1490049 (99m:03082)

[4] Erdős, Paul and Richard Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489. MR0081864 (18:458a)
[5] Dzhafarov, Damir D. and Jeffry L. Hirst, *The polarized Ramsey's theorem*, Arch. Math. Logic **48** (2009), 141–157. MR2487221
[6] Hájek, Petr and Pavel Pudlák, *Metamathematics of first-order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1993 (second printing, 1998). MR1748522 (2000m:03003)
[7] Hirschfeldt, Denis R. and Richard A. Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, J. Symbolic Logic **72** (2007), 171–206. MR2298478 (2007m:03115)
[8] Hirst, Jeffry L., *Combinatorics in subsystems of second order arithmetic*, Ph.D. Dissertation, Pennsylvania State University, 1987.
[9] Lerman, Manuel, *On suborderings of the α-recursively enumerable α-degrees*, Ann. Math. Logic **4** (1972), 369–392. MR0327493 (48:5835)
[10] Simpson, Stephen G., *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR1723993 (2001i:03126)
[11] Slaman, Theodore A., *Σ_n-bounding and Δ_n-induction*, Proc. Amer. Math. Soc. **132** (2004), 2449–2456. MR2052424 (2005b:03140)

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore
E-mail address: chongct@math.nus.edu.sg
URL: http://www.math.nus.edu.sg/~chongct

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin 53706-1388
E-mail address: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore
E-mail address: matyangy@math.nus.edu.sg
URL: http://www.math.nus.edu.sg/~matyangy