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On the Root-Class Residuality of Generalized Free Products with a Normal Amalgamation

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Abstract—We obtain both necessary and sufficient conditions for the free product of two groups with normal amalgamated subgroups to be a residually $\mathcal{C}$-group, where $\mathcal{C}$ is a root class of groups, which must be homomorphically closed in most cases.

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INTRODUCTION

Recall that a group X is said to be residually of class $\mathcal{L}$ (residually $\mathcal{L}$), if for any non-identity element $x \in X$ there exists a homomorphism σ of the group X onto some group from the class $\mathcal{L}$ such that the image of the element x under the homomorphism σ does not coincide with the identity.

Following K. Gruenberg [1], we define a root class of groups as a class of groups $\mathcal{K}$ containing at least one non-unit group, if the following three conditions are fulfilled:

1) if the group X belongs to the class $\mathcal{K}$ and Y is a subgroup of the group X, then the group Y also belongs to the class $\mathcal{K}$;

2) a direct product of any two groups from the class $\mathcal{K}$ belongs to the class $\mathcal{K}$;

3) the Gruenberg condition: If $1 \leqslant Z \leqslant Y \leqslant X$ is a subnormal series of the group X such that the factor groups X/Y and Y/Z belong to the class $\mathcal{K}$, then in the group X there exists a normal subgroup T such that $T \subseteq Z$ and the factor group X/T belongs to the class $\mathcal{K}$.

It is easy to see that many classes of groups that are under active investigation are root classes: the class of all finite groups; of finite p-groups, where p is a prime; of finite π-groups, where π is a nonempty set of primes; solvable groups; solvable torsion-free groups. Therefore, root-class residuality is a generalization of such intensively studied properties as residual finiteness, residual p-finiteness, residual solvability, and also allows to systematize and integrate into a single whole isolated results of residual theory.

In the above mentioned paper [1] K. Gruenberg showed that if $\mathcal{K}$ is a root class of groups such that each free group is residually $\mathcal{K}$, then the free product of an arbitrary family of the groups that are residually $\mathcal{K}$ is residually $\mathcal{K}$. Later D. N. Azarov and D. Tieudjo [2] established that each free group is residually of any root class, thus extending K. Gruenberg's statement onto an arbitrary root class of groups.

The root-class residuality of other free entities (generalized free products, HNN-extensions) was studied in [2–12]. Other properties of root classes of groups were considered in [12–15]. Among those E. V. Sokolov [12] obtained a characterization that allows to easily discriminate between root and non-root classes of groups. Namely, Sokolov showed that root classes are those and only those hereditary classes of groups that are closed under Cartesian junctions.

In this paper we consider the conditions for a group to be residually of a root class of groups that are closed under factorization, as well as arbitrary, of generalized free product of two groups, whose amalgamated subgroups are normal in respective free factors.

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